The N.P. and G.H.P. formalisms.

An application of *Spinors* and *xCoba.*

Alfonso García–Parrado Gómez–Lobo
Universidade do Minho, Portugal
September 2009

In this notebook we derive all the equations of the Newman–Penrose formalism from scratch, by using spinor calculus. We use the equations obtained to introduce the essentials of the GHP formalism (GHP operators, weighted quantities, etc) and then we obtain the set of GHP equations. This notebook can be used as starting point for any computation involving the N.P. and G.H.P. formalisms.

For a quick start, execute sections 1, 2 and 3 and then proceed to section 4 where the essential explanations about how to use this notebook are provided.

■ 1. Spin structure on the spacetime manifold

We start by loading `Spinors`.

In[1]:=
```
<< xAct`Spinors`
```

```
-----------------------------------------------------------

Package xAct`xPerm`  version 1.0.3, {2009, 9, 9}

CopyRight (C) 2003-2008, Jose M. Martin-Garcia, under the General Public License.

Connecting to external linux executable...

Connection established.

-----------------------------------------------------------

Package xAct`xTensor`  version 0.9.9, {2009, 9, 14}

CopyRight (C) 2002-2008, Jose M. Martin-Garcia, under the General Public License.

-----------------------------------------------------------

Package xAct`Spinors`  version 0.9.2, {2009, 9, 9}

CopyRight (C) 2006-2008, Alfonso Garcia-Parrado
  Gomez-Lobo and Jose M. Martin-Garcia, under the General Public License.
-----------------------------------------------------------

These packages come with ABSOLUTELY NO WARRANTY; for details type
  Disclaimer[]. This is free software, and you are welcome to redistribute
  it under certain conditions. See the General Public License for details.
-----------------------------------------------------------
```

Definition of a four dimensional manifold `M4`:

In[2]:=
```
DefManifold[M4, 4, {a, b, c, d, f, p, q, r, m, l, h, j, n, t, s}]
```

```
** DefManifold: Defining manifold M4.

** DefVBundle: Defining vbundle TangentM4.
```

Definition of a Lorentzian metric on the manifold `M4`:

In[3]:=
```
DefMetric[{3, 1, 0}, g[-a, -b], CD]
```

```
** DefTensor: Defining symmetric metric tensor g[-a, -b].

** DefTensor: Defining antisymmetric tensor epsilong[a, b, c, d].

** DefTensor: Defining tensor Tetrag[-a, -b, -c, -d].

** DefTensor: Defining tensor Tetrag†[-a, -b, -c, -d].

** DefCovD: Defining covariant derivative CD[-a].

** DefTensor: Defining vanishing torsion tensor TorsionCD[a, -b, -c].

** DefTensor: Defining symmetric Christoffel tensor ChristoffelCD[a, -b, -c].

** DefTensor: Defining Riemann tensor RiemannCD[-a, -b, -c, -d].

** DefTensor: Defining symmetric Ricci tensor RicciCD[-a, -b].

** DefCovD:  Contractions of Riemann automatically replaced by Ricci.

** DefTensor: Defining Ricci scalar RicciScalarCD[].

** DefCovD:  Contractions of Ricci automatically replaced by RicciScalar.

** DefTensor: Defining symmetric Einstein tensor EinsteinCD[-a, -b].

** DefTensor: Defining Weyl tensor WeylCD[-a, -b, -c, -d].

** DefTensor: Defining symmetric TFRicci tensor TFRicciCD[-a, -b].

** DefTensor: Defining Kretschmann scalar KretschmannCD[].

** DefCovD:  Computing RiemannToWeylRules for dim 4

** DefCovD:  Computing RicciToTFRicci for dim 4

** DefCovD:  Computing RicciToEinsteinRules for dim 4

** DefTensor: Defining weight +2 density Detg[]. Determinant.
```

We define a spin structure comprising the complex vector bundle `Spin` whose abstract indices are `{A, B, C, D, F, H, L, P, Q}`. We need to supply as arguments the name of the metric tensor of the background Lorentzian manifold `g[-a,-b]`, the name of the antisymmetric spin metric `∈[-A,-B]`, the name of the soldering form `σ[a,-A,-A†]` and the name of the spin covariant derivative `CDe[-a]`. The option `SpinorPrefix` is used to introduce a tag which will be prepended automatically to any spinor related to a spacetime tensor and the option `SpinorMark` introduces a formatting of the tag `SP` different to the default one. The long output of `DefSpinStructure` can be suppressed by adding the option `Info->False` which deactivates the information of the definition commands .

In[4]:=
```
DefSpinStructure[g, Spin, {A, B, C, D, F, H, L, P, Q},
 ∈, σ, CDe, SpinorPrefix -> SP, SpinorMark -> "S", Info -> False]
```

```
ValidateSymbol::capital : System name C is overloaded as an abstract index.

ValidateSymbol::capital : System name D is overloaded as an abstract index.
```

Pretty printing options for the spinor indices and the spin metric.

```
In[5]:= {PrintAs[A†] ^= "A′", PrintAs[B†] ^= "B′", PrintAs[C†] ^= "C′",
          PrintAs[D†] ^= "D′", PrintAs[F†] ^= "F′", PrintAs[H†] ^= "H′", PrintAs[L†] ^= "L′",
          PrintAs[M†] ^= "M′", PrintAs[P†] ^= "P′", PrintAs[Q†] ^= "Q′", PrintAs[ϵ ⧫] ^= "ϵ̄"};
```

2. The Newman–Penrose formalism

The Newman–Penrose formalism can be regarded as an application of the spinor calculus. In this section we show how all the equations of this formalism can be derived sytematically and efficiently using `Spinors`` in conjuntion with the package `xCoba``. Full explanations about the usage of `xCoba`` can be found in the package documentation.

We load `xCoba``.

```
In[6]:= << xAct`xCoba`
```

```
------------------------------------------------------------

Package xAct`xCoba`  version 0.7.0, {2009, 9, 9}

CopyRight (C) 2005-2009, David Yllanes
  and Jose M. Martin-Garcia, under the General Public License.
------------------------------------------------------------

These packages come with ABSOLUTELY NO WARRANTY; for details type
  Disclaimer[]. This is free software, and you are welcome to redistribute
  it under certain conditions. See the General Public License for details.
------------------------------------------------------------
```

We turn off the verbose messages issued by `ComponentValue`.

```
In[7]:= $CVVerbose = False;
```

In the following subsections we introduce the spin dyad and its associated null tetrad. We will use `xCoba`` to introduce two frames (basis), called `Dyad` and `NP` and we shall create a parallel set of vectors for the elements in each basis. See the corre–sponding subsections for full details.

Spin dyad

First of all, we define the elements of a spin dyad.

```
In[8]:= DefSpinor[#, M4] & /@ {o[A], ι[A]};
        PrintAs[o†] ^= "ō";
        PrintAs[ι ⧫] ^= "ῑ";
```

```
** DefTensor: Defining tensor o[A].

** DefTensor: Defining tensor o†[A†].

** DefTensor: Defining tensor ι[A].

** DefTensor: Defining tensor ι ⧫[A†].
```

These spinors belong to a spin basis which we define next.

```
In[11]:= DefBasis[Dyad, Spin, {0, 1}, BasisColor → RGBColor[1, 0, 0], Dagger → Complex]
```

```
** DefBasis: Defining basis Dyad.
** DefCovD: Defining parallel derivative PDDyad[-a].
** DefTensor: Defining torsion tensor TorsionPDDyad[a, -b, -c].
** DefTensor: Defining non-symmetric Christoffel tensor ChristoffelPDDyad[a, -b, -c].
** DefTensor: Defining vanishing Riemann tensor RiemannPDDyad[-a, -b, -c, d].
** DefTensor: Defining vanishing Ricci tensor RicciPDDyad[-a, -b].
** DefTensor: Defining nonsymmetric AChristoffel tensor AChristoffelPDDyad[A, -b, -C].
** DefTensor: Defining nonsymmetric AChristoffel tensor AChristoffelPDDyad†[A†, -b, -C†].
** DefTensor: Defining vanishing FRiemann tensor FRiemannPDDyad[-a, -b, -C, D].
** DefTensor: Defining vanishing FRiemann tensor FRiemannPDDyad†[-a, -b, -C†, D†].
** DefBasis: Defining basis Dyad†.
** DefTensor: Defining antisymmetric +1 density etaUpDyad[A, B].
** DefTensor: Defining antisymmetric +1 density etaUpDyad†[A†, B†].
** DefTensor: Defining antisymmetric -1 density etaDownDyad[-A, -B].
** DefTensor: Defining antisymmetric -1 density etaDownDyad†[-A†, -B†].
```

Component indices in the conjugated basis will be also primed :

In[12]:=
```
Dyad† /: CIndexForm[i_Integer, Dyad†] := StringJoin[ToString[i], "'"];
```

By definition, these are the equations relating both sets of objects:

In[13]:=
```
eqs = {o[A] == Basis[{0, -Dyad}, A], ι[A] == Basis[{1, -Dyad}, A]}
```

Out[13]= $\left\{o^{A} == e_{0}{}^{A},\ \iota^{A} == e_{1}{}^{A}\right\}$

From them, we read and store the components of the spinors o^{A}, ι^{A} .

In[14]:=
```
Map[ToBasis[Dyad], eqs, {2}]
```

Out[14]= $\left\{o^{A} == e_{0}{}^{A},\ \iota^{A} == e_{1}{}^{A}\right\}$

We now do the same for the conjugated equations:

In[15]:=
```
Map[ToBasis[Dyad†], Dagger[eqs], {2}]
```

Out[15]= $\left\{\overline{o}^{A'} == e_{0'}{}^{A'},\ \overline{\iota}^{A'} == e_{1'}{}^{A'}\right\}$

In[16]:=
```
ComponentArray@Join[%%, %]
```

Out[16]= $\left\{\left\{o^{0} == 1,\ o^{1} == 0\right\},\ \left\{\iota^{0} == 0,\ \iota^{1} == 1\right\},\ \left\{\overline{o}^{0'} == 1,\ \overline{o}^{1'} == 0\right\},\ \left\{\overline{\iota}^{0'} == 0,\ \overline{\iota}^{1'} == 1\right\}\right\}$

In[17]:=
```
% /. Equal -> ComponentValue;
```

Since $\{o^A, \iota^A\}$ is a spin dyad, the spin metric can be written in the form

In[18]:=
```
eqs = {ϵ[A, B] == o[A] ι[B] - o[B] ι[A]}
```

Out[18]= $\{\epsilon^{AB} == -o^B \iota^A + o^A \iota^B\}$

The components of the spin metric and its complex conjugate can now be computed and stored.

In[19]:=
```
Map[ToBasis[Dyad], eqs, {2}]
```

Out[19]= $\{\epsilon^{A\,B} == -o^B \iota^A + o^A \iota^B\}$

In[20]:=
```
Map[ToBasis[Dyad†], Dagger[eqs], {2}]
```

Out[20]= $\{\overline{\epsilon}^{A'\,B'} == -\overline{o}^{B'}\,\overline{\iota}^{A'} + \overline{o}^{A'}\,\overline{\iota}^{B'}\}$

In[21]:=
```
ToValues@ComponentArray@Join[%%, %]
```

Out[21]= $\{\{\{\epsilon^{0\,0} == 0, \epsilon^{0\,1} == 1\}, \{\epsilon^{1\,0} == -1, \epsilon^{1\,1} == 0\}\}, \{\{\overline{\epsilon}^{0'\,0'} == 0, \overline{\epsilon}^{0'\,1'} == 1\}, \{\overline{\epsilon}^{1'\,0'} == -1, \overline{\epsilon}^{1'\,1'} == 0\}\}\}$

In[22]:=
```
% /. Equal -> ComponentValue;
```

We deduce from here the components of ϵ_{AB} by means of the relation $\epsilon_{AB}\,\epsilon^{CB} = \delta_A{}^C$. To that end we need to find the inverse of the transposed of the matrix representation of the components of ϵ^{AB}.

In[23]:=
```
EP = ToValues@TableOfComponents[ϵ[A, B], Dyad]
```

Out[23]= {{0, 1}, {-1, 0}}

In[24]:=
```
Inverse@Transpose@EP
```

Out[24]= {{0, 1}, {-1, 0}}

Storage of the components of ϵ^{AB} and its complex conjugate.

In[25]:=
```
MetricInBasis[ϵ, -Dyad, %];
MetricInBasis[ϵ ↾ -Dyad†, %%];
```

We now compute and store the components of o_A, ι_A and their complex conjugate.

In[27]:=
```
eqs = {o[-A] == ϵ[-B, -A] o[B], ι[-A] == ϵ[-B, -A] ι[B]}
```

Out[27]= $\{o_A == o^B \epsilon_{BA}, \iota_A == \epsilon_{BA}\,\iota^B\}$

In[28]:=
```
Map[ToBasis[Dyad], eqs, {2}]
```

Out[28]= $\{o_A == o^B \epsilon_{B\,A}, \iota_A == \epsilon_{B\,A}\,\iota^B\}$

In[29]:=
```
Map[ToBasis[Dyad†], Dagger[eqs], {2}]
```

Out[29]= $\{\overline{o}_{A'} == \overline{o}^{B'}\,\overline{\epsilon}_{B',A'}, \overline{\iota}_{A'} == \overline{\epsilon}_{B',A'}\,\overline{\iota}^{B'}\}$

In[30]:=
```
Map[TraceBasisDummy, Join[%%, %], {2}]
```

Out[30]= $\{o_A == o^0 \epsilon_{0\,A} + o^1 \epsilon_{1\,A}, \iota_A == \epsilon_{0\,A}\,\iota^0 + \epsilon_{1\,A}\,\iota^1, \overline{o}_{A'} == \overline{o}^{0'}\,\overline{\epsilon}_{0',A'} + \overline{o}^{1'}\,\overline{\epsilon}_{1',A'}, \overline{\iota}_{A'} == \overline{\epsilon}_{0',A'}\,\overline{\iota}^{0'} + \overline{\epsilon}_{1',A'}\,\overline{\iota}^{1'}\}$

In[31]:=
```
ToValues@ComponentArray@%
```

Out[31]= $\{\{o_0 == 0, o_1 == 1\}, \{\iota_0 == -1, \iota_1 == 0\}, \{\overline{o}_{0'} == 0, \overline{o}_{1'} == 1\}, \{\overline{\iota}_{0'} == -1, \overline{\iota}_{1'} == 0\}\}$

In[32]:=
```
% /. Equal -> ComponentValue;
```

We change the `Rule` assignment to the `Set` assignment in each `FoldedRule` storing the components computed so far

In[33]:=
```
RuleToSet /@ {ϵ, ϵ †, o, o†, ι, ι †};
```

- **Null tetrad**

We define now four vectors which will represent the null tetrad basis elements.

In[34]:=
```
DefTensor[lNP[a], M4, PrintAs → "l"]
```

```
** DefTensor: Defining tensor lNP[a].
```

In[35]:=
```
DefTensor[nNP[a], M4, PrintAs → "n"]
```

```
** DefTensor: Defining tensor nNP[a].
```

In[36]:=
```
DefTensor[mNP[a], M4, Dagger → Complex, PrintAs → "m"]
```

```
** DefTensor: Defining tensor mNP[a].
```

```
** DefTensor: Defining tensor mNP†[a].
```

In[37]:=
```
PrintAs@mNP† ^= "m̅";
```

These vectors belong to a basis (null tetrad) which we define next.

In[38]:=
```
DefBasis[NP, TangentM4, {1, 2, 3, 4}, BasisColor -> RGBColor[0, 0, 1]]
```

```
** DefBasis: Defining basis NP.
```

```
** DefCovD: Defining parallel derivative PDNP[-a].
```

```
** DefTensor: Defining torsion tensor TorsionPDNP[a, -b, -c].
```

```
** DefTensor: Defining non-symmetric Christoffel tensor ChristoffelPDNP[a, -b, -c].
```

```
** DefTensor: Defining vanishing Riemann tensor RiemannPDNP[-a, -b, -c, d].
```

```
** DefTensor: Defining vanishing Ricci tensor RicciPDNP[-a, -b].
```

```
** DefTensor: Defining antisymmetric +1 density etaUpNP[a, b, c, d].
```

```
** DefTensor: Defining antisymmetric -1 density etaDownNP[-a, -b, -c, -d].
```

We specify the behaviour of the elements of this basis under complex conjugation. This is achieved with the command **`SetDaggerMatrix`**.

In[39]:=
```
SetDaggerMatrix[NP, {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}}]
```

We format a number of objects according to standard conventions. Due to the new way of sorting definitions in *Mathematica* 6.0, we need to stop the formatting engine and start it again after defining these new rules.

Stop formatting engine.

In[40]:=
```
xTensorFormStop[];
```

Formatting of the directional derivatives with the standard NP symbols.

In[41]:=
```
FormatBasis[PDNP[{1, -NP}], "D"];
FormatBasis[PDNP[{2, -NP}], "Δ"];
FormatBasis[PDNP[{3, -NP}], "δ"];
FormatBasis[PDNP[{4, -NP}], "δ̄"];
```

The Newman–Penrose directional derivatives can also act on spinors. For that case we need to use the covariant derivative `CDe` rather than `PDNP`. Therefore we need to set a formatting for the directional derivatives taken with respect to `CDe` (note that when `CDe` and `PDNP` act on scalars or tensors they coincide).

In[45]:=
```
FormatBasis[CDe[{1, -NP}], "D"];
FormatBasis[CDe[{2, -NP}], "Δ"];
FormatBasis[CDe[{3, -NP}], "δ"];
FormatBasis[CDe[{4, -NP}], "δ̄"];
```

Restart formatting engine:

In[49]:=
```
xTensorFormStart[];
```

These are the relations between the elements of the null tetrad and the elements of the basis `NP`.

In[50]:=
```
{lNP[a] == Basis[a, {1, -NP}], nNP[a] == Basis[a, {2, -NP}],
 mNP[a] == Basis[a, {3, -NP}], mNP†[a] == Basis[a, {4, -NP}]}
```

Out[50]=
$$\left\{l^a == e_1{}^a,\ n^a == e_2{}^a,\ m^a == e_3{}^a,\ \overline{m}^a == e_4{}^a\right\}$$

From this relation we compute the components of l^a, n^a, m^a, $\overline{m}^a$ in the basis `NP` .

In[51]:=
```
Map[ToBasis[NP], %, {2}]
```

Out[51]=
$$\left\{l^a == e_1{}^a,\ n^a == e_2{}^a,\ m^a == e_3{}^a,\ \overline{m}^a == e_4{}^a\right\}$$

In[52]:=
```
Flatten@ComponentArray@%
```

Out[52]=
$$\left\{l^1 == 1,\ l^2 == 0,\ l^3 == 0,\ l^4 == 0,\ n^1 == 0,\ n^2 == 1,\ n^3 == 0,\ n^4 == 0,\ m^1 == 0,\ m^2 == 0,\ m^3 == 1,\ m^4 == 0,\ \overline{m}^1 == 0,\ \overline{m}^2 == 0,\ \overline{m}^3 == 0,\ \overline{m}^4 == 1\right\}$$

In[53]:=
```
% /. Equal → ComponentValue;
```

As is well known, in the null tetrad the metric takes the form

In[54]:=
```
g[a, b] == lNP[a] nNP[b] + lNP[b] nNP[a] - (mNP[a] mNP†[b] + mNP[b] mNP†[a])
```

Out[54]=
$$g^{ab} == -m^b\,\overline{m}^a - m^a\,\overline{m}^b + l^b\,n^a + l^a\,n^b$$

We compute and store the components of g^{ab} in the basis `NP`.

In[55]:=
```
ToBasis[NP] /@ %
```

Out[55]=
$$g^{a\,b} == -m^b\,\overline{m}^a - m^a\,\overline{m}^b + l^b\,n^a + l^a\,n^b$$

In[56]:=
```
ToValues@ComponentArray@%
```

Out[56]=
$$\left\{\left\{g^{1\,1} == 0,\ g^{1\,2} == 1,\ g^{1\,3} == 0,\ g^{1\,4} == 0\right\},\ \left\{g^{2\,1} == 1,\ g^{2\,2} == 0,\ g^{2\,3} == 0,\ g^{2\,4} == 0\right\},\ \left\{g^{3\,1} == 0,\ g^{3\,2} == 0,\ g^{3\,3} == 0,\ g^{3\,4} == -1\right\},\ \left\{g^{4\,1} == 0,\ g^{4\,2} == 0,\ g^{4\,3} == -1,\ g^{4\,4} == 0\right\}\right\}$$

We store the components of the metric tensor

In[57]:=
```
% /. Equal -> ComponentValue
```

Out[57]=
$$\left\{\left\{g^{1\,1}\to 0,\ g^{1\,2}\to 1,\ g^{1\,3}\to 0,\ g^{1\,4}\to 0\right\},\ \left\{g^{2\,1}\to 1,\ g^{2\,2}\to 0,\ g^{2\,3}\to 0,\ g^{2\,4}\to 0\right\},\ \left\{g^{3\,1}\to 0,\ g^{3\,2}\to 0,\ g^{3\,3}\to 0,\ g^{3\,4}\to -1\right\},\ \left\{g^{4\,1}\to 0,\ g^{4\,2}\to 0,\ g^{4\,3}\to -1,\ g^{4\,4}\to 0\right\}\right\}$$

We compute the components of g_{ab} and store their values.

In[58]:=
```
MatrixForm@ToValues@TableOfComponents[g[a, b], NP]
```

Out[58]//MatrixForm=
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

In[59]:=
```
MatrixForm@Inverse@%
```

Out[59]//MatrixForm=
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

In[60]:=
```
MetricInBasis[g, -NP, %]
```

Out[60]=
$$\left\{\left\{g_{1\,1}\to 0,\ g_{1\,2}\to 1,\ g_{1\,3}\to 0,\ g_{1\,4}\to 0\right\},\ \left\{g_{2\,1}\to 1,\ g_{2\,2}\to 0,\ g_{2\,3}\to 0,\ g_{2\,4}\to 0\right\},\ \left\{g_{3\,1}\to 0,\ g_{3\,2}\to 0,\ g_{3\,3}\to 0,\ g_{3\,4}\to -1\right\},\ \left\{g_{4\,1}\to 0,\ g_{4\,2}\to 0,\ g_{4\,3}\to -1,\ g_{4\,4}\to 0\right\}\right\}$$

Finally we change the `Rule` assignment to the `Set` assignment in the `FoldedRule` storing all the components of the metric tensor.

In[61]:=
```
RuleToSet@g
```

We also need the components for the elements of the null tetrad with the indices downstairs. They are computed from those with the indices upstairs.

In[62]:=
```
{lNP[-a] == g[-a, -b] lNP[b], nNP[-a] == g[-a, -b] nNP[b],
 mNP[-a] == g[-a, -b] mNP[b], mNP†[-a] == g[-a, -b] mNP†[b]}
```

Out[62]=
$$\left\{l_a == g_{ab}\,l^b,\ n_a == g_{ab}\,n^b,\ m_a == g_{ab}\,m^b,\ \overline{m}_a == g_{ab}\,\overline{m}^b\right\}$$

In[63]:=
```
Map[ToBasis[NP], %, {2}]
```

Out[63]=
$$\left\{l_a == g_{a\,b}\,l^b,\ n_a == g_{a\,b}\,n^b,\ m_a == g_{a\,b}\,m^b,\ \overline{m}_a == g_{a\,b}\,\overline{m}^b\right\}$$

In[64]:=
```
Map[TraceBasisDummy, %, {2}]
```

Out[64]=
$$\left\{l_a == g_{a\,1}\,l^1 + g_{a\,2}\,l^2 + g_{a\,3}\,l^3 + g_{a\,4}\,l^4,\ n_a == g_{a\,1}\,n^1 + g_{a\,2}\,n^2 + g_{a\,3}\,n^3 + g_{a\,4}\,n^4,\ m_a == g_{a\,1}\,m^1 + g_{a\,2}\,m^2 + g_{a\,3}\,m^3 + g_{a\,4}\,m^4,\ \overline{m}_a == g_{a\,1}\,\overline{m}^1 + g_{a\,2}\,\overline{m}^2 + g_{a\,3}\,\overline{m}^3 + g_{a\,4}\,\overline{m}^4\right\}$$

In[65]:=
```
ToValues@Flatten@ComponentArray@%
```

Out[65]=
$$\{l_1 == 0,\ l_2 == 1,\ l_3 == 0,\ l_4 == 0,\ n_1 == 1,\ n_2 == 0,\ n_3 == 0,\ n_4 == 0,\ m_1 == 0,\ m_2 == 0,\ m_3 == 0,\ m_4 == -1,\ \overline{m}_1 == 0,\ \overline{m}_2 == 0,\ \overline{m}_3 == -1,\ \overline{m}_4 == 0\}$$

In[66]:=
```
% /. Equal → ComponentValue;
```

Automatic replacement of the values of the components of the elements of the null tetrad.

In[67]:=
```
RuleToSet /@ {lNP, nNP, mNP, mNP†};
```

It is convenient to have a rule which relates l^a, n^a, m^a, $\overline{m}^a$ and the elements of the basis NP. We compute such a rule next.

In[68]:=
```
{lNP[a] == SeparateBasis[NP]@lNP[a], nNP[a] == SeparateBasis[NP]@nNP[a],
 mNP[a] == SeparateBasis[NP]@mNP[a], mNP†[a] == SeparateBasis[NP]@mNP†[a]}
```

Out[68]=
$$\left\{l^a == e_b{}^a\, l^b,\ n^a == e_b{}^a\, n^b,\ m^a == e_b{}^a\, m^b,\ \overline{m}^a == e_b{}^a\, \overline{m}^b\right\}$$

In[69]:=
```
Map[TraceBasisDummy, %, {2}]
```

Out[69]=
$$\left\{l^a == e_1{}^a,\ n^a == e_2{}^a,\ m^a == e_3{}^a,\ \overline{m}^a == e_4{}^a\right\}$$

In[70]:=
```
Solve[%, Cases[%, Basis[{__}, a_], Infinity]] // Flatten
```

Out[70]=
$$\left\{e_1{}^a \to l^a,\ e_2{}^a \to n^a,\ e_3{}^a \to m^a,\ e_4{}^a \to \overline{m}^a\right\}$$

In[71]:=
```
% /. HoldPattern@Rule[X_, Y_] :> MakeRule[{X, Y}] // Flatten // Union
```

Out[71]=
$$\left\{\text{HoldPattern}\left[e_1^{\overset{a}{=}}\right] :\to \text{Module}[\{\}, l^a],\ \text{HoldPattern}\left[e_2^{\overset{a}{=}}\right] :\to \text{Module}[\{\}, n^a],\right.$$
$$\left.\text{HoldPattern}\left[e_3^{\overset{a}{=}}\right] :\to \text{Module}[\{\}, m^a],\ \text{HoldPattern}\left[e_4^{\overset{a}{=}}\right] :\to \text{Module}[\{\}, \overline{m}^a]\right\}$$

In[72]:=
```
BasisToVectors = %;
```

We need the same rule with the indices downstairs.

In[73]:=
```
{lNP[-a] == SeparateBasis[NP]@lNP[-a], nNP[-a] == SeparateBasis[NP]@nNP[-a],
 mNP[-a] == SeparateBasis[NP]@mNP[-a], mNP†[-a] == SeparateBasis[NP]@mNP†[-a]}
```

Out[73]=
$$\left\{l_a == e_a{}^b\, l_b,\ n_a == e_a{}^b\, n_b,\ m_a == e_a{}^b\, m_b,\ \overline{m}_a == e_a{}^b\, \overline{m}_b\right\}$$

In[74]:=
```
Map[TraceBasisDummy, %, {2}]
```

Out[74]=
$$\left\{l_a == e_a{}^2,\ n_a == e_a{}^1,\ m_a == -e_a{}^4,\ \overline{m}_a == -e_a{}^3\right\}$$

In[75]:=
```
Solve[%, Cases[%, Basis[__], Infinity]] // Flatten
```

Out[75]=
$$\left\{e_a{}^2 \to l_a,\ e_a{}^1 \to n_a,\ e_a{}^4 \to -m_a,\ e_a{}^3 \to -\overline{m}_a\right\}$$

In[76]:=
```
% /. HoldPattern@Rule[X_, Y_] :> MakeRule[{X, Y}] // Flatten // Union
```

Out[76]=
$$\left\{\text{HoldPattern}\left[e^{\overset{a}{=}1}\right] :\to \text{Module}[\{\}, n^a],\ \text{HoldPattern}\left[e^{\overset{a}{=}2}\right] :\to \text{Module}[\{\}, l^a],\right.$$
$$\left.\text{HoldPattern}\left[e^{\overset{a}{=}3}\right] :\to \text{Module}[\{\}, -\overline{m}^a],\ \text{HoldPattern}\left[e^{\overset{a}{=}4}\right] :\to \text{Module}[\{\}, -m^a]\right\}$$

We combine these relations with the relations found above for the indices upstairs.

In[77]:=
```
AppendTo[BasisToVectors, %]
```

Out[77]=
$$\left\{\text{HoldPattern}\left[e_1^{\overset{a}{=}}\right] :\to \text{Module}[\{\}, l^a],\ \text{HoldPattern}\left[e_2^{\overset{a}{=}}\right] :\to \text{Module}[\{\}, n^a],\right.$$
$$\text{HoldPattern}\left[e_3^{\overset{a}{=}}\right] :\to \text{Module}[\{\}, m^a],\ \text{HoldPattern}\left[e_4^{\overset{a}{=}}\right] :\to \text{Module}[\{\}, \overline{m}^a],$$
$$\left\{\text{HoldPattern}\left[e^{\overset{a}{=}1}\right] :\to \text{Module}[\{\}, n^a],\ \text{HoldPattern}\left[e^{\overset{a}{=}2}\right] :\to \text{Module}[\{\}, l^a],\right.$$
$$\left.\left.\text{HoldPattern}\left[e^{\overset{a}{=}3}\right] :\to \text{Module}[\{\}, -\overline{m}^a],\ \text{HoldPattern}\left[e^{\overset{a}{=}4}\right] :\to \text{Module}[\{\}, -m^a]\right\}\right\}$$

Relation between the spin dyad and the null tetrad

The relation between the elements of the spinor dyad and the elements of the null tetrad is as follows

In[78]:=
```
SpinDyadToNullTetrad =
 {σ[a, -A, -A†] o[A] Dagger@o[A] == lNP[a], σ[a, -A, -A†] ι[A] Dagger@ι[A] == nNP[a],
  σ[a, -A, -A†] o[A] Dagger@ι[A] == mNP[a], σ[a, -A, -A†] Dagger[o[A] Dagger@ι[A]] == mNP†[a]}
```

Out[78]= $\left\{o^{A}\,\overline{o}^{A'}\,\sigma^{a}{}_{AA'} == l^{a},\ \iota^{A}\,\overline{\iota}^{A'}\,\sigma^{a}{}_{AA'} == n^{a},\ o^{A}\,\overline{\iota}^{A'}\,\sigma^{a}{}_{AA'} == m^{a},\ \overline{o}^{A'}\,\iota^{A}\,\sigma^{a}{}_{AA'} == \overline{m}^{a}\right\}$

Using this relation we find and store the components of the soldering form $\sigma^{a}{}_{AA'}$.

In[79]:=
```
Map[ToBasis[Dyad], %, {2}]
```

Out[79]= $\left\{o^{A}\,\overline{o}^{A'}\,\sigma^{a}{}_{A\,A'} == l^{a},\ \iota^{A}\,\overline{\iota}^{A'}\,\sigma^{a}{}_{A\,A'} == n^{a},\ o^{A}\,\overline{\iota}^{A'}\,\sigma^{a}{}_{A\,A'} == m^{a},\ \overline{o}^{A'}\,\iota^{A}\,\sigma^{a}{}_{A\,A'} == \overline{m}^{a}\right\}$

In[80]:=
```
Map[ToBasis[Dyad†], %, {2}]
```

Out[80]= $\left\{o^{A}\,\overline{o}^{A'}\,\sigma^{a}{}_{A\,A'} == l^{a},\ \iota^{A}\,\overline{\iota}^{A'}\,\sigma^{a}{}_{A\,A'} == n^{a},\ o^{A}\,\overline{\iota}^{A'}\,\sigma^{a}{}_{A\,A'} == m^{a},\ \overline{o}^{A'}\,\iota^{A}\,\sigma^{a}{}_{A\,A'} == \overline{m}^{a}\right\}$

In[81]:=
```
Map[ToBasis[NP], %, {2}]
```

Out[81]= $\left\{o^{A}\,\overline{o}^{A'}\,\sigma^{a}{}_{A\,A'} == l^{a},\ \iota^{A}\,\overline{\iota}^{A'}\,\sigma^{a}{}_{A\,A'} == n^{a},\ o^{A}\,\overline{\iota}^{A'}\,\sigma^{a}{}_{A\,A'} == m^{a},\ \overline{o}^{A'}\,\iota^{A}\,\sigma^{a}{}_{A\,A'} == \overline{m}^{a}\right\}$

In[82]:=
```
Map[TraceBasisDummy, %, {2}]
```

Out[82]= $\left\{\sigma^{a}{}_{0\,0'} == l^{a},\ \sigma^{a}{}_{1\,1'} == n^{a},\ \sigma^{a}{}_{0\,1'} == m^{a},\ \sigma^{a}{}_{1\,0'} == \overline{m}^{a}\right\}$

In[83]:=
```
ComponentArray@%
```

Out[83]= $\left\{\left\{\sigma^{1}{}_{0\,0'} == 1,\ \sigma^{2}{}_{0\,0'} == 0,\ \sigma^{3}{}_{0\,0'} == 0,\ \sigma^{4}{}_{0\,0'} == 0\right\},\ \left\{\sigma^{1}{}_{1\,1'} == 0,\ \sigma^{2}{}_{1\,1'} == 1,\ \sigma^{3}{}_{1\,1'} == 0,\ \sigma^{4}{}_{1\,1'} == 0\right\},\right.$
$\left.\left\{\sigma^{1}{}_{0\,1'} == 0,\ \sigma^{2}{}_{0\,1'} == 0,\ \sigma^{3}{}_{0\,1'} == 1,\ \sigma^{4}{}_{0\,1'} == 0\right\},\ \left\{\sigma^{1}{}_{1\,0'} == 0,\ \sigma^{2}{}_{1\,0'} == 0,\ \sigma^{3}{}_{1\,0'} == 0,\ \sigma^{4}{}_{1\,0'} == 1\right\}\right\}$

In[84]:=
```
% /. Equal → ComponentValue
```

Out[84]= $\left\{\left\{\sigma^{1}{}_{0\,0'} \to 1,\ \sigma^{2}{}_{0\,0'} \to 0,\ \sigma^{3}{}_{0\,0'} \to 0,\ \sigma^{4}{}_{0\,0'} \to 0\right\},\ \left\{\sigma^{1}{}_{1\,1'} \to 0,\ \sigma^{2}{}_{1\,1'} \to 1,\ \sigma^{3}{}_{1\,1'} \to 0,\ \sigma^{4}{}_{1\,1'} \to 0\right\},\right.$
$\left.\left\{\sigma^{1}{}_{0\,1'} \to 0,\ \sigma^{2}{}_{0\,1'} \to 0,\ \sigma^{3}{}_{0\,1'} \to 1,\ \sigma^{4}{}_{0\,1'} \to 0\right\},\ \left\{\sigma^{1}{}_{1\,0'} \to 0,\ \sigma^{2}{}_{1\,0'} \to 0,\ \sigma^{3}{}_{1\,0'} \to 0,\ \sigma^{4}{}_{1\,0'} \to 1\right\}\right\}$

We also need the components of the soldering form with the spacetime index downstairs and the spin indices upstairs.

In[85]:=
```
NullTetradToSpinDyad = Map[PutSolderingForm, SpinDyadToNullTetrad, {2}] /. B -> A /. B† -> A†
```

Out[85]= $\left\{o^{A}\,\overline{o}^{A'} == l^{a}\,\sigma_{a}{}^{AA'},\ \iota^{A}\,\overline{\iota}^{A'} == n^{a}\,\sigma_{a}{}^{AA'},\ o^{A}\,\overline{\iota}^{A'} == m^{a}\,\sigma_{a}{}^{AA'},\ \overline{o}^{A'}\,\iota^{A} == \overline{m}^{a}\,\sigma_{a}{}^{AA'}\right\}$

In[86]:=
```
Map[ToBasis[Dyad], %, {2}]
```

Out[86]= $\left\{o^{A}\,\overline{o}^{A'} == l^{a}\,\sigma_{a}{}^{A\,A'},\ \iota^{A}\,\overline{\iota}^{A'} == n^{a}\,\sigma_{a}{}^{A\,A'},\ o^{A}\,\overline{\iota}^{A'} == m^{a}\,\sigma_{a}{}^{A\,A'},\ \overline{o}^{A'}\,\iota^{A} == \overline{m}^{a}\,\sigma_{a}{}^{A\,A'}\right\}$

In[87]:=
```
Map[ToBasis[Dyad†], %, {2}]
```

Out[87]= $\left\{o^{A}\,\overline{o}^{A'} == l^{a}\,\sigma_{a}{}^{A\,A'},\ \iota^{A}\,\overline{\iota}^{A'} == n^{a}\,\sigma_{a}{}^{A\,A'},\ o^{A}\,\overline{\iota}^{A'} == m^{a}\,\sigma_{a}{}^{A\,A'},\ \overline{o}^{A'}\,\iota^{A} == \overline{m}^{a}\,\sigma_{a}{}^{A\,A'}\right\}$

In[88]:=
```
Map[ToBasis[NP], %, {2}]
```

Out[88]= $\left\{o^{A}\,\overline{o}^{A'} == l^{a}\,\sigma_{a}{}^{A\,A'},\ \iota^{A}\,\overline{\iota}^{A'} == n^{a}\,\sigma_{a}{}^{A\,A'},\ o^{A}\,\overline{\iota}^{A'} == m^{a}\,\sigma_{a}{}^{A\,A'},\ \overline{o}^{A'}\,\iota^{A} == \overline{m}^{a}\,\sigma_{a}{}^{A\,A'}\right\}$

In[89]:= **Map[TraceBasisDummy, %, {2}]**

Out[89]= $\{o^{A}\,\overline{o}^{A'} == \sigma_{1}{}^{A\,A'},\ \iota^{A}\,\overline{\iota}^{A'} == \sigma_{2}{}^{A\,A'},\ o^{A}\,\overline{\iota}^{A'} == \sigma_{3}{}^{A\,A'},\ \overline{o}^{A'}\,\iota^{A} == \sigma_{4}{}^{A\,A'}\}$

In[90]:= **Flatten@ComponentArray@%**

Out[90]= $\{1 == \sigma_{1}{}^{0\,0'},\ 0 == \sigma_{1}{}^{0\,1'},\ 0 == \sigma_{1}{}^{1\,0'},\ 0 == \sigma_{1}{}^{1\,1'},\ 0 == \sigma_{2}{}^{0\,0'},\ 0 == \sigma_{2}{}^{0\,1'},\ 0 == \sigma_{2}{}^{1\,0'},\ 1 == \sigma_{2}{}^{1\,1'},$
$0 == \sigma_{3}{}^{0\,0'},\ 1 == \sigma_{3}{}^{0\,1'},\ 0 == \sigma_{3}{}^{1\,0'},\ 0 == \sigma_{3}{}^{1\,1'},\ 0 == \sigma_{4}{}^{0\,0'},\ 0 == \sigma_{4}{}^{0\,1'},\ 1 == \sigma_{4}{}^{1\,0'},\ 0 == \sigma_{4}{}^{1\,1'}\}$

In[91]:= **Solve[%, Cases[%, _σ, Infinity]]**

Out[91]= $\{\{\sigma_{1}{}^{0\,0'} \to 1,\ \sigma_{1}{}^{0\,1'} \to 0,\ \sigma_{1}{}^{1\,0'} \to 0,\ \sigma_{1}{}^{1\,1'} \to 0,\ \sigma_{2}{}^{0\,0'} \to 0,\ \sigma_{2}{}^{0\,1'} \to 0,\ \sigma_{2}{}^{1\,0'} \to 0,\ \sigma_{2}{}^{1\,1'} \to 1,$
$\sigma_{3}{}^{0\,0'} \to 0,\ \sigma_{3}{}^{0\,1'} \to 1,\ \sigma_{3}{}^{1\,0'} \to 0,\ \sigma_{3}{}^{1\,1'} \to 0,\ \sigma_{4}{}^{0\,0'} \to 0,\ \sigma_{4}{}^{0\,1'} \to 0,\ \sigma_{4}{}^{1\,0'} \to 1,\ \sigma_{4}{}^{1\,1'} \to 0\}\}$

We store these values.

In[92]:= **% /. Rule → ComponentValue**

Out[92]= $\{\{\sigma_{1}{}^{0\,0'} \to 1,\ \sigma_{1}{}^{0\,1'} \to 0,\ \sigma_{1}{}^{1\,0'} \to 0,\ \sigma_{1}{}^{1\,1'} \to 0,\ \sigma_{2}{}^{0\,0'} \to 0,\ \sigma_{2}{}^{0\,1'} \to 0,\ \sigma_{2}{}^{1\,0'} \to 0,\ \sigma_{2}{}^{1\,1'} \to 1,$
$\sigma_{3}{}^{0\,0'} \to 0,\ \sigma_{3}{}^{0\,1'} \to 1,\ \sigma_{3}{}^{1\,0'} \to 0,\ \sigma_{3}{}^{1\,1'} \to 0,\ \sigma_{4}{}^{0\,0'} \to 0,\ \sigma_{4}{}^{0\,1'} \to 0,\ \sigma_{4}{}^{1\,0'} \to 1,\ \sigma_{4}{}^{1\,1'} \to 0\}\}$

Change of the `Rule` assignment to the `Set` assignment in the folded rule.

In[93]:= **RuleToSet@σ**

■ Newman–Penrose spin coefficients

We define the 12 Newmann–Penrose spin coefficients, and their complex conjugates:

In[94]:=
```
DefSpinor[κ Dyad[], M4];
PrintAs[κ Dyad] ^= "κ";
PrintAs[κ Dyad†] ^= "κ̄";
```

```
** DefTensor: Defining tensor κ Dyad[].

** DefTensor: Defining tensor κ Dyad†[].
```

In[97]:=
```
DefSpinor[ϵ Dyad[], M4];
PrintAs[ϵ Dyad] ^= "ϵ";
PrintAs[ϵ Dyad†] ^= "ϵ̄";
```

```
** DefTensor: Defining tensor ϵ Dyad[].

** DefTensor: Defining tensor ϵ Dyad†[].
```

In[100]:=
```
DefSpinor[π Dyad[], M4];
PrintAs[π Dyad] ^= "π";
PrintAs[π Dyad†] ^= "π̄";
```

```
** DefTensor: Defining tensor π Dyad[].

** DefTensor: Defining tensor π Dyad†[].
```

In[103]:=
```
DefSpinor[τ Dyad[], M4];
PrintAs[τ Dyad] ^= "τ";
PrintAs[τ Dyad†] ^= "τ̄";
```

```
** DefTensor: Defining tensor τ Dyad[].

** DefTensor: Defining tensor τ Dyad†[].
```

In[106]:=
```
DefSpinor[γ Dyad[], M4];
PrintAs[γ Dyad] ^= "γ";
PrintAs[γ Dyad†] ^= "γ̄";
```

```
** DefTensor: Defining tensor γ Dyad[].

** DefTensor: Defining tensor γ Dyad†[].
```

In[109]:=
```
DefSpinor[ν Dyad[], M4];
PrintAs[ν Dyad] ^= "ν";
PrintAs[ν Dyad†] ^= "ν̄";
```

```
** DefTensor: Defining tensor ν Dyad[].

** DefTensor: Defining tensor ν Dyad†[].
```

In[112]:=
```
DefSpinor[σ Dyad[], M4];
PrintAs[σ Dyad] ^= "σ";
PrintAs[σ Dyad†] ^= "σ̄";
```

```
** DefTensor: Defining tensor σ Dyad[].

** DefTensor: Defining tensor σ Dyad†[].
```

In[115]:=
```
DefSpinor[β Dyad[], M4];
PrintAs[β Dyad] ^= "β";
PrintAs[β Dyad†] ^= "β̄";
```

```
** DefTensor: Defining tensor β Dyad[].

** DefTensor: Defining tensor β Dyad†[].
```

In[118]:=
```
DefSpinor[μ Dyad[], M4];
PrintAs[μ Dyad] ^= "μ";
PrintAs[μ Dyad†] ^= "μ̄";
```

```
** DefTensor: Defining tensor μ Dyad[].

** DefTensor: Defining tensor μ Dyad†[].
```

In[121]:=
```
DefSpinor[ρ Dyad[], M4];
PrintAs[ρ Dyad] ^= "ρ";
PrintAs[ρ Dyad†] ^= "ρ̄";
```

```
** DefTensor: Defining tensor ρ Dyad[].

** DefTensor: Defining tensor ρ Dyad†[].
```

In[124]:=
```
DefSpinor[α Dyad[], M4];
PrintAs[α Dyad] ^= "α";
PrintAs[α Dyad†] ^= "ᾱ";
```

```
** DefTensor: Defining tensor α Dyad[].

** DefTensor: Defining tensor α Dyad†[].
```

In[127]:=
```
DefSpinor[λ Dyad[], M4];
PrintAs[λ Dyad] ^= "λ";
PrintAs[λ Dyad†] ^= "λ̄";
```

```
** DefTensor: Defining tensor λ Dyad[].

** DefTensor: Defining tensor λ Dyad†[].
```

This is full the list of Newman–Penrose spin coefficients.

In[130]:=
```
NPSpinCoefficients = {α Dyad[], β Dyad[], γ Dyad[], ϵ Dyad[],
   κ Dyad[], λ Dyad[], μ Dyad[], ν Dyad[], π Dyad[], ρ Dyad[], σ Dyad[], τ Dyad[]}
```

Out[130]= $\{\alpha , \beta , \gamma , \epsilon , \kappa , \lambda , \mu , \nu , \pi , \rho , \sigma , \tau \}$

In[131]:=
```
% // Length
```

Out[131]= 12

The spin coefficients are defined through the derivations of the elements of the spin tetrad by means of the following relations.

In[132]:=
```
DyadDerivatives = {
   CDe[{1, -NP}][o[-A]] ==  o[-A] ϵ Dyad[] - ι[-A] κ Dyad[],
   CDe[{1, -NP}][ι[-A]] == -ι[-A] ϵ Dyad[] + o[-A] π Dyad[],
   CDe[{2, -NP}][o[-A]] ==  o[-A] γ Dyad[] - ι[-A] τ Dyad[],
   CDe[{2, -NP}][ι[-A]] == -ι[-A] γ Dyad[] + o[-A] ν Dyad[],
   CDe[{3, -NP}][o[-A]] ==  o[-A] β Dyad[] - ι[-A] σ Dyad[],
   CDe[{3, -NP}][ι[-A]] == -ι[-A] β Dyad[] + o[-A] μ Dyad[],
   CDe[{4, -NP}][o[-A]] ==  o[-A] α Dyad[] - ι[-A] ρ Dyad[],
   CDe[{4, -NP}][ι[-A]] == -ι[-A] α Dyad[] + o[-A] λ Dyad[]
  }
```

Out[132]= $\{D\, o_A == o_A \epsilon - \iota_A \kappa , D\, \iota_A == -\epsilon\ \iota_A + o_A \pi , \Delta\, o_A == o_A \gamma - \iota_A \tau , \Delta\, \iota_A == -\gamma\ \iota_A + o_A \nu ,$
$\delta\, o_A == o_A \beta - \iota_A \sigma , \delta\, \iota_A == -\beta\ \iota_A + o_A \mu , \overline{\delta}\, o_A == o_A \alpha - \iota_A \rho , \overline{\delta}\, \iota_A == -\alpha\ \iota_A + o_A \lambda \}$

From this expression we compute and store the components of the spin covariant derivative "intrinsic" connection (these are the components of the "inner" Christoffel symbols of the covariant derivative `CDe`).

In[133]:=
```
Map[ToBasis[Dyad], %, {2}]
```

Out[133]= $\{e_A{}^B\, (D\, o_B) == o_A \epsilon - \iota_A \kappa , e_A{}^B\, (D\, \iota_B) == -\epsilon\ \iota_A + o_A \pi ,$
$e_A{}^B\, (\Delta\, o_B) == o_A \gamma - \iota_A \tau , e_A{}^B\, (\Delta\, \iota_B) == -\gamma\ \iota_A + o_A \nu , e_A{}^B\, (\delta\, o_B) == o_A \beta - \iota_A \sigma ,$
$e_A{}^B\, (\delta\, \iota_B) == -\beta\ \iota_A + o_A \mu , e_A{}^B\, (\overline{\delta}\, o_B) == o_A \alpha - \iota_A \rho , e_A{}^B\, (\overline{\delta}\, \iota_B) == -\alpha\ \iota_A + o_A \lambda \}$

In[134]:=
```
Map[ToBasis[Dyad], %, {2}]
```

```
** DefTensor: Defining tensor AChristoffelCDePDDyad[A, -b, -C].

** DefTensor: Defining tensor AChristoffelCDePDDyad†[A†, -b, -C†].
```

Out[134]= $\{-A[\nabla,]^B{}_{1\,A}\, o_B + D\, o_A == o_A \epsilon - \iota_A \kappa , -A[\nabla,]^B{}_{1\,A}\, \iota_B + D\, \iota_A == -\epsilon\ \iota_A + o_A \pi ,$
$-A[\nabla,]^B{}_{2\,A}\, o_B + \Delta\, o_A == o_A \gamma - \iota_A \tau , -A[\nabla,]^B{}_{2\,A}\, \iota_B + \Delta\, \iota_A == -\gamma\ \iota_A + o_A \nu ,$
$-A[\nabla,]^B{}_{3\,A}\, o_B + \delta\, o_A == o_A \beta - \iota_A \sigma , -A[\nabla,]^B{}_{3\,A}\, \iota_B + \delta\, \iota_A == -\beta\ \iota_A + o_A \mu ,$
$-A[\nabla,]^B{}_{4\,A}\, o_B + \overline{\delta}\, o_A == o_A \alpha - \iota_A \rho , -A[\nabla,]^B{}_{4\,A}\, \iota_B + \overline{\delta}\, \iota_A == -\alpha\ \iota_A + o_A \lambda \}$

In[135]:=
```
Map[TraceBasisDummy, %, {2}]
```

Out[135]= $\{-A[\nabla,]^1{}_{1\,A} + D\, o_A == o_A \epsilon - \iota_A \kappa , A[\nabla,]^0{}_{1\,A} + D\, \iota_A == -\epsilon\ \iota_A + o_A \pi ,$
$-A[\nabla,]^1{}_{2\,A} + \Delta\, o_A == o_A \gamma - \iota_A \tau , A[\nabla,]^0{}_{2\,A} + \Delta\, \iota_A == -\gamma\ \iota_A + o_A \nu , -A[\nabla,]^1{}_{3\,A} + \delta\, o_A == o_A \beta - \iota_A \sigma ,$
$A[\nabla,]^0{}_{3\,A} + \delta\, \iota_A == -\beta\ \iota_A + o_A \mu , -A[\nabla,]^1{}_{4\,A} + \overline{\delta}\, o_A == o_A \alpha - \iota_A \rho , A[\nabla,]^0{}_{4\,A} + \overline{\delta}\, \iota_A == -\alpha\ \iota_A + o_A \lambda \}$

In[136]:=
```
Flatten@ComponentArray@%
```

Out[136]= $\{-A[\nabla,]^1{}_{1\,0} == \kappa , -A[\nabla,]^1{}_{1\,1} == \epsilon , A[\nabla,]^0{}_{1\,0} == \epsilon , A[\nabla,]^0{}_{1\,1} == \pi , -A[\nabla,]^1{}_{2\,0} == \tau ,$
$-A[\nabla,]^1{}_{2\,1} == \gamma , A[\nabla,]^0{}_{2\,0} == \gamma , A[\nabla,]^0{}_{2\,1} == \nu , -A[\nabla,]^1{}_{3\,0} == \sigma , -A[\nabla,]^1{}_{3\,1} == \beta ,$
$A[\nabla,]^0{}_{3\,0} == \beta , A[\nabla,]^0{}_{3\,1} == \mu , -A[\nabla,]^1{}_{4\,0} == \rho , -A[\nabla,]^1{}_{4\,1} == \alpha , A[\nabla,]^0{}_{4\,0} == \alpha , A[\nabla,]^0{}_{4\,1} == \lambda \}$

In[137]:= **Solve[%, Cases[%, _AChristoffelCDePDDyad, Infinity]] // Flatten**

Out[137]= $\{A[\nabla,\ ]^{1}_{1\,0} \to -\varkappa\ ,\ A[\nabla,\ ]^{1}_{1\,1} \to -\epsilon\ ,\ A[\nabla,\ ]^{0}_{1\,0} \to \epsilon\ ,\ A[\nabla,\ ]^{0}_{1\,1} \to \pi\ ,\ A[\nabla,\ ]^{1}_{2\,0} \to -\tau\ ,$
$A[\nabla,\ ]^{1}_{2\,1} \to -\gamma\ ,\ A[\nabla,\ ]^{0}_{2\,0} \to \gamma\ ,\ A[\nabla,\ ]^{0}_{2\,1} \to \nu\ ,\ A[\nabla,\ ]^{1}_{3\,0} \to -\sigma\ ,\ A[\nabla,\ ]^{1}_{3\,1} \to -\beta\ ,$
$A[\nabla,\ ]^{0}_{3\,0} \to \beta\ ,\ A[\nabla,\ ]^{0}_{3\,1} \to \mu\ ,\ A[\nabla,\ ]^{1}_{4\,0} \to -\rho\ ,\ A[\nabla,\ ]^{1}_{4\,1} \to -\alpha\ ,\ A[\nabla,\ ]^{0}_{4\,0} \to \alpha\ ,\ A[\nabla,\ ]^{0}_{4\,1} \to \lambda\ \}$

In[138]:= **% /. Rule -> ComponentValue**

Out[138]= $\{A[\nabla,\ ]^{1}_{1\,0} \to -\varkappa\ ,\ A[\nabla,\ ]^{1}_{1\,1} \to -\epsilon\ ,\ A[\nabla,\ ]^{0}_{1\,0} \to \epsilon\ ,\ A[\nabla,\ ]^{0}_{1\,1} \to \pi\ ,\ A[\nabla,\ ]^{1}_{2\,0} \to -\tau\ ,$
$A[\nabla,\ ]^{1}_{2\,1} \to -\gamma\ ,\ A[\nabla,\ ]^{0}_{2\,0} \to \gamma\ ,\ A[\nabla,\ ]^{0}_{2\,1} \to \nu\ ,\ A[\nabla,\ ]^{1}_{3\,0} \to -\sigma\ ,\ A[\nabla,\ ]^{1}_{3\,1} \to -\beta\ ,$
$A[\nabla,\ ]^{0}_{3\,0} \to \beta\ ,\ A[\nabla,\ ]^{0}_{3\,1} \to \mu\ ,\ A[\nabla,\ ]^{1}_{4\,0} \to -\rho\ ,\ A[\nabla,\ ]^{1}_{4\,1} \to -\alpha\ ,\ A[\nabla,\ ]^{0}_{4\,0} \to \alpha\ ,\ A[\nabla,\ ]^{0}_{4\,1} \to \lambda\ \}$

We need the complex conjugate of all these relations.

In[139]:= **Map[Dagger, %, {2}]**

Out[139]= $\{A[\nabla,\ ]\dagger^{1'}_{1\,0'} \to -\overline{\varkappa}\ ,\ A[\nabla,\ ]\dagger^{1'}_{1\,1'} \to -\overline{\epsilon}\ ,\ A[\nabla,\ ]\dagger^{0'}_{1\,0'} \to \overline{\epsilon}\ ,\ A[\nabla,\ ]\dagger^{0'}_{1\,1'} \to \overline{\pi}\ ,$
$A[\nabla,\ ]\dagger^{1'}_{2\,0'} \to -\overline{\tau}\ ,\ A[\nabla,\ ]\dagger^{1'}_{2\,1'} \to -\overline{\gamma}\ ,\ A[\nabla,\ ]\dagger^{0'}_{2\,0'} \to \overline{\gamma}\ ,\ A[\nabla,\ ]\dagger^{0'}_{2\,1'} \to \overline{\nu}\ ,$
$A[\nabla,\ ]\dagger^{1'}_{4\,0'} \to -\overline{\sigma}\ ,\ A[\nabla,\ ]\dagger^{1'}_{4\,1'} \to -\overline{\beta}\ ,\ A[\nabla,\ ]\dagger^{0'}_{4\,0'} \to \overline{\beta}\ ,\ A[\nabla,\ ]\dagger^{0'}_{4\,1'} \to \overline{\mu}\ ,$
$A[\nabla,\ ]\dagger^{1'}_{3\,0'} \to -\overline{\rho}\ ,\ A[\nabla,\ ]\dagger^{1'}_{3\,1'} \to -\overline{\alpha}\ ,\ A[\nabla,\ ]\dagger^{0'}_{3\,0'} \to \overline{\alpha}\ ,\ A[\nabla,\ ]\dagger^{0'}_{3\,1'} \to \overline{\lambda}\ \}$

In[140]:= **% /. Rule -> ComponentValue**

Out[140]= $\{A[\nabla,\ ]\dagger^{1'}_{1\,0'} \to -\overline{\varkappa}\ ,\ A[\nabla,\ ]\dagger^{1'}_{1\,1'} \to -\overline{\epsilon}\ ,\ A[\nabla,\ ]\dagger^{0'}_{1\,0'} \to \overline{\epsilon}\ ,\ A[\nabla,\ ]\dagger^{0'}_{1\,1'} \to \overline{\pi}\ ,$
$A[\nabla,\ ]\dagger^{1'}_{2\,0'} \to -\overline{\tau}\ ,\ A[\nabla,\ ]\dagger^{1'}_{2\,1'} \to -\overline{\gamma}\ ,\ A[\nabla,\ ]\dagger^{0'}_{2\,0'} \to \overline{\gamma}\ ,\ A[\nabla,\ ]\dagger^{0'}_{2\,1'} \to \overline{\nu}\ ,$
$A[\nabla,\ ]\dagger^{1'}_{4\,0'} \to -\overline{\sigma}\ ,\ A[\nabla,\ ]\dagger^{1'}_{4\,1'} \to -\overline{\beta}\ ,\ A[\nabla,\ ]\dagger^{0'}_{4\,0'} \to \overline{\beta}\ ,\ A[\nabla,\ ]\dagger^{0'}_{4\,1'} \to \overline{\mu}\ ,$
$A[\nabla,\ ]\dagger^{1'}_{3\,0'} \to -\overline{\rho}\ ,\ A[\nabla,\ ]\dagger^{1'}_{3\,1'} \to -\overline{\alpha}\ ,\ A[\nabla,\ ]\dagger^{0'}_{3\,0'} \to \overline{\alpha}\ ,\ A[\nabla,\ ]\dagger^{0'}_{3\,1'} \to \overline{\lambda}\ \}$

Automatic replacement of the components of $A[\nabla,\ ]^{A}_{bC}$ and $A[\nabla,\ ]\dagger^{A'}_{bC'}$.

In[141]:= **RuleToSet@AChristoffelCDePDDyad;**

In[142]:= **RuleToSet@AChristoffelCDePDDyad†;**

■ Components of the Levi–Civita connection in the null tetrad

We need to obtain the components of the Levi–Civita connection (Ricci rotation coefficients) in the null tetrad. This is done by taking the covariant derivative of the set of equations relating the null tetrad elements and the spin dyad elements and using the expressions for the derivatives of the latter in terms of the Newman–Penrose spin coefficients.

As we know the relation between the spinor dyad and the null tetrad is

In[143]:= **SpinDyadToNullTetrad**

Out[143]= $\{o^{A}\,\overline{o}^{A'}\,\sigma^{a}_{AA'} == l^{a},\ \iota^{A}\,\overline{\iota}^{A'}\,\sigma^{a}_{AA'} == n^{a},\ o^{A}\,\overline{\iota}^{A'}\,\sigma^{a}_{AA'} == m^{a},\ \overline{o}^{A'}\,\iota^{A}\,\sigma^{a}_{AA'} == \overline{m}^{a}\}$

We take now the derivative of each of these equations and expand them in components.

In[144]:= **Map[CDe[-b], %, {2}]**

Out[144]= $\{\overline{o}^{A'}\,\sigma^{a}_{AA'}\,(\nabla_{b}\,o^{A}) + o^{A}\,\sigma^{a}_{AA'}\,(\nabla_{b}\,\overline{o}^{A'}) == \nabla_{b}\,l^{a},\ \overline{\iota}^{A'}\,\sigma^{a}_{AA'}\,(\nabla_{b}\,\iota^{A}) + \iota^{A}\,\sigma^{a}_{AA'}\,(\nabla_{b}\,\overline{\iota}^{A'}) == \nabla_{b}\,n^{a},$
$\overline{\iota}^{A'}\,\sigma^{a}_{AA'}\,(\nabla_{b}\,o^{A}) + o^{A}\,\sigma^{a}_{AA'}\,(\nabla_{b}\,\overline{\iota}^{A'}) == \nabla_{b}\,m^{a},\ \iota^{A}\,\sigma^{a}_{AA'}\,(\nabla_{b}\,\overline{o}^{A'}) + \overline{o}^{A'}\,\sigma^{a}_{AA'}\,(\nabla_{b}\,\iota^{A}) == \nabla_{b}\,\overline{m}^{a}\}$

In[145]:= **Map[ToBasis[NP], %, {2}]**

Out[145]= $\{\overline{o}^{A'}\,\sigma^{a}{}_{AA'}\,(\nabla_b\, o^A) + o^A\,\sigma^{a}{}_{AA'}\,(\nabla_b\,\overline{o}^{A'}) == e_c{}^a\,(\nabla_b\, l^c),\ \overline{\iota}^{A'}\,\sigma^{a}{}_{AA'}\,(\nabla_b\,\iota^A) + \iota^A\,\sigma^{a}{}_{AA'}\,(\nabla_b\,\overline{\iota}^{A'}) == e_c{}^a\,(\nabla_b\, n^c),$
$\overline{\iota}^{A'}\,\sigma^{a}{}_{AA'}\,(\nabla_b\, o^A) + o^A\,\sigma^{a}{}_{AA'}\,(\nabla_b\,\overline{\iota}^{A'}) == e_c{}^a\,(\nabla_b\, m^c),\ \iota^A\,\sigma^{a}{}_{AA'}\,(\nabla_b\,\overline{o}^{A'}) + \overline{o}^{A'}\,\sigma^{a}{}_{AA'}\,(\nabla_b\,\iota^A) == e_c{}^a\,(\nabla_b\,\overline{m}^c)\}$

In[146]:= **Map[ToBasis[NP], %, {2}]**

```
** DefTensor: Defining tensor ChristoffelCDePDNP[a, -b, -c].
```

Out[146]= $\{\overline{o}^{A'}\,\sigma^{a}{}_{AA'}\,(\nabla_b\, o^A) + o^A\,\sigma^{a}{}_{AA'}\,(\nabla_b\,\overline{o}^{A'}) == \Gamma[\nabla,\ ]^{a}{}_{bc}\, l^c + \nabla_b\, l^a,$
$\overline{\iota}^{A'}\,\sigma^{a}{}_{AA'}\,(\nabla_b\,\iota^A) + \iota^A\,\sigma^{a}{}_{AA'}\,(\nabla_b\,\overline{\iota}^{A'}) == \Gamma[\nabla,\ ]^{a}{}_{bc}\, n^c + \nabla_b\, n^a,$
$\overline{\iota}^{A'}\,\sigma^{a}{}_{AA'}\,(\nabla_b\, o^A) + o^A\,\sigma^{a}{}_{AA'}\,(\nabla_b\,\overline{\iota}^{A'}) == \Gamma[\nabla,\ ]^{a}{}_{bc}\, m^c + \nabla_b\, m^a,$
$\iota^A\,\sigma^{a}{}_{AA'}\,(\nabla_b\,\overline{o}^{A'}) + \overline{o}^{A'}\,\sigma^{a}{}_{AA'}\,(\nabla_b\,\iota^A) == \Gamma[\nabla,\ ]^{a}{}_{bc}\,\overline{m}^c + \nabla_b\,\overline{m}^a\}$

In[147]:= **Map[ToBasis[Dyad], %, {2}]**

Out[147]= $\{A[\nabla,\ ]^{A}{}_{bB}\, o^B\,\overline{o}^{A'}\,\sigma^{a}{}_{AA'} + \overline{o}^{A'}\,\sigma^{a}{}_{AA'}\,(\nabla_b\, o^A) + o^A\,\sigma^{a}{}_{AA'}\,(\nabla_b\,\overline{o}^{A'}) == \Gamma[\nabla,\ ]^{a}{}_{bc}\, l^c + \nabla_b\, l^a,$
$A[\nabla,\ ]^{A}{}_{bB}\,\iota^B\,\overline{\iota}^{A'}\,\sigma^{a}{}_{AA'} + \overline{\iota}^{A'}\,\sigma^{a}{}_{AA'}\,(\nabla_b\,\iota^A) + \iota^A\,\sigma^{a}{}_{AA'}\,(\nabla_b\,\overline{\iota}^{A'}) == \Gamma[\nabla,\ ]^{a}{}_{bc}\, n^c + \nabla_b\, n^a,$
$A[\nabla,\ ]^{A}{}_{bB}\, o^B\,\overline{\iota}^{A'}\,\sigma^{a}{}_{AA'} + \overline{\iota}^{A'}\,\sigma^{a}{}_{AA'}\,(\nabla_b\, o^A) + o^A\,\sigma^{a}{}_{AA'}\,(\nabla_b\,\overline{\iota}^{A'}) == \Gamma[\nabla,\ ]^{a}{}_{bc}\, m^c + \nabla_b\, m^a,$
$A[\nabla,\ ]^{A}{}_{bB}\,\overline{o}^{A'}\,\iota^B\,\sigma^{a}{}_{AA'} + \iota^A\,\sigma^{a}{}_{AA'}\,(\nabla_b\,\overline{o}^{A'}) + \overline{o}^{A'}\,\sigma^{a}{}_{AA'}\,(\nabla_b\,\iota^A) == \Gamma[\nabla,\ ]^{a}{}_{bc}\,\overline{m}^c + \nabla_b\,\overline{m}^a\}$

In[148]:= **Map[ToBasis[Dyad†], %, {2}]**

Out[148]= $\{A[\nabla,\ ]^{A}{}_{bB}\, o^B\,\overline{o}^{A'}\,\sigma^{a}{}_{AA'} + A[\nabla,\ ]\,\dagger^{A'}{}_{bB'}\, o^A\,\overline{o}^{B'}\,\sigma^{a}{}_{AA'} + \overline{o}^{A'}\,\sigma^{a}{}_{AA'}\,(\nabla_b\, o^A) + o^A\,\sigma^{a}{}_{AA'}\,(\nabla_b\,\overline{o}^{A'}) ==$
$\Gamma[\nabla,\ ]^{a}{}_{bc}\, l^c + \nabla_b\, l^a,\ A[\nabla,\ ]^{A}{}_{bB}\,\iota^B\,\overline{\iota}^{A'}\,\sigma^{a}{}_{AA'} + A[\nabla,\ ]\,\dagger^{A'}{}_{bB'}\,\iota^A\,\overline{\iota}^{B'}\,\sigma^{a}{}_{AA'} +$
$\overline{\iota}^{A'}\,\sigma^{a}{}_{AA'}\,(\nabla_b\,\iota^A) + \iota^A\,\sigma^{a}{}_{AA'}\,(\nabla_b\,\overline{\iota}^{A'}) == \Gamma[\nabla,\ ]^{a}{}_{bc}\, n^c + \nabla_b\, n^a,$
$A[\nabla,\ ]^{A}{}_{bB}\, o^B\,\overline{\iota}^{A'}\,\sigma^{a}{}_{AA'} + A[\nabla,\ ]\,\dagger^{A'}{}_{bB'}\, o^A\,\overline{\iota}^{B'}\,\sigma^{a}{}_{AA'} + \overline{\iota}^{A'}\,\sigma^{a}{}_{AA'}\,(\nabla_b\, o^A) + o^A\,\sigma^{a}{}_{AA'}\,(\nabla_b\,\overline{\iota}^{A'}) ==$
$\Gamma[\nabla,\ ]^{a}{}_{bc}\, m^c + \nabla_b\, m^a,\ A[\nabla,\ ]\,\dagger^{A'}{}_{bB'}\,\overline{o}^{B'}\,\iota^A\,\sigma^{a}{}_{AA'} + A[\nabla,\ ]^{A}{}_{bB}\,\overline{o}^{A'}\,\iota^B\,\sigma^{a}{}_{AA'} +$
$\iota^A\,\sigma^{a}{}_{AA'}\,(\nabla_b\,\overline{o}^{A'}) + \overline{o}^{A'}\,\sigma^{a}{}_{AA'}\,(\nabla_b\,\iota^A) == \Gamma[\nabla,\ ]^{a}{}_{bc}\,\overline{m}^c + \nabla_b\,\overline{m}^a\}$

All the ocurrences of the spin covariant derivative are now acting on scalar functions. Therefore we may replace them by any other covariant derivative, in particular `PDNP`.

In[149]:= **% /. CDe → PDNP**

Out[149]= $\{A[\nabla,\ ]^{A}{}_{bB}\, o^B\,\overline{o}^{A'}\,\sigma^{a}{}_{AA'} + A[\nabla,\ ]\,\dagger^{A'}{}_{bB'}\, o^A\,\overline{o}^{B'}\,\sigma^{a}{}_{AA'} + \overline{o}^{A'}\,\sigma^{a}{}_{AA'}\,(\ {}_b\, o^A) + o^A\,\sigma^{a}{}_{AA'}\,(\ {}_b\,\overline{o}^{A'}) ==$
$\Gamma[\nabla,\ ]^{a}{}_{bc}\, l^c +\ {}_b\, l^a,\ A[\nabla,\ ]^{A}{}_{bB}\,\iota^B\,\overline{\iota}^{A'}\,\sigma^{a}{}_{AA'} + A[\nabla,\ ]\,\dagger^{A'}{}_{bB'}\,\iota^A\,\overline{\iota}^{B'}\,\sigma^{a}{}_{AA'} +$
$\overline{\iota}^{A'}\,\sigma^{a}{}_{AA'}\,(\ {}_b\,\iota^A) + \iota^A\,\sigma^{a}{}_{AA'}\,(\ {}_b\,\overline{\iota}^{A'}) == \Gamma[\nabla,\ ]^{a}{}_{bc}\, n^c +\ {}_b\, n^a,$
$A[\nabla,\ ]^{A}{}_{bB}\, o^B\,\overline{\iota}^{A'}\,\sigma^{a}{}_{AA'} + A[\nabla,\ ]\,\dagger^{A'}{}_{bB'}\, o^A\,\overline{\iota}^{B'}\,\sigma^{a}{}_{AA'} + \overline{\iota}^{A'}\,\sigma^{a}{}_{AA'}\,(\ {}_b\, o^A) + o^A\,\sigma^{a}{}_{AA'}\,(\ {}_b\,\overline{\iota}^{A'}) ==$
$\Gamma[\nabla,\ ]^{a}{}_{bc}\, m^c +\ {}_b\, m^a,\ A[\nabla,\ ]\,\dagger^{A'}{}_{bB'}\,\overline{o}^{B'}\,\iota^A\,\sigma^{a}{}_{AA'} + A[\nabla,\ ]^{A}{}_{bB}\,\overline{o}^{A'}\,\iota^B\,\sigma^{a}{}_{AA'} +$
$\iota^A\,\sigma^{a}{}_{AA'}\,(\ {}_b\,\overline{o}^{A'}) + \overline{o}^{A'}\,\sigma^{a}{}_{AA'}\,(\ {}_b\,\iota^A) == \Gamma[\nabla,\ ]^{a}{}_{bc}\,\overline{m}^c +\ {}_b\,\overline{m}^a\}$

In[150]:= **Map[TraceBasisDummy, %, {2}];**

In[151]:=
```
Flatten@ComponentArray@%
```

Out[151]=
$$\{\epsilon + \overline{\epsilon} == \Gamma[\nabla,]^1_{11}, \gamma + \overline{\gamma} == \Gamma[\nabla,]^1_{21}, \overline{\alpha} + \beta == \Gamma[\nabla,]^1_{31}, \alpha + \overline{\beta} == \Gamma[\nabla,]^1_{41}, 0 == \Gamma[\nabla,]^2_{11},$$
$$0 == \Gamma[\nabla,]^2_{21}, 0 == \Gamma[\nabla,]^2_{31}, 0 == \Gamma[\nabla,]^2_{41}, -\overline{\kappa} == \Gamma[\nabla,]^3_{11}, -\overline{\tau} == \Gamma[\nabla,]^3_{21}, -\overline{\rho} == \Gamma[\nabla,]^3_{31},$$
$$-\overline{\sigma} == \Gamma[\nabla,]^3_{41}, -\kappa == \Gamma[\nabla,]^4_{11}, -\tau == \Gamma[\nabla,]^4_{21}, -\sigma == \Gamma[\nabla,]^4_{31}, -\rho == \Gamma[\nabla,]^4_{41},$$
$$0 == \Gamma[\nabla,]^1_{12}, 0 == \Gamma[\nabla,]^1_{22}, 0 == \Gamma[\nabla,]^1_{32}, 0 == \Gamma[\nabla,]^1_{42}, -\epsilon - \overline{\epsilon} == \Gamma[\nabla,]^2_{12},$$
$$-\gamma - \overline{\gamma} == \Gamma[\nabla,]^2_{22}, -\overline{\alpha} - \beta == \Gamma[\nabla,]^2_{32}, -\alpha - \overline{\beta} == \Gamma[\nabla,]^2_{42}, \pi == \Gamma[\nabla,]^3_{12},$$
$$\nu == \Gamma[\nabla,]^3_{22}, \mu == \Gamma[\nabla,]^3_{32}, \lambda == \Gamma[\nabla,]^3_{42}, \overline{\pi} == \Gamma[\nabla,]^4_{12}, \overline{\nu} == \Gamma[\nabla,]^4_{22},$$
$$\overline{\lambda} == \Gamma[\nabla,]^4_{32}, \overline{\mu} == \Gamma[\nabla,]^4_{42}, \overline{\pi} == \Gamma[\nabla,]^1_{13}, \overline{\nu} == \Gamma[\nabla,]^1_{23}, \overline{\lambda} == \Gamma[\nabla,]^1_{33},$$
$$\overline{\mu} == \Gamma[\nabla,]^1_{43}, -\kappa == \Gamma[\nabla,]^2_{13}, -\tau == \Gamma[\nabla,]^2_{23}, -\sigma == \Gamma[\nabla,]^2_{33}, -\rho == \Gamma[\nabla,]^2_{43},$$
$$\epsilon - \overline{\epsilon} == \Gamma[\nabla,]^3_{13}, \gamma - \overline{\gamma} == \Gamma[\nabla,]^3_{23}, -\overline{\alpha} + \beta == \Gamma[\nabla,]^3_{33}, \alpha - \overline{\beta} == \Gamma[\nabla,]^3_{43},$$
$$0 == \Gamma[\nabla,]^4_{13}, 0 == \Gamma[\nabla,]^4_{23}, 0 == \Gamma[\nabla,]^4_{33}, 0 == \Gamma[\nabla,]^4_{43}, \pi == \Gamma[\nabla,]^1_{14}, \nu == \Gamma[\nabla,]^1_{24},$$
$$\mu == \Gamma[\nabla,]^1_{34}, \lambda == \Gamma[\nabla,]^1_{44}, -\overline{\kappa} == \Gamma[\nabla,]^2_{14}, -\overline{\tau} == \Gamma[\nabla,]^2_{24}, -\overline{\rho} == \Gamma[\nabla,]^2_{34},$$
$$-\overline{\sigma} == \Gamma[\nabla,]^2_{44}, 0 == \Gamma[\nabla,]^3_{14}, 0 == \Gamma[\nabla,]^3_{24}, 0 == \Gamma[\nabla,]^3_{34}, 0 == \Gamma[\nabla,]^3_{44},$$
$$-\epsilon + \overline{\epsilon} == \Gamma[\nabla,]^4_{14}, -\gamma + \overline{\gamma} == \Gamma[\nabla,]^4_{24}, \overline{\alpha} - \beta == \Gamma[\nabla,]^4_{34}, -\alpha + \overline{\beta} == \Gamma[\nabla,]^4_{44}\}$$

In[152]:=
```
Solve[%, Union@Cases[%, _ChristoffelCDePDNP, Infinity]] // Flatten // Simplify
```

Out[152]=
$$\{\Gamma[\nabla,]^1_{11} \to \epsilon + \overline{\epsilon}, \Gamma[\nabla,]^1_{21} \to \gamma + \overline{\gamma}, \Gamma[\nabla,]^1_{31} \to \overline{\alpha} + \beta, \Gamma[\nabla,]^1_{41} \to \alpha + \overline{\beta}, \Gamma[\nabla,]^2_{11} \to 0,$$
$$\Gamma[\nabla,]^2_{21} \to 0, \Gamma[\nabla,]^2_{31} \to 0, \Gamma[\nabla,]^2_{41} \to 0, \Gamma[\nabla,]^3_{11} \to -\overline{\kappa}, \Gamma[\nabla,]^3_{21} \to -\overline{\tau}, \Gamma[\nabla,]^3_{31} \to -\overline{\rho},$$
$$\Gamma[\nabla,]^3_{41} \to -\overline{\sigma}, \Gamma[\nabla,]^4_{11} \to -\kappa, \Gamma[\nabla,]^4_{21} \to -\tau, \Gamma[\nabla,]^4_{31} \to -\sigma, \Gamma[\nabla,]^4_{41} \to -\rho,$$
$$\Gamma[\nabla,]^1_{12} \to 0, \Gamma[\nabla,]^1_{22} \to 0, \Gamma[\nabla,]^1_{32} \to 0, \Gamma[\nabla,]^1_{42} \to 0, \Gamma[\nabla,]^2_{12} \to -\epsilon - \overline{\epsilon},$$
$$\Gamma[\nabla,]^2_{22} \to -\gamma - \overline{\gamma}, \Gamma[\nabla,]^2_{32} \to -\overline{\alpha} - \beta, \Gamma[\nabla,]^2_{42} \to -\alpha - \overline{\beta}, \Gamma[\nabla,]^3_{12} \to \pi, \Gamma[\nabla,]^3_{22} \to \nu,$$
$$\Gamma[\nabla,]^3_{32} \to \mu, \Gamma[\nabla,]^3_{42} \to \lambda, \Gamma[\nabla,]^4_{12} \to \overline{\pi}, \Gamma[\nabla,]^4_{22} \to \overline{\nu}, \Gamma[\nabla,]^4_{32} \to \overline{\lambda}, \Gamma[\nabla,]^4_{42} \to \overline{\mu},$$
$$\Gamma[\nabla,]^1_{13} \to \overline{\pi}, \Gamma[\nabla,]^1_{23} \to \overline{\nu}, \Gamma[\nabla,]^1_{33} \to \overline{\lambda}, \Gamma[\nabla,]^1_{43} \to \overline{\mu}, \Gamma[\nabla,]^2_{13} \to -\kappa, \Gamma[\nabla,]^2_{23} \to -\tau,$$
$$\Gamma[\nabla,]^2_{33} \to -\sigma, \Gamma[\nabla,]^2_{43} \to -\rho, \Gamma[\nabla,]^3_{13} \to \epsilon - \overline{\epsilon}, \Gamma[\nabla,]^3_{23} \to \gamma - \overline{\gamma}, \Gamma[\nabla,]^3_{33} \to -\overline{\alpha} + \beta,$$
$$\Gamma[\nabla,]^3_{43} \to \alpha - \overline{\beta}, \Gamma[\nabla,]^4_{13} \to 0, \Gamma[\nabla,]^4_{23} \to 0, \Gamma[\nabla,]^4_{33} \to 0, \Gamma[\nabla,]^4_{43} \to 0, \Gamma[\nabla,]^1_{14} \to \pi,$$
$$\Gamma[\nabla,]^1_{24} \to \nu, \Gamma[\nabla,]^1_{34} \to \mu, \Gamma[\nabla,]^1_{44} \to \lambda, \Gamma[\nabla,]^2_{14} \to -\overline{\kappa}, \Gamma[\nabla,]^2_{24} \to -\overline{\tau},$$
$$\Gamma[\nabla,]^2_{34} \to -\overline{\rho}, \Gamma[\nabla,]^2_{44} \to -\overline{\sigma}, \Gamma[\nabla,]^3_{14} \to 0, \Gamma[\nabla,]^3_{24} \to 0, \Gamma[\nabla,]^3_{34} \to 0, \Gamma[\nabla,]^3_{44} \to 0,$$
$$\Gamma[\nabla,]^4_{14} \to -\epsilon + \overline{\epsilon}, \Gamma[\nabla,]^4_{24} \to -\gamma + \overline{\gamma}, \Gamma[\nabla,]^4_{34} \to \overline{\alpha} - \beta, \Gamma[\nabla,]^4_{44} \to -\alpha + \overline{\beta}\}$$

These are the components of the Levi–Civita connection in the null tetrad written in terms of the spin coefficients. We store them.

In[153]:=
```
% /. Rule → ComponentValue;
```

We show the list of Ricci rotation coefficients.

In[154]:=
```
TensorValues@ChristoffelCDePDNP
```

Out[154]=
```
FoldedRule[{}, {Γ[∇, ]^1_11 → ϵ + ϵ̄ , Γ[∇, ]^1_21 → γ + γ̄ , Γ[∇, ]^1_31 → ᾱ + β , Γ[∇, ]^1_41 → α + β̄ ,
  Γ[∇, ]^2_11 → 0, Γ[∇, ]^2_21 → 0, Γ[∇, ]^2_31 → 0, Γ[∇, ]^2_41 → 0, Γ[∇, ]^3_11 → -κ̄ , Γ[∇, ]^3_21 → -τ̄ ,
  Γ[∇, ]^3_31 → -ρ̄ , Γ[∇, ]^3_41 → -σ̄ , Γ[∇, ]^4_11 → -κ , Γ[∇, ]^4_21 → -τ , Γ[∇, ]^4_31 → -σ ,
  Γ[∇, ]^4_41 → -ρ , Γ[∇, ]^1_12 → 0, Γ[∇, ]^1_22 → 0, Γ[∇, ]^1_32 → 0, Γ[∇, ]^1_42 → 0,
  Γ[∇, ]^2_12 → -ϵ - ϵ̄ , Γ[∇, ]^2_22 → -γ - γ̄ , Γ[∇, ]^2_32 → -ᾱ - β , Γ[∇, ]^2_42 → -α - β̄ ,
  Γ[∇, ]^3_12 → π , Γ[∇, ]^3_22 → ν , Γ[∇, ]^3_32 → μ , Γ[∇, ]^3_42 → λ , Γ[∇, ]^4_12 → π̄ , Γ[∇, ]^4_22 → ν̄ ,
  Γ[∇, ]^4_32 → λ̄ , Γ[∇, ]^4_42 → μ̄ , Γ[∇, ]^1_13 → π̄ , Γ[∇, ]^1_23 → ν̄ , Γ[∇, ]^1_33 → λ̄ ,
  Γ[∇, ]^1_43 → μ̄ , Γ[∇, ]^2_13 → -κ , Γ[∇, ]^2_23 → -τ , Γ[∇, ]^2_33 → -σ , Γ[∇, ]^2_43 → -ρ ,
  Γ[∇, ]^3_13 → ϵ - ϵ̄ , Γ[∇, ]^3_23 → γ - γ̄ , Γ[∇, ]^3_33 → -ᾱ + β , Γ[∇, ]^3_43 → α - β̄ ,
  Γ[∇, ]^4_13 → 0, Γ[∇, ]^4_23 → 0, Γ[∇, ]^4_33 → 0, Γ[∇, ]^4_43 → 0, Γ[∇, ]^1_14 → π , Γ[∇, ]^1_24 → ν ,
  Γ[∇, ]^1_34 → μ , Γ[∇, ]^1_44 → λ , Γ[∇, ]^2_14 → -κ̄ , Γ[∇, ]^2_24 → -τ̄ , Γ[∇, ]^2_34 → -ρ̄ ,
  Γ[∇, ]^2_44 → -σ̄ , Γ[∇, ]^3_14 → 0, Γ[∇, ]^3_24 → 0, Γ[∇, ]^3_34 → 0, Γ[∇, ]^3_44 → 0,
  Γ[∇, ]^4_14 → -ϵ + ϵ̄ , Γ[∇, ]^4_24 → -γ + γ̄ , Γ[∇, ]^4_34 → ᾱ - β , Γ[∇, ]^4_44 → -α + β̄ }]
```

From now on, each Ricci rotation coefficient will be automatically replaced by its value in terms of the Newman–Penrose spin coefficients.

In[155]:=
```
RuleToSet@ChristoffelCDePDNP;
```

■ Commutation relations

In this subsection we calculate the commutation coefficients of the elements of the null tetrad in terms of the spin coefficients. This is achieved by calculating the brackets of the elements of the null tetrad which in turn requires the commutation coefficients of the null tetrad elements. These coeffients are the components of the tensor `TorsionPDNP`.

The first step is the creation of a folded rule containing the independent components of `TorsionPDNP` in the null tetrad.

In[156]:=
```
AllComponentValues@ToBasis[NP]@TorsionPDNP[a, -b, -c]
```

Out[156]=
```
FoldedRule[{T^1_11 → 0, T^1_21 → -T^1_12, T^1_22 → 0, T^1_31 → -T^1_13, T^1_32 → -T^1_23, T^1_33 → 0,
  T^1_41 → -T^1_14, T^1_42 → -T^1_24, T^1_43 → -T^1_34, T^1_44 → 0, T^2_11 → 0, T^2_21 → -T^2_12, T^2_22 → 0,
  T^2_31 → -T^2_13, T^2_32 → -T^2_23, T^2_33 → 0, T^2_41 → -T^2_14, T^2_42 → -T^2_24, T^2_43 → -T^2_34,
  T^2_44 → 0, T^3_11 → 0, T^3_21 → -T^3_12, T^3_22 → 0, T^3_31 → -T^3_13, T^3_32 → -T^3_23, T^3_33 → 0,
  T^3_41 → -T^3_14, T^3_42 → -T^3_24, T^3_43 → -T^3_34, T^3_44 → 0, T^4_11 → 0, T^4_21 → -T^4_12, T^4_22 → 0,
  T^4_31 → -T^4_13, T^4_32 → -T^4_23, T^4_33 → 0, T^4_41 → -T^4_14, T^4_42 → -T^4_24, T^4_43 → -T^4_34, T^4_44 → 0},
 {T^1_12 → T^1_12, T^1_13 → T^1_13, T^1_14 → T^1_14, T^1_23 → T^1_23, T^1_24 → T^1_24, T^1_34 → T^1_34, T^2_12 → T^2_12, T^2_13 → T^2_13,
  T^2_14 → T^2_14, T^2_23 → T^2_23, T^2_24 → T^2_24, T^2_34 → T^2_34, T^3_12 → T^3_12, T^3_13 → T^3_13, T^3_14 → T^3_14, T^3_23 → T^3_23,
  T^3_24 → T^3_24, T^3_34 → T^3_34, T^4_12 → T^4_12, T^4_13 → T^4_13, T^4_14 → T^4_14, T^4_23 → T^4_23, T^4_24 → T^4_24, T^4_34 → T^4_34}]
```

Next we compute the explicit expression for the bracket of each pair of elements of the null tetrad.

In[157]:=
```
Bracket[b][Basis[{-c, -NP}, a], Basis[{-d, -NP}, a]] ==
 ChangeTorsion[Bracket[b][Basis[{-c, -NP}, a], Basis[{-d, -NP}, a]], PDNP, CDe]
```

Out[157]=
```
-T^b_cd == Γ[∇, ]^b_cd - Γ[∇, ]^b_dc
```

In[158]:=
```
ToBasis[NP] /@ %
```

Out[158]=
```
-T^b_cd == Γ[∇, ]^b_cd - Γ[∇, ]^b_dc
```

Finally we compute and store the components of `TorsionPDNP` in the null tetrad.

In[159]:=
```
Flatten@ComponentArray@%
```

Out[159]=
$$\{-T^1{}_{11}==0,\ -T^1{}_{12}==-\gamma\ -\overline{\gamma},\ -T^1{}_{13}==-\overline{\alpha}\ -\beta\ +\overline{\pi},\ -T^1{}_{14}==-\alpha\ -\overline{\beta}\ +\pi,\ -T^1{}_{21}==\gamma\ +\overline{\gamma},\ -T^1{}_{22}==0,$$
$$-T^1{}_{23}==\overline{\nu},\ -T^1{}_{24}==\nu,\ -T^1{}_{31}==\overline{\alpha}\ +\beta\ -\overline{\pi},\ -T^1{}_{32}==-\overline{\nu},\ -T^1{}_{33}==0,\ -T^1{}_{34}==\mu\ -\overline{\mu},$$
$$-T^1{}_{41}==\alpha\ +\overline{\beta}\ -\pi,\ -T^1{}_{42}==-\nu,\ -T^1{}_{43}==-\mu\ +\overline{\mu},\ -T^1{}_{44}==0,\ -T^2{}_{11}==0,\ -T^2{}_{12}==-\epsilon\ -\overline{\epsilon},$$
$$-T^2{}_{13}==-\kappa,\ -T^2{}_{14}==-\overline{\kappa},\ -T^2{}_{21}==\epsilon\ +\overline{\epsilon},\ -T^2{}_{22}==0,\ -T^2{}_{23}==\overline{\alpha}\ +\beta\ -\tau,\ -T^2{}_{24}==\alpha\ +\overline{\beta}\ -\overline{\tau},$$
$$-T^2{}_{31}==\kappa,\ -T^2{}_{32}==-\overline{\alpha}\ -\beta\ +\tau,\ -T^2{}_{33}==0,\ -T^2{}_{34}==\rho\ -\overline{\rho},\ -T^2{}_{41}==\overline{\kappa},\ -T^2{}_{42}==-\alpha\ -\overline{\beta}\ +\overline{\tau},$$
$$-T^2{}_{43}==-\rho\ +\overline{\rho},\ -T^2{}_{44}==0,\ -T^3{}_{11}==0,\ -T^3{}_{12}==\pi\ +\overline{\tau},\ -T^3{}_{13}==\epsilon\ -\overline{\epsilon}\ +\overline{\rho},\ -T^3{}_{14}==\overline{\sigma},$$
$$-T^3{}_{21}==-\pi\ -\overline{\tau},\ -T^3{}_{22}==0,\ -T^3{}_{23}==\gamma\ -\overline{\gamma}\ -\mu,\ -T^3{}_{24}==-\lambda,\ -T^3{}_{31}==-\epsilon\ +\overline{\epsilon}\ -\overline{\rho},$$
$$-T^3{}_{32}==-\gamma\ +\overline{\gamma}\ +\mu,\ -T^3{}_{33}==0,\ -T^3{}_{34}==-\alpha\ +\overline{\beta},\ -T^3{}_{41}==-\overline{\sigma},\ -T^3{}_{42}==\lambda,\ -T^3{}_{43}==\alpha\ -\overline{\beta},$$
$$-T^3{}_{44}==0,\ -T^4{}_{11}==0,\ -T^4{}_{12}==\overline{\pi}\ +\tau,\ -T^4{}_{13}==\sigma,\ -T^4{}_{14}==-\epsilon\ +\overline{\epsilon}\ +\rho,\ -T^4{}_{21}==-\overline{\pi}\ -\tau,$$
$$-T^4{}_{22}==0,\ -T^4{}_{23}==-\overline{\lambda},\ -T^4{}_{24}==-\gamma\ +\overline{\gamma}\ -\overline{\mu},\ -T^4{}_{31}==-\sigma,\ -T^4{}_{32}==\overline{\lambda},\ -T^4{}_{33}==0,$$
$$-T^4{}_{34}==\overline{\alpha}\ -\beta,\ -T^4{}_{41}==\epsilon\ -\overline{\epsilon}\ -\rho,\ -T^4{}_{42}==\gamma\ -\overline{\gamma}\ +\overline{\mu},\ -T^4{}_{43}==-\overline{\alpha}\ +\beta,\ -T^4{}_{44}==0\}$$

In[160]:=
```
Union@ToValues@%
```

Out[160]=
$$\{\text{True},\ -T^1{}_{12}==-\gamma\ -\overline{\gamma},\ T^1{}_{12}==\gamma\ +\overline{\gamma},\ -T^1{}_{13}==-\overline{\alpha}\ -\beta\ +\overline{\pi},\ T^1{}_{13}==\overline{\alpha}\ +\beta\ -\overline{\pi},$$
$$-T^1{}_{14}==-\alpha\ -\overline{\beta}\ +\pi,\ T^1{}_{14}==\alpha\ +\overline{\beta}\ -\pi,\ -T^1{}_{23}==\overline{\nu},\ T^1{}_{23}==-\overline{\nu},\ -T^1{}_{24}==\nu,\ T^1{}_{24}==-\nu,$$
$$-T^1{}_{34}==\mu\ -\overline{\mu},\ T^1{}_{34}==-\mu\ +\overline{\mu},\ -T^2{}_{12}==-\epsilon\ -\overline{\epsilon},\ T^2{}_{12}==\epsilon\ +\overline{\epsilon},\ -T^2{}_{13}==-\kappa,\ T^2{}_{13}==\kappa,$$
$$-T^2{}_{14}==-\overline{\kappa},\ T^2{}_{14}==\overline{\kappa},\ -T^2{}_{23}==\overline{\alpha}\ +\beta\ -\tau,\ T^2{}_{23}==-\overline{\alpha}\ -\beta\ +\tau,\ -T^2{}_{24}==\alpha\ +\overline{\beta}\ -\overline{\tau},$$
$$T^2{}_{24}==-\alpha\ -\overline{\beta}\ +\overline{\tau},\ -T^2{}_{34}==\rho\ -\overline{\rho},\ T^2{}_{34}==-\rho\ +\overline{\rho},\ -T^3{}_{12}==\pi\ +\overline{\tau},\ T^3{}_{12}==-\pi\ -\overline{\tau},$$
$$-T^3{}_{13}==\epsilon\ -\overline{\epsilon}\ +\overline{\rho},\ T^3{}_{13}==-\epsilon\ +\overline{\epsilon}\ -\overline{\rho},\ -T^3{}_{14}==\overline{\sigma},\ T^3{}_{14}==-\overline{\sigma},\ -T^3{}_{23}==\gamma\ -\overline{\gamma}\ -\mu,$$
$$T^3{}_{23}==-\gamma\ +\overline{\gamma}\ +\mu,\ -T^3{}_{24}==-\lambda,\ T^3{}_{24}==\lambda,\ -T^3{}_{34}==-\alpha\ +\overline{\beta},\ T^3{}_{34}==\alpha\ -\overline{\beta},\ -T^4{}_{12}==\overline{\pi}\ +\tau,$$
$$T^4{}_{12}==-\overline{\pi}\ -\tau,\ -T^4{}_{13}==\sigma,\ T^4{}_{13}==-\sigma,\ -T^4{}_{14}==-\epsilon\ +\overline{\epsilon}\ +\rho,\ T^4{}_{14}==\epsilon\ -\overline{\epsilon}\ -\rho,\ -T^4{}_{23}==-\overline{\lambda},$$
$$T^4{}_{23}==\overline{\lambda},\ -T^4{}_{24}==-\gamma\ +\overline{\gamma}\ -\overline{\mu},\ T^4{}_{24}==\gamma\ -\overline{\gamma}\ +\overline{\mu},\ -T^4{}_{34}==\overline{\alpha}\ -\beta,\ T^4{}_{34}==-\overline{\alpha}\ +\beta\ \}$$

In[161]:=
```
Solve[%, Union@Cases[%, _TorsionPDNP, Infinity]] // Flatten
```

Out[161]=
$$\{T^1{}_{12}\to\gamma\ +\overline{\gamma},\ T^1{}_{13}\to\overline{\alpha}\ +\beta\ -\overline{\pi},\ T^1{}_{14}\to\alpha\ +\overline{\beta}\ -\pi,\ T^1{}_{23}\to-\overline{\nu},\ T^1{}_{24}\to-\nu,\ T^1{}_{34}\to-\mu\ +\overline{\mu},$$
$$T^2{}_{12}\to\epsilon\ +\overline{\epsilon},\ T^2{}_{13}\to\kappa,\ T^2{}_{14}\to\overline{\kappa},\ T^2{}_{23}\to-\overline{\alpha}\ -\beta\ +\tau,\ T^2{}_{24}\to-\alpha\ -\overline{\beta}\ +\overline{\tau},\ T^2{}_{34}\to-\rho\ +\overline{\rho},$$
$$T^3{}_{12}\to-\pi\ -\overline{\tau},\ T^3{}_{13}\to-\epsilon\ +\overline{\epsilon}\ -\overline{\rho},\ T^3{}_{14}\to-\overline{\sigma},\ T^3{}_{23}\to-\gamma\ +\overline{\gamma}\ +\mu,\ T^3{}_{24}\to\lambda,\ T^3{}_{34}\to\alpha\ -\overline{\beta},$$
$$T^4{}_{12}\to-\overline{\pi}\ -\tau,\ T^4{}_{13}\to-\sigma,\ T^4{}_{14}\to\epsilon\ -\overline{\epsilon}\ -\rho,\ T^4{}_{23}\to\overline{\lambda},\ T^4{}_{24}\to\gamma\ -\overline{\gamma}\ +\overline{\mu},\ T^4{}_{34}\to-\overline{\alpha}\ +\beta\ \}$$

We store these values.

In[162]:=
```
% /. Rule → ComponentValue;
```

In[163]:=
```
RuleToSet@TorsionPDNP;
```

We can now write down the equations arising from the commutation relations. First we define a scalar function upon which the differential operators will act.

In[164]:=
```
DefTensor[ψ[], M4]
```

```
** DefTensor: Defining tensor ψ[].
```

By default, `xTensor`` commutes automatically pairs of covariant derivatives when they act on scalars.

In[165]:=
```
PDNP[-a]@PDNP[-b]@ψ[]
```

Out[165]=
$$_b\ {}_a\psi\ +T^c{}_{ba}\ (\ {}_c\psi\ )$$

We need to remove this automatic commutation in our calculations. This is done by means of the next line of code.

In[166]:=
```
xAct`xTensor`Private`b$  xAct`xTensor`Private`a$ xAct`xTensor`Private`scalar$_Symbol?xTensorQ[] =.
```

Commutation of the operator PDNP on the scalar ψ.

In[167]:=
```
PDNP[-a]@PDNP[-b]@ψ[] - PDNP[-b]@PDNP[-a]@ψ[] ==
 SortCovDs[PDNP[-a]@PDNP[-b]@ψ[] - PDNP[-b]@PDNP[-a]@ψ[]]
```

Out[167]=
$$\nabla_a \nabla_b \psi - \nabla_b \nabla_a \psi == T^{c}{}_{ba} \left(\nabla_c \psi \right)$$

The components of this relation in the null tetrad yield the Newman–Penrose commutation relations.

In[168]:=
```
ToBasis[NP] /@ %
```

Out[168]=
$$e_b{}^{c} \left(\nabla_a \nabla_c \psi \right) - e_a{}^{c} \left(\nabla_b \nabla_c \psi \right) == T^{c}{}_{ba} \left(\nabla_c \psi \right)$$

In[169]:=
```
ToBasis[NP] /@ %
```

Out[169]=
$$\nabla_a \nabla_b \psi - \nabla_b \nabla_a \psi == T^{c}{}_{ba} \left(\nabla_c \psi \right)$$

In[170]:=
```
TraceBasisDummy /@ %
```

Out[170]=
$$\nabla_a \nabla_b \psi - \nabla_b \nabla_a \psi == T^{1}{}_{ba} \left(\mathrm{D} \psi \right) + T^{2}{}_{ba} \left(\Delta \psi \right) + T^{3}{}_{ba} \left(\delta \psi \right) + T^{4}{}_{ba} \left(\overline{\delta} \psi \right)$$

In[171]:=
```
Union@Flatten@ComponentArray@%
```

Out[171]=
$$\Big\{ \text{True},\ \mathrm{D}\Delta\psi - \Delta\mathrm{D}\psi == (-\gamma - \overline{\gamma}) (\mathrm{D}\psi) + (-\epsilon - \overline{\epsilon}) (\Delta\psi) + (\pi + \overline{\tau}) (\delta\psi) + (\overline{\pi} + \tau) \left(\overline{\delta}\psi\right),$$
$$-(\mathrm{D}\Delta\psi) + \Delta\mathrm{D}\psi == (\gamma + \overline{\gamma}) (\mathrm{D}\psi) + (\epsilon + \overline{\epsilon}) (\Delta\psi) + (-\pi - \overline{\tau}) (\delta\psi) + (-\overline{\pi} - \tau) \left(\overline{\delta}\psi\right),$$
$$\mathrm{D}\delta\psi - \delta\mathrm{D}\psi == (-\overline{\alpha} - \beta + \overline{\pi}) (\mathrm{D}\psi) - \kappa (\Delta\psi) + (\epsilon - \overline{\epsilon} + \overline{\rho}) (\delta\psi) + \sigma \left(\overline{\delta}\psi\right),$$
$$-(\mathrm{D}\delta\psi) + \delta\mathrm{D}\psi == (\overline{\alpha} + \beta - \overline{\pi}) (\mathrm{D}\psi) + \kappa (\Delta\psi) + (-\epsilon + \overline{\epsilon} - \overline{\rho}) (\delta\psi) - \sigma \left(\overline{\delta}\psi\right),$$
$$\Delta\delta\psi - \delta\Delta\psi == \overline{\nu} (\mathrm{D}\psi) + (\overline{\alpha} + \beta - \tau) (\Delta\psi) + (\gamma - \overline{\gamma} - \mu) (\delta\psi) - \overline{\lambda} \left(\overline{\delta}\psi\right),$$
$$-(\Delta\delta\psi) + \delta\Delta\psi == -\overline{\nu} (\mathrm{D}\psi) + (-\overline{\alpha} - \beta + \tau) (\Delta\psi) + (-\gamma + \overline{\gamma} + \mu) (\delta\psi) + \overline{\lambda} \left(\overline{\delta}\psi\right),$$
$$\mathrm{D}\overline{\delta}\psi - \overline{\delta}\mathrm{D}\psi == \left(-\alpha - \overline{\beta} + \pi\right) (\mathrm{D}\psi) - \overline{\kappa} (\Delta\psi) + \overline{\sigma} (\delta\psi) + (-\epsilon + \overline{\epsilon} + \rho) \left(\overline{\delta}\psi\right),$$
$$-\left(\mathrm{D}\overline{\delta}\psi\right) + \overline{\delta}\mathrm{D}\psi == \left(\alpha + \overline{\beta} - \pi\right) (\mathrm{D}\psi) + \overline{\kappa} (\Delta\psi) - \overline{\sigma} (\delta\psi) + (\epsilon - \overline{\epsilon} - \rho) \left(\overline{\delta}\psi\right),$$
$$\Delta\overline{\delta}\psi - \overline{\delta}\Delta\psi == \nu (\mathrm{D}\psi) + \left(\alpha + \overline{\beta} - \overline{\tau}\right) (\Delta\psi) - \lambda (\delta\psi) + (-\gamma + \overline{\gamma} - \overline{\mu}) \left(\overline{\delta}\psi\right),$$
$$-\left(\Delta\overline{\delta}\psi\right) + \overline{\delta}\Delta\psi == -\nu (\mathrm{D}\psi) + \left(-\alpha - \overline{\beta} + \overline{\tau}\right) (\Delta\psi) + \lambda (\delta\psi) + (\gamma - \overline{\gamma} + \overline{\mu}) \left(\overline{\delta}\psi\right),$$
$$\delta\overline{\delta}\psi - \overline{\delta}\delta\psi == (\mu - \overline{\mu}) (\mathrm{D}\psi) + (\rho - \overline{\rho}) (\Delta\psi) + \left(-\alpha + \overline{\beta}\right) (\delta\psi) + (\overline{\alpha} - \beta) \left(\overline{\delta}\psi\right),$$
$$-\left(\delta\overline{\delta}\psi\right) + \overline{\delta}\delta\psi == (-\mu + \overline{\mu}) (\mathrm{D}\psi) + (-\rho + \overline{\rho}) (\Delta\psi) + \left(\alpha - \overline{\beta}\right) (\delta\psi) + (-\overline{\alpha} + \beta) \left(\overline{\delta}\psi\right) \Big\}$$

We extract the independent equations from this set.

In[172]:=
```
Solve[%, Union@Cases[%, PDNP[_]@PDNP[_]@ψ[], Infinity]] // Flatten
```

```
Solve::svars : Equations may not give solutions for all "solve" variables. >>
```

Out[172]=
$$\Big\{ \mathrm{D}\Delta\psi \to (-\gamma - \overline{\gamma}) (\mathrm{D}\psi) + (-\epsilon - \overline{\epsilon}) (\Delta\psi) + \Delta\mathrm{D}\psi + (\pi + \overline{\tau}) (\delta\psi) + (\overline{\pi} + \tau) \left(\overline{\delta}\psi\right),$$
$$\mathrm{D}\delta\psi \to (-\overline{\alpha} - \beta + \overline{\pi}) (\mathrm{D}\psi) - \kappa (\Delta\psi) + (\epsilon - \overline{\epsilon} + \overline{\rho}) (\delta\psi) + \delta\mathrm{D}\psi + \sigma \left(\overline{\delta}\psi\right),$$
$$\mathrm{D}\overline{\delta}\psi \to \left(-\alpha - \overline{\beta} + \pi\right) (\mathrm{D}\psi) - \overline{\kappa} (\Delta\psi) + \overline{\sigma} (\delta\psi) + (-\epsilon + \overline{\epsilon} + \rho) \left(\overline{\delta}\psi\right) + \overline{\delta}\mathrm{D}\psi,$$
$$\Delta\delta\psi \to \overline{\nu} (\mathrm{D}\psi) + (\overline{\alpha} + \beta - \tau) (\Delta\psi) + (\gamma - \overline{\gamma} - \mu) (\delta\psi) + \delta\Delta\psi - \overline{\lambda} \left(\overline{\delta}\psi\right),$$
$$\Delta\overline{\delta}\psi \to \nu (\mathrm{D}\psi) + \left(\alpha + \overline{\beta} - \overline{\tau}\right) (\Delta\psi) - \lambda (\delta\psi) + (-\gamma + \overline{\gamma} - \overline{\mu}) \left(\overline{\delta}\psi\right) + \overline{\delta}\Delta\psi,$$
$$\delta\overline{\delta}\psi \to (\mu - \overline{\mu}) (\mathrm{D}\psi) + (\rho - \overline{\rho}) (\Delta\psi) + \left(-\alpha + \overline{\beta}\right) (\delta\psi) + (\overline{\alpha} - \beta) \left(\overline{\delta}\psi\right) + \overline{\delta}\delta\psi \Big\}$$

These equations come in complex–conjugate pairs. Therefore we only need to consider a subset of them.

In[173]:=
```
Expand //@ %;
```

In[174]:=
```
Union[%, SameTest → (Dagger /@ #1 === #2 &)];
```

In[175]:=
```
FullSimplify //@ %
```

Out[175]=
$$\begin{aligned}
&\{\mathrm{D}\,\Delta\,\psi \to -(\gamma+\overline{\gamma})\,(\mathrm{D}\,\psi) - (\epsilon+\overline{\epsilon})\,(\Delta\,\psi) + \Delta\,\mathrm{D}\,\psi + (\pi+\overline{\tau})\,(\delta\,\psi) + (\overline{\pi}+\tau)\,\left(\overline{\delta}\,\psi\right),\\
&\mathrm{D}\,\delta\,\psi \to -(\overline{\alpha}+\beta-\overline{\pi})\,(\mathrm{D}\,\psi) - \kappa\,(\Delta\,\psi) + (\epsilon-\overline{\epsilon}+\overline{\rho})\,(\delta\,\psi) + \delta\,\mathrm{D}\,\psi + \sigma\,\left(\overline{\delta}\,\psi\right),\\
&\Delta\,\delta\,\psi \to \overline{\nu}\,(\mathrm{D}\,\psi) + (\overline{\alpha}+\beta-\tau)\,(\Delta\,\psi) + (\gamma-\overline{\gamma}-\mu)\,(\delta\,\psi) + \delta\,\Delta\,\psi - \overline{\lambda}\,\left(\overline{\delta}\,\psi\right),\\
&\delta\,\overline{\delta}\,\psi \to (\mu-\overline{\mu})\,(\mathrm{D}\,\psi) + (\rho-\overline{\rho})\,(\Delta\,\psi) + \left(-\alpha+\overline{\beta}\right)(\delta\,\psi) + (\overline{\alpha}-\beta)\,\left(\overline{\delta}\,\psi\right) + \overline{\delta}\,\delta\,\psi\,\}
\end{aligned}$$

Therefore, the full set of independent commutation relations is

In[176]:=
```
NPCommutators = % /. Rule -> Equal
```

Out[176]=
$$\begin{aligned}
&\{\mathrm{D}\,\Delta\,\psi == -(\gamma+\overline{\gamma})\,(\mathrm{D}\,\psi) - (\epsilon+\overline{\epsilon})\,(\Delta\,\psi) + \Delta\,\mathrm{D}\,\psi + (\pi+\overline{\tau})\,(\delta\,\psi) + (\overline{\pi}+\tau)\,\left(\overline{\delta}\,\psi\right),\\
&\mathrm{D}\,\delta\,\psi == -(\overline{\alpha}+\beta-\overline{\pi})\,(\mathrm{D}\,\psi) - \kappa\,(\Delta\,\psi) + (\epsilon-\overline{\epsilon}+\overline{\rho})\,(\delta\,\psi) + \delta\,\mathrm{D}\,\psi + \sigma\,\left(\overline{\delta}\,\psi\right),\\
&\Delta\,\delta\,\psi == \overline{\nu}\,(\mathrm{D}\,\psi) + (\overline{\alpha}+\beta-\tau)\,(\Delta\,\psi) + (\gamma-\overline{\gamma}-\mu)\,(\delta\,\psi) + \delta\,\Delta\,\psi - \overline{\lambda}\,\left(\overline{\delta}\,\psi\right),\\
&\delta\,\overline{\delta}\,\psi == (\mu-\overline{\mu})\,(\mathrm{D}\,\psi) + (\rho-\overline{\rho})\,(\Delta\,\psi) + \left(-\alpha+\overline{\beta}\right)(\delta\,\psi) + (\overline{\alpha}-\beta)\,\left(\overline{\delta}\,\psi\right) + \overline{\delta}\,\delta\,\psi\,\}
\end{aligned}$$

■ Components of the Riemann tensor in terms of the components of the curvature spinors

We are going to write the components of the Riemann tensor in the null tetrad in terms of the spinor curvature components expressed in the spin dyad. First we compute folded rules containing the independent components of each of the quantities involved.

In[177]:=
```
AllComponentValues@ToBasis[Dyad]@PsiCDe[-A, -B, -C, -D]
```

Out[177]=
$$\begin{aligned}
&\text{FoldedRule}[\{\Psi[\nabla]_{0010} \to \Psi[\nabla]_{0001},\ \Psi[\nabla]_{0100} \to \Psi[\nabla]_{0001},\ \Psi[\nabla]_{0101} \to \Psi[\nabla]_{0011},\ \Psi[\nabla]_{0110} \to \Psi[\nabla]_{0011},\\
&\ \Psi[\nabla]_{1000} \to \Psi[\nabla]_{0001},\ \Psi[\nabla]_{1001} \to \Psi[\nabla]_{0011},\ \Psi[\nabla]_{1010} \to \Psi[\nabla]_{0011},\ \Psi[\nabla]_{1011} \to \Psi[\nabla]_{0111},\\
&\ \Psi[\nabla]_{1100} \to \Psi[\nabla]_{0011},\ \Psi[\nabla]_{1101} \to \Psi[\nabla]_{0111},\ \Psi[\nabla]_{1110} \to \Psi[\nabla]_{0111}\},\ \{\Psi[\nabla]_{0000} \to \Psi[\nabla]_{0000},\\
&\ \Psi[\nabla]_{0001} \to \Psi[\nabla]_{0001},\ \Psi[\nabla]_{0011} \to \Psi[\nabla]_{0011},\ \Psi[\nabla]_{0111} \to \Psi[\nabla]_{0111},\ \Psi[\nabla]_{1111} \to \Psi[\nabla]_{1111}\}]
\end{aligned}$$

In[178]:=
```
RuleToSet@PsiCDe
```

In[179]:=
```
AllComponentValues@ToBasis[Dyad†]@PsiCDe†[-A†, -B†, -C†, -D†]
```

Out[179]=
$$\begin{aligned}
&\text{FoldedRule}[\{\Psi[\nabla]\,\dagger_{0',0',1',0'} \to \Psi[\nabla]\,\dagger_{0',0',0',1'},\ \Psi[\nabla]\,\dagger_{0',1',0',0'} \to \Psi[\nabla]\,\dagger_{0',0',0',1'},\\
&\ \Psi[\nabla]\,\dagger_{0',1',0',1'} \to \Psi[\nabla]\,\dagger_{0',0',1',1'},\ \Psi[\nabla]\,\dagger_{0',1',1',0'} \to \Psi[\nabla]\,\dagger_{0',0',1',1'},\ \Psi[\nabla]\,\dagger_{1',0',0',0'} \to \Psi[\nabla]\,\dagger_{0',0',0',1'},\\
&\ \Psi[\nabla]\,\dagger_{1',0',0',1'} \to \Psi[\nabla]\,\dagger_{0',0',1',1'},\ \Psi[\nabla]\,\dagger_{1',0',1',0'} \to \Psi[\nabla]\,\dagger_{0',0',1',1'},\ \Psi[\nabla]\,\dagger_{1',0',1',1'} \to \Psi[\nabla]\,\dagger_{0',1',1',1'},\\
&\ \Psi[\nabla]\,\dagger_{1',1',0',0'} \to \Psi[\nabla]\,\dagger_{0',0',1',1'},\ \Psi[\nabla]\,\dagger_{1',1',0',1'} \to \Psi[\nabla]\,\dagger_{0',1',1',1'},\ \Psi[\nabla]\,\dagger_{1',1',1',0'} \to \Psi[\nabla]\,\dagger_{0',1',1',1'}\},\\
&\{\Psi[\nabla]\,\dagger_{0',0',0',0'} \to \Psi[\nabla]\,\dagger_{0',0',0',0'},\ \Psi[\nabla]\,\dagger_{0',0',0',1'} \to \Psi[\nabla]\,\dagger_{0',0',0',1'},\ \Psi[\nabla]\,\dagger_{0',0',1',1'} \to \Psi[\nabla]\,\dagger_{0',0',1',1'},\\
&\ \Psi[\nabla]\,\dagger_{0',1',1',1'} \to \Psi[\nabla]\,\dagger_{0',1',1',1'},\ \Psi[\nabla]\,\dagger_{1',1',1',1'} \to \Psi[\nabla]\,\dagger_{1',1',1',1'}\}]
\end{aligned}$$

In[180]:=
```
RuleToSet@PsiCDe†
```

In[181]:=
```
AllComponentValues@ToBasis[Dyad†]@ToBasis[Dyad]@PhiCDe[-A, -B, -A†, -B†]
```

Out[181]=
$$\begin{aligned}
&\text{FoldedRule}[\{\Phi[\nabla]_{001'0'} \to \Phi[\nabla]_{000'1'},\\
&\ \Phi[\nabla]_{011'0'} \to \Phi[\nabla]_{010'1'},\ \Phi[\nabla]_{100'0'} \to \Phi[\nabla]_{010'0'},\ \Phi[\nabla]_{100'1'} \to \Phi[\nabla]_{010'1'},\\
&\ \Phi[\nabla]_{101'0'} \to \Phi[\nabla]_{010'1'},\ \Phi[\nabla]_{101'1'} \to \Phi[\nabla]_{011'1'},\ \Phi[\nabla]_{111'0'} \to \Phi[\nabla]_{110'1'}\},\\
&\{\Phi[\nabla]_{000'0'} \to \Phi[\nabla]_{000'0'},\ \Phi[\nabla]_{000'1'} \to \Phi[\nabla]_{000'1'},\ \Phi[\nabla]_{001'1'} \to \Phi[\nabla]_{001'1'},\\
&\ \Phi[\nabla]_{010'0'} \to \Phi[\nabla]_{010'0'},\ \Phi[\nabla]_{010'1'} \to \Phi[\nabla]_{010'1'},\ \Phi[\nabla]_{011'1'} \to \Phi[\nabla]_{011'1'},\\
&\ \Phi[\nabla]_{110'0'} \to \Phi[\nabla]_{110'0'},\ \Phi[\nabla]_{110'1'} \to \Phi[\nabla]_{110'1'},\ \Phi[\nabla]_{111'1'} \to \Phi[\nabla]_{111'1'}\}]
\end{aligned}$$

In[182]:=
```
RuleToSet@PhiCDe
```

We introduce the scalars which will represent the components of the Weyl spinor

In[183]:=
```
DefTensor[#, M4, Dagger → Complex] & /@
  {PsiCDe0Dyad[], PsiCDe1Dyad[], PsiCDe2Dyad[], PsiCDe3Dyad[], PsiCDe4Dyad[]};
```

```
** DefTensor: Defining tensor PsiCDe0Dyad[].

** DefTensor: Defining tensor PsiCDe0Dyad†[].

** DefTensor: Defining tensor PsiCDe1Dyad[].

** DefTensor: Defining tensor PsiCDe1Dyad†[].

** DefTensor: Defining tensor PsiCDe2Dyad[].

** DefTensor: Defining tensor PsiCDe2Dyad†[].

** DefTensor: Defining tensor PsiCDe3Dyad[].

** DefTensor: Defining tensor PsiCDe3Dyad†[].

** DefTensor: Defining tensor PsiCDe4Dyad[].

** DefTensor: Defining tensor PsiCDe4Dyad†[].
```

The different components of the Weyl spinor are formatted with their conventional names.

In[184]:=
```
PrintAs@PsiCDe0Dyad ^= "Ψ0";
PrintAs@PsiCDe1Dyad ^= "Ψ1";
PrintAs@PsiCDe2Dyad ^= "Ψ2";
PrintAs@PsiCDe3Dyad ^= "Ψ3";
PrintAs@PsiCDe4Dyad ^= "Ψ4";
PrintAs@PsiCDe0Dyad† ^= "Ψ̄0";
PrintAs@PsiCDe1Dyad† ^= "Ψ̄1";
PrintAs@PsiCDe2Dyad† ^= "Ψ̄2";
PrintAs@PsiCDe3Dyad† ^= "Ψ̄3";
PrintAs@PsiCDe4Dyad† ^= "Ψ̄4";
```

In[194]:=
```
SetToRule@PsiCDe
```

In[195]:=
```
SetToRule@PsiCDe†
```

Explicit definition of the components of the Weyl spinor.

In[196]:=
```
{PsiCDe[{0, -Dyad}, {0, -Dyad}, {0, -Dyad}, {0, -Dyad}] -> PsiCDe0Dyad[],
 PsiCDe[{0, -Dyad}, {0, -Dyad}, {0, -Dyad}, {1, -Dyad}] -> PsiCDe1Dyad[],
 PsiCDe[{0, -Dyad}, {0, -Dyad}, {1, -Dyad}, {1, -Dyad}] -> PsiCDe2Dyad[],
 PsiCDe[{0, -Dyad}, {1, -Dyad}, {1, -Dyad}, {1, -Dyad}] -> PsiCDe3Dyad[],
 PsiCDe[{1, -Dyad}, {1, -Dyad}, {1, -Dyad}, {1, -Dyad}] -> PsiCDe4Dyad[]}
```

Out[196]=
$$\{\Psi[\nabla]_{0000} \to \Psi_0 \ , \ \Psi[\nabla]_{0001} \to \Psi_1 \ , \ \Psi[\nabla]_{0011} \to \Psi_2 \ , \ \Psi[\nabla]_{0111} \to \Psi_3 \ , \ \Psi[\nabla]_{1111} \to \Psi_4 \ \}$$

In[197]:=
```
% /. Rule → ComponentValue
```

Out[197]=
$$\{\Psi[\nabla]_{0000} \to \Psi_0 \ , \ \Psi[\nabla]_{0001} \to \Psi_1 \ , \ \Psi[\nabla]_{0011} \to \Psi_2 \ , \ \Psi[\nabla]_{0111} \to \Psi_3 \ , \ \Psi[\nabla]_{1111} \to \Psi_4 \ \}$$

In[198]:=
```
Map[Dagger, %, {2}]
```

Out[198]=
$$\{\Psi[\nabla]\,\dagger_{0',0',0',0'} \to \overline{\Psi}_0 \ , \ \Psi[\nabla]\,\dagger_{0',0',0',1'} \to \overline{\Psi}_1 \ ,$$
$$\Psi[\nabla]\,\dagger_{0',0',1',1'} \to \overline{\Psi}_2 \ , \ \Psi[\nabla]\,\dagger_{0',1',1',1'} \to \overline{\Psi}_3 \ , \ \Psi[\nabla]\,\dagger_{1',1',1',1'} \to \overline{\Psi}_4 \ \}$$

In[199]:=
```
% /. Rule → ComponentValue
```

Out[199]=
$$\{\Psi[\nabla]\,\dagger_{0',0',0',0'} \to \overline{\Psi_0}\ ,\ \Psi[\nabla]\,\dagger_{0',0',0',1'} \to \overline{\Psi_1}\ ,\ \Psi[\nabla]\,\dagger_{0',0',1',1'} \to \overline{\Psi_2}\ ,\ \Psi[\nabla]\,\dagger_{0',1',1',1'} \to \overline{\Psi_3}\ ,\ \Psi[\nabla]\,\dagger_{1',1',1',1'} \to \overline{\Psi_4}\ \}$$

In[200]:=
```
RuleToSet@PsiCDe
```

In[201]:=
```
RuleToSet@PsiCDe†
```

We introduce the scalars which will represent the components of the Ricci spinor.

In[202]:=
```
DefTensor[#, M4, Dagger → Complex] & /@
   {PhiCDe00Dyad[], PhiCDe01Dyad[], PhiCDe02Dyad[], PhiCDe10Dyad[],
    PhiCDe11Dyad[], PhiCDe12Dyad[], PhiCDe20Dyad[], PhiCDe21Dyad[], PhiCDe22Dyad[]};
```

```
** DefTensor: Defining tensor PhiCDe00Dyad[].

** DefTensor: Defining tensor PhiCDe00Dyad†[].

** DefTensor: Defining tensor PhiCDe01Dyad[].

** DefTensor: Defining tensor PhiCDe01Dyad†[].

** DefTensor: Defining tensor PhiCDe02Dyad[].

** DefTensor: Defining tensor PhiCDe02Dyad†[].

** DefTensor: Defining tensor PhiCDe10Dyad[].

** DefTensor: Defining tensor PhiCDe10Dyad†[].

** DefTensor: Defining tensor PhiCDe11Dyad[].

** DefTensor: Defining tensor PhiCDe11Dyad†[].

** DefTensor: Defining tensor PhiCDe12Dyad[].

** DefTensor: Defining tensor PhiCDe12Dyad†[].

** DefTensor: Defining tensor PhiCDe20Dyad[].

** DefTensor: Defining tensor PhiCDe20Dyad†[].

** DefTensor: Defining tensor PhiCDe21Dyad[].

** DefTensor: Defining tensor PhiCDe21Dyad†[].

** DefTensor: Defining tensor PhiCDe22Dyad[].

** DefTensor: Defining tensor PhiCDe22Dyad†[].
```

The components of the Ricci spinor are formatted with their conventional names.

In[203]:=
```
PrintAs[PhiCDe00Dyad] ^= "Φ00";
PrintAs[PhiCDe01Dyad] ^= "Φ01";
PrintAs[PhiCDe02Dyad] ^= "Φ02";
PrintAs[PhiCDe10Dyad] ^= "Φ10";
PrintAs[PhiCDe11Dyad] ^= "Φ11";
PrintAs[PhiCDe12Dyad] ^= "Φ12";
PrintAs[PhiCDe20Dyad] ^= "Φ20";
PrintAs[PhiCDe21Dyad] ^= "Φ21";
PrintAs[PhiCDe22Dyad] ^= "Φ22";
```

In[212]:=
```
SetToRule@PhiCDe
```

Explicit definition of the components of the Ricci spinor.

In[213]:=
```
{PhiCDe[{0, -Dyad}, {0, -Dyad}, {0, -Dyad†}, {0, -Dyad†}] → PhiCDe00Dyad[],
 PhiCDe[{0, -Dyad}, {0, -Dyad}, {0, -Dyad†}, {1, -Dyad†}] → PhiCDe01Dyad[],
 PhiCDe[{0, -Dyad}, {0, -Dyad}, {1, -Dyad†}, {1, -Dyad†}] → PhiCDe02Dyad[],
 PhiCDe[{0, -Dyad}, {1, -Dyad}, {0, -Dyad†}, {0, -Dyad†}] → PhiCDe10Dyad[],
 PhiCDe[{1, -Dyad}, {1, -Dyad}, {0, -Dyad†}, {0, -Dyad†}] → PhiCDe20Dyad[],
 PhiCDe[{0, -Dyad}, {1, -Dyad}, {0, -Dyad†}, {1, -Dyad†}] → PhiCDe11Dyad[],
 PhiCDe[{1, -Dyad}, {1, -Dyad}, {0, -Dyad†}, {1, -Dyad†}] → PhiCDe21Dyad[],
 PhiCDe[{0, -Dyad}, {1, -Dyad}, {1, -Dyad†}, {1, -Dyad†}] → PhiCDe12Dyad[],
 PhiCDe[{1, -Dyad}, {1, -Dyad}, {1, -Dyad†}, {1, -Dyad†}] → PhiCDe22Dyad[]}
```

Out[213]=
$$\{\Phi[\nabla]_{000'0'} \to \Phi_{00}\ ,\ \Phi[\nabla]_{000'1'} \to \Phi_{01}\ ,\ \Phi[\nabla]_{001'1'} \to \Phi_{02}\ ,\ \Phi[\nabla]_{010'0'} \to \Phi_{10}\ ,$$
$$\Phi[\nabla]_{110'0'} \to \Phi_{20}\ ,\ \Phi[\nabla]_{010'1'} \to \Phi_{11}\ ,\ \Phi[\nabla]_{110'1'} \to \Phi_{21}\ ,\ \Phi[\nabla]_{011'1'} \to \Phi_{12}\ ,\ \Phi[\nabla]_{111'1'} \to \Phi_{22}\ \}$$

In[214]:=
```
% /. Rule → ComponentValue;
```

In[215]:=
```
ToValues@Map[Dagger, %, {2}]
```

Out[215]=
$$\{\Phi_{00} \to \text{PhiCDe00Dyad}\dagger\ ,\ \Phi_{10} \to \text{PhiCDe01Dyad}\dagger\ ,\ \Phi_{20} \to \text{PhiCDe02Dyad}\dagger\ ,$$
$$\Phi_{01} \to \text{PhiCDe10Dyad}\dagger\ ,\ \Phi_{02} \to \text{PhiCDe20Dyad}\dagger\ ,\ \Phi_{11} \to \text{PhiCDe11Dyad}\dagger\ ,$$
$$\Phi_{12} \to \text{PhiCDe21Dyad}\dagger\ ,\ \Phi_{21} \to \text{PhiCDe12Dyad}\dagger\ ,\ \Phi_{22} \to \text{PhiCDe22Dyad}\dagger\ \}$$

In[216]:=
```
% /. Rule → Equal
```

Out[216]=
$$\{\Phi_{00} == \text{PhiCDe00Dyad}\dagger\ ,\ \Phi_{10} == \text{PhiCDe01Dyad}\dagger\ ,\ \Phi_{20} == \text{PhiCDe02Dyad}\dagger\ ,$$
$$\Phi_{01} == \text{PhiCDe10Dyad}\dagger\ ,\ \Phi_{02} == \text{PhiCDe20Dyad}\dagger\ ,\ \Phi_{11} == \text{PhiCDe11Dyad}\dagger\ ,$$
$$\Phi_{12} == \text{PhiCDe21Dyad}\dagger\ ,\ \Phi_{21} == \text{PhiCDe12Dyad}\dagger\ ,\ \Phi_{22} == \text{PhiCDe22Dyad}\dagger\ \}$$

In[217]:=
```
Solve[%,
  {PhiCDe00Dyad†[], PhiCDe01Dyad†[], PhiCDe02Dyad†[], PhiCDe10Dyad†[], PhiCDe11Dyad†[],
   PhiCDe12Dyad†[], PhiCDe20Dyad†[], PhiCDe21Dyad†[], PhiCDe22Dyad†[]}] // Flatten
```

Out[217]=
$$\{\text{PhiCDe00Dyad}\dagger \to \Phi_{00}\ ,\ \text{PhiCDe01Dyad}\dagger \to \Phi_{10}\ ,\ \text{PhiCDe02Dyad}\dagger \to \Phi_{20}\ ,$$
$$\text{PhiCDe10Dyad}\dagger \to \Phi_{01}\ ,\ \text{PhiCDe20Dyad}\dagger \to \Phi_{02}\ ,\ \text{PhiCDe11Dyad}\dagger \to \Phi_{11}\ ,$$
$$\text{PhiCDe21Dyad}\dagger \to \Phi_{12}\ ,\ \text{PhiCDe12Dyad}\dagger \to \Phi_{21}\ ,\ \text{PhiCDe22Dyad}\dagger \to \Phi_{22}\ \}$$

In[218]:=
```
% /. Rule → Set;
```

In[219]:=
```
RuleToSet@PhiCDe
```

In[220]:=
```
PrintAs[LambdaCDe] ^= "Λ";
```

Computation of the folded rule for the Riemann tensor.

In[221]:=
```
AllComponentValues@ToBasis[NP]@RiemannCD[-a, -b, -c, -d];
```

In[222]:=
```
RuleToSet@RiemannCD
```

Next we show how to write Riemann tensor components in terms of the components of curvature spinors

In[223]:=
```
RiemannCD[-a, -b, -c, -d]
```

Out[223]=
$$R[\nabla]_{abcd}$$

In[224]:=
```
% == Expand@Decomposition[SeparateSolderingForm[][%], Riemann]
```

Out[224]=
$$R[\nabla]_{abcd} == \Psi[\nabla]\dagger_{A'B'D'C'}\,\epsilon_{AB}\,\epsilon_{DC}\,\sigma_a{}^{AA'}\,\sigma_b{}^{BB'}\,\sigma_c{}^{CC'}\,\sigma_d{}^{DD'} - \Phi[\nabla]_{ABD'C'}\,\epsilon_{DC}\,\overline{\epsilon}_{A'B'}\,\sigma_a{}^{AA'}\,\sigma_b{}^{BB'}\,\sigma_c{}^{CC'}\,\sigma_d{}^{DD'} -$$
$$2\,\Lambda\ \epsilon_{AD}\,\epsilon_{BC}\,\overline{\epsilon}_{A'D'}\,\overline{\epsilon}_{B'C'}\,\sigma_a{}^{AA'}\,\sigma_b{}^{BB'}\,\sigma_c{}^{CC'}\,\sigma_d{}^{DD'} + 2\,\Lambda\ \epsilon_{AC}\,\epsilon_{BD}\,\overline{\epsilon}_{A'C'}\,\overline{\epsilon}_{B'D'}\,\sigma_a{}^{AA'}\,\sigma_b{}^{BB'}\,\sigma_c{}^{CC'}\,\sigma_d{}^{DD'} -$$
$$\Phi[\nabla]_{DCA'B'}\,\epsilon_{AB}\,\overline{\epsilon}_{D'C'}\,\sigma_a{}^{AA'}\,\sigma_b{}^{BB'}\,\sigma_c{}^{CC'}\,\sigma_d{}^{DD'} + \Psi[\nabla]_{ABDC}\,\overline{\epsilon}_{A'B'}\,\overline{\epsilon}_{D'C'}\,\sigma_a{}^{AA'}\,\sigma_b{}^{BB'}\,\sigma_c{}^{CC'}\,\sigma_d{}^{DD'}$$

In[225]:= **ToBasis[Dyad†] /@ ToBasis[Dyad] /@ ToBasis[NP] /@ %**

Out[225]= $R[\nabla]_{abcd} == \Psi[\nabla]\,\dagger_{A'B'D'C'}\,\epsilon_{AB}\,\epsilon_{DC}\,\sigma_a{}^{AA'}\,\sigma_b{}^{BB'}\,\sigma_c{}^{CC'}\,\sigma_d{}^{DD'} - \Phi[\nabla]_{ABD'C'}\,\epsilon_{DC}\,\overline{\epsilon}_{A'B'}\,\sigma_a{}^{AA'}\,\sigma_b{}^{BB'}\,\sigma_c{}^{CC'}\,\sigma_d{}^{DD'} - 2\,\Lambda\,\epsilon_{AD}\,\epsilon_{BC}\,\overline{\epsilon}_{A'D'}\,\overline{\epsilon}_{B'C'}\,\sigma_a{}^{AA'}\,\sigma_b{}^{BB'}\,\sigma_c{}^{CC'}\,\sigma_d{}^{DD'} + 2\,\Lambda\,\epsilon_{AC}\,\epsilon_{BD}\,\overline{\epsilon}_{A'C'}\,\overline{\epsilon}_{B'D'}\,\sigma_a{}^{AA'}\,\sigma_b{}^{BB'}\,\sigma_c{}^{CC'}\,\sigma_d{}^{DD'} - \Phi[\nabla]_{DCA'B'}\,\epsilon_{AB}\,\overline{\epsilon}_{D'C'}\,\sigma_a{}^{AA'}\,\sigma_b{}^{BB'}\,\sigma_c{}^{CC'}\,\sigma_d{}^{DD'} + \Psi[\nabla]_{ABDC}\,\overline{\epsilon}_{A'B'}\,\overline{\epsilon}_{D'C'}\,\sigma_a{}^{AA'}\,\sigma_b{}^{BB'}\,\sigma_c{}^{CC'}\,\sigma_d{}^{DD'}$

In[226]:= **TraceBasisDummy /@ %;**

In[227]:= **Flatten@ComponentArray@%;**

In[228]:= **Union@%**

Out[228]= $\{\text{True}, -R[\nabla]_{1212} == 2\Lambda - 2\Phi_{11} + \Psi_2 + \overline{\Psi}_2,\ R[\nabla]_{1212} == -2\Lambda + 2\Phi_{11} - \Psi_2 - \overline{\Psi}_2,\ -R[\nabla]_{1213} == -\Phi_{01} + \Psi_1,\ R[\nabla]_{1213} == \Phi_{01} - \Psi_1,\ -R[\nabla]_{1214} == -\Phi_{10} + \overline{\Psi}_1,\ R[\nabla]_{1214} == \Phi_{10} - \overline{\Psi}_1,\ -R[\nabla]_{1223} == \Phi_{12} - \overline{\Psi}_3,\ R[\nabla]_{1223} == -\Phi_{12} + \overline{\Psi}_3,\ -R[\nabla]_{1224} == \Phi_{21} - \Psi_3,\ R[\nabla]_{1224} == -\Phi_{21} + \Psi_3,\ -R[\nabla]_{1234} == -\Psi_2 + \overline{\Psi}_2,\ R[\nabla]_{1234} == \Psi_2 - \overline{\Psi}_2,\ -R[\nabla]_{1313} == \Psi_0,\ R[\nabla]_{1313} == -\Psi_0,\ -R[\nabla]_{1314} == -\Phi_{00},\ R[\nabla]_{1314} == \Phi_{00},\ -R[\nabla]_{1323} == \Phi_{02},\ R[\nabla]_{1323} == -\Phi_{02},\ -R[\nabla]_{1324} == 2\Lambda - \Psi_2,\ R[\nabla]_{1324} == -2\Lambda + \Psi_2,\ -R[\nabla]_{1334} == -\Phi_{01} - \Psi_1,\ R[\nabla]_{1334} == \Phi_{01} + \Psi_1,\ -R[\nabla]_{1414} == \overline{\Psi}_0,\ R[\nabla]_{1414} == -\overline{\Psi}_0,\ -R[\nabla]_{1423} == 2\Lambda - \overline{\Psi}_2,\ R[\nabla]_{1423} == -2\Lambda + \overline{\Psi}_2,\ -R[\nabla]_{1424} == \Phi_{20},\ R[\nabla]_{1424} == -\Phi_{20},\ -R[\nabla]_{1434} == \Phi_{10} + \overline{\Psi}_1,\ R[\nabla]_{1434} == -\Phi_{10} - \overline{\Psi}_1,\ -R[\nabla]_{2323} == \overline{\Psi}_4,\ R[\nabla]_{2323} == -\overline{\Psi}_4,\ -R[\nabla]_{2324} == -\Phi_{22},\ R[\nabla]_{2324} == \Phi_{22},\ -R[\nabla]_{2334} == -\Phi_{12} - \overline{\Psi}_3,\ R[\nabla]_{2334} == \Phi_{12} + \overline{\Psi}_3,\ -R[\nabla]_{2424} == \Psi_4,\ R[\nabla]_{2424} == -\Psi_4,\ -R[\nabla]_{2434} == \Phi_{21} + \Psi_3,\ R[\nabla]_{2434} == -\Phi_{21} - \Psi_3,\ -R[\nabla]_{3434} == 2\Lambda + 2\Phi_{11} + \Psi_2 + \overline{\Psi}_2,\ R[\nabla]_{3434} == -2\Lambda - 2\Phi_{11} - \Psi_2 - \overline{\Psi}_2\}$

In[229]:= **Solve[%, Union@Cases[%, _RiemannCD, Infinity]] // Flatten**

Out[229]= $\{R[\nabla]_{1212} \to -2\Lambda + 2\Phi_{11} - \Psi_2 - \overline{\Psi}_2,\ R[\nabla]_{1213} \to \Phi_{01} - \Psi_1,\ R[\nabla]_{1214} \to \Phi_{10} - \overline{\Psi}_1,\ R[\nabla]_{1223} \to -\Phi_{12} + \overline{\Psi}_3,\ R[\nabla]_{1224} \to -\Phi_{21} + \Psi_3,\ R[\nabla]_{1234} \to \Psi_2 - \overline{\Psi}_2,\ R[\nabla]_{1313} \to -\Psi_0,\ R[\nabla]_{1314} \to \Phi_{00},\ R[\nabla]_{1323} \to -\Phi_{02},\ R[\nabla]_{1324} \to -2\Lambda + \Psi_2,\ R[\nabla]_{1334} \to \Phi_{01} + \Psi_1,\ R[\nabla]_{1414} \to -\overline{\Psi}_0,\ R[\nabla]_{1423} \to -2\Lambda + \overline{\Psi}_2,\ R[\nabla]_{1424} \to -\Phi_{20},\ R[\nabla]_{1434} \to -\Phi_{10} - \overline{\Psi}_1,\ R[\nabla]_{2323} \to -\overline{\Psi}_4,\ R[\nabla]_{2324} \to \Phi_{22},\ R[\nabla]_{2334} \to \Phi_{12} + \overline{\Psi}_3,\ R[\nabla]_{2424} \to -\Psi_4,\ R[\nabla]_{2434} \to -\Phi_{21} - \Psi_3,\ R[\nabla]_{3434} \to -2\Lambda - 2\Phi_{11} - \Psi_2 - \overline{\Psi}_2\}$

We add the values just found to the folded rule containg the components of the Riemann tensor.

In[230]:= **SetToRule@RiemannCD**

In[231]:= **%% /. Rule -> ComponentValue;**

In[232]:= **RuleToSet@RiemannCD**

From previous computations we can obtain and store the values for the components of the Ricci tensor.

In[233]:= **AllComponentValues@ToBasis[NP]@RicciCD[-a, -b];**

In[234]:= **RicciCD[-a, -b] = g[p, q] RiemannCD[-a, -p, -b, -q]**

Out[234]= $R[\nabla]_{ab} == g^{pq}\,R[\nabla]_{apbq}$

In[235]:= **ToBasis[NP] /@ %**

Out[235]= $R[\nabla]_{ab} == g^{cd}\,R[\nabla]_{acbd}$

In[236]:= **TraceBasisDummy /@ %**

Out[236]= $R[\nabla]_{ab} == R[\nabla]_{a1b2} + R[\nabla]_{a2b1} - R[\nabla]_{a3b4} - R[\nabla]_{a4b3}$

In[237]:= **Flatten@ComponentArray@%**

Out[237]= $\{R[\nabla]_{1\,1} == -2\,\Phi_{00}\ ,\ R[\nabla]_{1\,2} == 6\,\Lambda\ -2\,\Phi_{11}\ ,\ R[\nabla]_{1\,3} == -2\,\Phi_{01}\ ,\ R[\nabla]_{1\,4} == -2\,\Phi_{10}\ ,$
$R[\nabla]_{2\,1} == 6\,\Lambda\ -2\,\Phi_{11}\ ,\ R[\nabla]_{2\,2} == -2\,\Phi_{22}\ ,\ R[\nabla]_{2\,3} == -2\,\Phi_{12}\ ,\ R[\nabla]_{2\,4} == -2\,\Phi_{21}\ ,$
$R[\nabla]_{3\,1} == -2\,\Phi_{01}\ ,\ R[\nabla]_{3\,2} == -2\,\Phi_{12}\ ,\ R[\nabla]_{3\,3} == -2\,\Phi_{02}\ ,\ R[\nabla]_{3\,4} == -6\,\Lambda\ -2\,\Phi_{11}\ ,$
$R[\nabla]_{4\,1} == -2\,\Phi_{10}\ ,\ R[\nabla]_{4\,2} == -2\,\Phi_{21}\ ,\ R[\nabla]_{4\,3} == -6\,\Lambda\ -2\,\Phi_{11}\ ,\ R[\nabla]_{4\,4} == -2\,\Phi_{20}\ \}$

In[238]:= **Union@%**

Out[238]= $\{R[\nabla]_{1\,1} == -2\,\Phi_{00}\ ,\ R[\nabla]_{1\,2} == 6\,\Lambda\ -2\,\Phi_{11}\ ,\ R[\nabla]_{1\,3} == -2\,\Phi_{01}\ ,\ R[\nabla]_{1\,4} == -2\,\Phi_{10}\ ,$
$R[\nabla]_{2\,1} == 6\,\Lambda\ -2\,\Phi_{11}\ ,\ R[\nabla]_{2\,2} == -2\,\Phi_{22}\ ,\ R[\nabla]_{2\,3} == -2\,\Phi_{12}\ ,\ R[\nabla]_{2\,4} == -2\,\Phi_{21}\ ,$
$R[\nabla]_{3\,1} == -2\,\Phi_{01}\ ,\ R[\nabla]_{3\,2} == -2\,\Phi_{12}\ ,\ R[\nabla]_{3\,3} == -2\,\Phi_{02}\ ,\ R[\nabla]_{3\,4} == -6\,\Lambda\ -2\,\Phi_{11}\ ,$
$R[\nabla]_{4\,1} == -2\,\Phi_{10}\ ,\ R[\nabla]_{4\,2} == -2\,\Phi_{21}\ ,\ R[\nabla]_{4\,3} == -6\,\Lambda\ -2\,\Phi_{11}\ ,\ R[\nabla]_{4\,4} == -2\,\Phi_{20}\ \}$

In[239]:= **% /. Equal → ComponentValue;**

In[240]:= **RuleToSet@RicciCD**

The scalar curvature is worked out in a similar fashion.

In[241]:= **RicciScalarCD[] = ToBasis[NP]@g[a, b] RicciCD[-a, -b]**

Out[241]= $R[\nabla] == g^{a\,b}\,R[\nabla]_{a\,b}$

In[242]:= **ToBasis[NP] /@ %**

Out[242]= $R[\nabla] == g^{a\,b}\,R[\nabla]_{a\,b}$

In[243]:= **TraceBasisDummy /@ %**

Out[243]= $R[\nabla] == 24\,\Lambda$

In[244]:= **% /. Equal → ComponentValue**

Out[244]= $R[\nabla] \to 24\,\Lambda$

In[245]:= **RuleToSet@RicciScalarCD**

We may also need the components of the traceless part of the Ricci tensor.

In[246]:= **AllComponentValues@ToBasis[NP]@TFRicciCD[-a, -b];**

In[247]:= **TFRicciCD[-a, -b] == (TFRicciCD[-a, -b] // TFRicciToRicci)**

Out[247]= $S[\nabla]_{a\,b} == -6\,g_{b\,a}\,\Lambda\ + R[\nabla]_{a\,b}$

In[248]:= **ToBasis[NP] /@ %**

Out[248]= $S[\nabla]_{a\,b} == -6\,g_{b\,a}\,\Lambda\ + R[\nabla]_{a\,b}$

In[249]:= **ToValues@Flatten@ComponentArray@%**

Out[249]= $\{S[\nabla]_{1\,1} == -2\,\Phi_{00}\ ,\ S[\nabla]_{1\,2} == -2\,\Phi_{11}\ ,\ S[\nabla]_{1\,3} == -2\,\Phi_{01}\ ,\ S[\nabla]_{1\,4} == -2\,\Phi_{10}\ ,$
$S[\nabla]_{1\,2} == -2\,\Phi_{11}\ ,\ S[\nabla]_{2\,2} == -2\,\Phi_{22}\ ,\ S[\nabla]_{2\,3} == -2\,\Phi_{12}\ ,\ S[\nabla]_{2\,4} == -2\,\Phi_{21}\ ,$
$S[\nabla]_{1\,3} == -2\,\Phi_{01}\ ,\ S[\nabla]_{2\,3} == -2\,\Phi_{12}\ ,\ S[\nabla]_{3\,3} == -2\,\Phi_{02}\ ,\ S[\nabla]_{3\,4} == -2\,\Phi_{11}\ ,$
$S[\nabla]_{1\,4} == -2\,\Phi_{10}\ ,\ S[\nabla]_{2\,4} == -2\,\Phi_{21}\ ,\ S[\nabla]_{3\,4} == -2\,\Phi_{11}\ ,\ S[\nabla]_{4\,4} == -2\,\Phi_{20}\ \}$

In[250]:= **Union@%**

Out[250]= $\{S[\nabla]_{1\,1} == -2\,\Phi_{00}\ ,\ S[\nabla]_{1\,2} == -2\,\Phi_{11}\ ,\ S[\nabla]_{1\,3} == -2\,\Phi_{01}\ ,\ S[\nabla]_{1\,4} == -2\,\Phi_{10}\ ,\ S[\nabla]_{2\,2} == -2\,\Phi_{22}\ ,$
$S[\nabla]_{2\,3} == -2\,\Phi_{12}\ ,\ S[\nabla]_{2\,4} == -2\,\Phi_{21}\ ,\ S[\nabla]_{3\,3} == -2\,\Phi_{02}\ ,\ S[\nabla]_{3\,4} == -2\,\Phi_{11}\ ,\ S[\nabla]_{4\,4} == -2\,\Phi_{20}\ \}$

In[251]:= **% /. Equal -> ComponentValue**

Out[251]= $\{S[\nabla]_{11} \to -2\,\Phi_{00}\ ,\ S[\nabla]_{12} \to -2\,\Phi_{11}\ ,\ S[\nabla]_{13} \to -2\,\Phi_{01}\ ,\ S[\nabla]_{14} \to -2\,\Phi_{10}\ ,\ S[\nabla]_{22} \to -2\,\Phi_{22}\ ,$
$S[\nabla]_{23} \to -2\,\Phi_{12}\ ,\ S[\nabla]_{24} \to -2\,\Phi_{21}\ ,\ S[\nabla]_{33} \to -2\,\Phi_{02}\ ,\ S[\nabla]_{34} \to -2\,\Phi_{11}\ ,\ S[\nabla]_{44} \to -2\,\Phi_{20}\ \}$

In[252]:= **RuleToSet@TFRicciCD**

This is the complete list of the components of the Riemann tensor.

In[253]:=
```
NPCVComponents = {PsiCDe0Dyad[], PsiCDe1Dyad[], PsiCDe2Dyad[], PsiCDe3Dyad[], PsiCDe4Dyad[],
  PhiCDe00Dyad[], PhiCDe01Dyad[], PhiCDe02Dyad[], PhiCDe10Dyad[], PhiCDe11Dyad[],
  PhiCDe12Dyad[], PhiCDe20Dyad[], PhiCDe21Dyad[], PhiCDe22Dyad[], LambdaCDe[]}
```

Out[253]= $\{\Psi_0\ ,\ \Psi_1\ ,\ \Psi_2\ ,\ \Psi_3\ ,\ \Psi_4\ ,\ \Phi_{00}\ ,\ \Phi_{01}\ ,\ \Phi_{02}\ ,\ \Phi_{10}\ ,\ \Phi_{11}\ ,\ \Phi_{12}\ ,\ \Phi_{20}\ ,\ \Phi_{21}\ ,\ \Phi_{22}\ ,\ \Lambda\ \}$

■ The Newman–Penrose equations

We now calculate the Riemann tensor with respect to the null tetrad

In[254]:= **RiemannCD[-a, -b, -c, -d] == ChangeCurvature[RiemannCD[-a, -b, -c, -d], CD, PDNP]**

```
** DefTensor: Defining tensor ChristoffelCDPDNP[a, -b, -c].
```

Out[254]= $R[\nabla]_{abcd} == g_{df}\left(\Gamma[\nabla,\ ]^{f}{}_{bh}\,\Gamma[\nabla,\ ]^{h}{}_{ac} - \Gamma[\nabla,\ ]^{f}{}_{ah}\,\Gamma[\nabla,\ ]^{h}{}_{bc} - \Gamma[\nabla,\ ]^{f}{}_{hc}\,T^{h}{}_{ab} - {}_{a}\Gamma[\nabla,\ ]^{f}{}_{bc} + {}_{b}\Gamma[\nabla,\ ]^{f}{}_{ac}\right)$

In[255]:= **% /. ChristoffelCDPDNP -> ChristoffelCDePDNP**

Out[255]= $R[\nabla]_{abcd} == g_{df}\left(\Gamma[\nabla,\ ]^{f}{}_{bh}\,\Gamma[\nabla,\ ]^{h}{}_{ac} - \Gamma[\nabla,\ ]^{f}{}_{ah}\,\Gamma[\nabla,\ ]^{h}{}_{bc} - \Gamma[\nabla,\ ]^{f}{}_{hc}\,T^{h}{}_{ab} - {}_{a}\Gamma[\nabla,\ ]^{f}{}_{bc} + {}_{b}\Gamma[\nabla,\ ]^{f}{}_{ac}\right)$

In[256]:= **ToBasis[NP] /@ %**

Out[256]= $R[\nabla]_{abcd} == \Gamma[\nabla,\ ]^{f}{}_{bh}\,\Gamma[\nabla,\ ]^{h}{}_{ac}\,g_{df} - \Gamma[\nabla,\ ]^{f}{}_{ah}\,\Gamma[\nabla,\ ]^{h}{}_{bc}\,g_{df} -$
$\Gamma[\nabla,\ ]^{f}{}_{hc}\,g_{df}\,T^{h}{}_{ab} - e_{b}{}^{h}\,e_{c}{}^{j}\,g_{df}\left({}_{a}\Gamma[\nabla,\ ]^{f}{}_{hj}\right) + e_{a}{}^{h}\,e_{c}{}^{j}\,g_{df}\left({}_{b}\Gamma[\nabla,\ ]^{f}{}_{hj}\right)$

In[257]:= **ToBasis[NP] /@ %**

Out[257]= $R[\nabla]_{abcd} == \Gamma[\nabla,\ ]^{f}{}_{bh}\,\Gamma[\nabla,\ ]^{h}{}_{ac}\,g_{df} - \Gamma[\nabla,\ ]^{f}{}_{ah}\,\Gamma[\nabla,\ ]^{h}{}_{bc}\,g_{df} -$
$\Gamma[\nabla,\ ]^{f}{}_{hc}\,g_{df}\,T^{h}{}_{ab} - g_{df}\left({}_{a}\Gamma[\nabla,\ ]^{f}{}_{bc}\right) + g_{df}\left({}_{b}\Gamma[\nabla,\ ]^{f}{}_{ac}\right)$

In[258]:= **TraceBasisDummy /@ %;**

In[259]:= **Flatten@ComponentArray@%;**

From here we obtain directly the Newman–Penrose equations

In[260]:= **Union[Simplify //@ Union[%]];**

In[261]:= **NPRicci = %**

Out[261]= $$\Big\{\text{True},\ \Phi_{01} + 2\,\overline{\alpha}\ \overline{\epsilon} + 2\,\beta\ \overline{\epsilon} + \gamma\ \kappa + \overline{\gamma}\ \kappa + \overline{\kappa}\ \overline{\lambda} + \kappa\ \mu + \mathrm{D}\,\overline{\alpha} + \mathrm{D}\,\beta ==$$
$$\Psi_1 + \epsilon\ \overline{\pi} + \overline{\epsilon}\ \overline{\pi} + \overline{\alpha}\ \overline{\rho} + \beta\ \overline{\rho} + \overline{\pi}\ \overline{\rho} + \alpha\ \sigma + \overline{\beta}\ \sigma + \pi\ \sigma + \delta\,\epsilon + \delta\,\overline{\epsilon},$$
$$\Phi_{10} + 2\,\alpha\ \epsilon + 2\,\overline{\beta}\ \epsilon + \gamma\ \overline{\kappa} + \overline{\gamma}\ \overline{\kappa} + \kappa\ \lambda + \overline{\kappa}\ \overline{\mu} + \mathrm{D}\,\alpha + \mathrm{D}\,\overline{\beta} ==$$
$$\overline{\Psi}_1 + \epsilon\ \pi + \overline{\epsilon}\ \pi + \alpha\ \rho + \overline{\beta}\ \rho + \pi\ \rho + \overline{\alpha}\ \overline{\sigma} + \beta\ \overline{\sigma} + \overline{\pi}\ \overline{\sigma} + \overline{\delta}\,\epsilon + \overline{\delta}\,\overline{\epsilon},$$
$$2\,\Phi_{11} + 2\,(\gamma + \overline{\gamma})\,(\epsilon + \overline{\epsilon}) + \kappa\ \nu + \overline{\kappa}\ \overline{\nu} + \mathrm{D}\,\gamma + \mathrm{D}\,\overline{\gamma} ==$$
$$2\,\Lambda + \Psi_2 + \overline{\Psi}_2 + \pi\ \tau + \left(\alpha + \overline{\beta}\right)(\overline{\pi} + \tau) + \overline{\pi}\ \overline{\tau} + (\overline{\alpha} + \beta)(\pi + \overline{\tau}) + \Delta\,\epsilon + \Delta\,\overline{\epsilon},$$
$$\Phi_{20} + 3\,\epsilon\ \lambda + \overline{\kappa}\ \nu + \overline{\beta}\ \pi + \mathrm{D}\,\lambda == \overline{\epsilon}\ \lambda + \alpha\ \pi + \pi^2 + \lambda\ \rho + \mu\ \overline{\sigma} + \overline{\delta}\,\pi,$$
$$\Phi_{02} + 3\,\overline{\epsilon}\ \overline{\lambda} + \kappa\ \overline{\nu} + \beta\ \overline{\pi} + \mathrm{D}\,\overline{\lambda} == \epsilon\ \overline{\lambda} + \overline{\alpha}\ \overline{\pi} + \overline{\pi}^2 + \overline{\lambda}\ \overline{\rho} + \overline{\mu}\ \sigma + \delta\,\overline{\pi},$$
$$2\,\Lambda + \epsilon\ \mu + \overline{\epsilon}\ \mu + \kappa\ \nu + \overline{\alpha}\ \pi + \mathrm{D}\,\mu == \Psi_2 + \beta\ \pi + \pi\ \overline{\pi} + \mu\ \overline{\rho} + \lambda\ \sigma + \delta\,\pi,$$
$$2\,\Lambda + \epsilon\ \overline{\mu} + \overline{\epsilon}\ \overline{\mu} + \overline{\kappa}\ \overline{\nu} + \alpha\ \overline{\pi} + \mathrm{D}\,\overline{\mu} == \overline{\Psi}_2 + \overline{\beta}\ \overline{\pi} + \pi\ \overline{\pi} + \overline{\mu}\ \rho + \overline{\lambda}\ \overline{\sigma} + \overline{\delta}\,\overline{\pi},$$
$$\Phi_{21} + 3\,\epsilon\ \nu + \overline{\epsilon}\ \nu + \overline{\gamma}\ \pi + \mathrm{D}\,\nu == \Psi_3 + \gamma\ \pi + \mu\ \pi + \lambda\ \overline{\pi} + \lambda\ \tau + \mu\ \overline{\tau} + \Delta\,\pi,$$
$$\Phi_{12} + \epsilon\ \overline{\nu} + 3\,\overline{\epsilon}\ \overline{\nu} + \gamma\ \overline{\pi} + \mathrm{D}\,\overline{\nu} == \overline{\Psi}_3 + \overline{\lambda}\ \pi + \overline{\gamma}\ \overline{\pi} + \overline{\mu}\ \overline{\pi} + \overline{\mu}\ \tau + \overline{\lambda}\ \overline{\tau} + \Delta\,\overline{\pi},$$
$$\Phi_{00} + 3\,\alpha\ \kappa + \overline{\beta}\ \kappa + \overline{\kappa}\ \tau + \mathrm{D}\,\rho == \kappa\ \pi + \epsilon\ \rho + \overline{\epsilon}\ \rho + \rho^2 + \sigma\ \overline{\sigma} + \overline{\delta}\,\kappa,$$
$$\Phi_{00} + 3\,\overline{\alpha}\ \overline{\kappa} + \beta\ \overline{\kappa} + \kappa\ \overline{\tau} + \mathrm{D}\,\overline{\rho} == \overline{\kappa}\ \overline{\pi} + \epsilon\ \overline{\rho} + \overline{\epsilon}\ \overline{\rho} + \overline{\rho}^2 + \sigma\ \overline{\sigma} + \delta\,\overline{\kappa},$$
$$\Phi_{01} + 3\,\gamma\ \kappa + \overline{\gamma}\ \kappa + \overline{\epsilon}\ \tau + \mathrm{D}\,\tau == \Psi_1 + \overline{\pi}\ \rho + \pi\ \sigma + \epsilon\ \tau + \rho\ \tau + \sigma\ \overline{\tau} + \Delta\,\kappa,$$
$$\Phi_{10} + \gamma\ \overline{\kappa} + 3\,\overline{\gamma}\ \overline{\kappa} + \epsilon\ \overline{\tau} + \mathrm{D}\,\overline{\tau} == \overline{\Psi}_1 + \pi\ \overline{\rho} + \overline{\pi}\ \overline{\sigma} + \overline{\sigma}\ \tau + \overline{\epsilon}\ \overline{\tau} + \overline{\rho}\ \overline{\tau} + \Delta\,\overline{\kappa},$$
$$\Psi_2 + 2\,\overline{\gamma}\ \overline{\epsilon} + \overline{\kappa}\ \overline{\nu} + \beta\ \pi + \alpha\ \overline{\pi} + \alpha\ \tau + \pi\ \tau + \beta\ \overline{\tau} + \mathrm{D}\,\overline{\gamma} + \Delta\,\epsilon ==$$
$$\overline{\Psi}_2 + 2\,\gamma\ \epsilon + \kappa\ \nu + \overline{\alpha}\ \pi + \overline{\beta}\ \overline{\pi} + \overline{\beta}\ \tau + \overline{\alpha}\ \overline{\tau} + \overline{\pi}\ \overline{\tau} + \mathrm{D}\,\gamma + \Delta\,\overline{\epsilon},$$
$$2\,\Lambda + \Psi_2 + \overline{\Psi}_2 + \pi\ \tau + \left(\alpha + \overline{\beta}\right)(\overline{\pi} + \tau) + \overline{\pi}\ \overline{\tau} + (\overline{\alpha} + \beta)(\pi + \overline{\tau}) + \Delta\,\epsilon + \Delta\,\overline{\epsilon} ==$$
$$2\,\Phi_{11} + 2\,(\gamma + \overline{\gamma})\,(\epsilon + \overline{\epsilon}) + \kappa\ \nu + \overline{\kappa}\ \overline{\nu} + \mathrm{D}\,\gamma + \mathrm{D}\,\overline{\gamma},$$
$$\Psi_4 + 3\,\gamma\ \lambda + \lambda\ \mu + \lambda\ \overline{\mu} + \nu\ \overline{\tau} + \Delta\,\lambda == \overline{\gamma}\ \lambda + \nu\ \left(3\,\alpha + \overline{\beta} + \pi\right) + \overline{\delta}\,\nu,$$
$$\overline{\Psi}_4 + 3\,\overline{\gamma}\ \overline{\lambda} + \overline{\lambda}\ \mu + \overline{\lambda}\ \overline{\mu} + \overline{\nu}\ \tau + \Delta\,\overline{\lambda} == \gamma\ \overline{\lambda} + \overline{\nu}\ (3\,\overline{\alpha} + \beta + \overline{\pi}) + \delta\,\overline{\nu},$$
$$\Psi_2 + 2\,\alpha\ \beta + \epsilon\ \overline{\mu} + \overline{\epsilon}\ \overline{\mu} + \gamma\ \overline{\rho} + \overline{\gamma}\ \overline{\rho} + \overline{\mu}\ \overline{\rho} + \lambda\ \sigma + \delta\,\alpha + \delta\,\overline{\beta} ==$$
$$\overline{\Psi}_2 + 2\,\overline{\alpha}\ \overline{\beta} + \epsilon\ \mu + \overline{\epsilon}\ \mu + \gamma\ \rho + \overline{\gamma}\ \rho + \mu\ \rho + \overline{\lambda}\ \overline{\sigma} + \overline{\delta}\,\overline{\alpha} + \overline{\delta}\,\beta,$$
$$\Phi_{12} + \overline{\Psi}_3 + 2\,\beta\ \gamma + \overline{\beta}\ \overline{\lambda} + \overline{\alpha}\ \mu + \epsilon\ \overline{\nu} + \nu\ \sigma + \overline{\gamma}\ \tau + \overline{\lambda}\ \overline{\tau} + \Delta\,\overline{\alpha} + \delta\,\gamma ==$$
$$2\,\beta\ \overline{\gamma} + \alpha\ \overline{\lambda} + \beta\ \mu + \overline{\epsilon}\ \overline{\nu} + \overline{\nu}\ \overline{\rho} + \gamma\ \tau + \mu\ \tau + \Delta\,\beta + \delta\,\overline{\gamma},$$
$$\Phi_{12} + 2\,\overline{\alpha}\ \gamma + 2\,\beta\ \gamma + \epsilon\ \overline{\nu} + \overline{\epsilon}\ \overline{\nu} + \overline{\nu}\ \overline{\rho} + \nu\ \sigma + \delta\,\gamma + \delta\,\overline{\gamma} ==$$
$$\overline{\Psi}_3 + \alpha\ \overline{\lambda} + \overline{\beta}\ \overline{\lambda} + \overline{\alpha}\ \mu + \beta\ \mu + \gamma\ \tau + \overline{\gamma}\ \tau + \mu\ \tau + \overline{\lambda}\ \overline{\tau} + \Delta\,\overline{\alpha} + \Delta\,\beta,$$
$$\Phi_{01} + \Psi_1 + 2\,\overline{\alpha}\ \overline{\epsilon} + \overline{\gamma}\ \kappa + \overline{\kappa}\ \overline{\lambda} + \epsilon\ \overline{\pi} + \beta\ \overline{\rho} + \alpha\ \sigma + \pi\ \sigma + \mathrm{D}\,\overline{\alpha} + \delta\,\epsilon ==$$
$$2\,\overline{\alpha}\ \epsilon + \gamma\ \kappa + \kappa\ \mu + \overline{\epsilon}\ \overline{\pi} + \overline{\alpha}\ \overline{\rho} + \overline{\pi}\ \overline{\rho} + \overline{\beta}\ \sigma + \mathrm{D}\,\beta + \delta\,\overline{\epsilon},$$
$$\Psi_0 + \kappa\ \overline{\pi} + 3\,\epsilon\ \sigma + \rho\ \sigma + \overline{\rho}\ \sigma + \delta\,\kappa == \overline{\alpha}\ \kappa + 3\,\beta\ \kappa + \overline{\epsilon}\ \sigma + \kappa\ \tau + \mathrm{D}\,\sigma,$$
$$\Phi_{21} + \Psi_3 + 3\,\beta\ \lambda + \overline{\mu}\ \pi + \nu\ \overline{\rho} + \delta\,\lambda == \overline{\alpha}\ \lambda + \alpha\ \mu + \overline{\beta}\ \mu + \mu\ \pi + \nu\ \rho + \overline{\delta}\,\mu,$$
$$\Phi_{22} + \overline{\alpha}\ \nu + 3\,\beta\ \nu + \overline{\nu}\ \pi + \delta\,\nu == \lambda\ \overline{\lambda} + \gamma\ \mu + \overline{\gamma}\ \mu + \mu^2 + \nu\ \tau + \Delta\,\mu,$$
$$\gamma\ \overline{\lambda} + \overline{\nu}\ (3\,\overline{\alpha} + \beta + \overline{\pi}) + \delta\,\overline{\nu} == \overline{\Psi}_4 + 3\,\overline{\gamma}\ \overline{\lambda} + \overline{\lambda}\ \mu + \overline{\lambda}\ \overline{\mu} + \overline{\nu}\ \tau + \Delta\,\overline{\lambda},$$
$$\Phi_{01} + \Psi_1 + \kappa\ \overline{\mu} + 3\,\alpha\ \sigma + \overline{\rho}\ \tau + \delta\,\rho == \kappa\ \mu + \overline{\alpha}\ \rho + \beta\ \rho + \overline{\beta}\ \sigma + \rho\ \tau + \overline{\delta}\,\sigma,$$
$$\Phi_{02} + \kappa\ \overline{\nu} + 3\,\gamma\ \sigma + \overline{\alpha}\ \tau + \delta\,\tau == \overline{\lambda}\ \rho + \overline{\gamma}\ \sigma + \mu\ \sigma + \beta\ \tau + \tau^2 + \Delta\,\sigma,$$
$$2\,\Lambda + \overline{\kappa}\ \overline{\nu} + \gamma\ \overline{\rho} + \overline{\gamma}\ \overline{\rho} + \beta\ \overline{\tau} + \delta\,\overline{\tau} == \overline{\Psi}_2 + \mu\ \overline{\rho} + \overline{\lambda}\ \overline{\sigma} + \overline{\alpha}\ \overline{\tau} + \tau\ \overline{\tau} + \Delta\,\overline{\rho},$$
$$2\,\Lambda + 2\,\Phi_{11} + \Psi_2 + \overline{\Psi}_2 + 2\,\alpha\ \beta + 2\,\overline{\alpha}\ \overline{\beta} + \overline{\epsilon}\ \mu + \epsilon\ \overline{\mu} + \overline{\gamma}\ \rho + \gamma\ \overline{\rho} + \lambda\ \sigma + \overline{\lambda}\ \overline{\sigma} + \delta\,\alpha + \overline{\delta}\,\overline{\alpha} ==$$
$$2\,\alpha\ \overline{\alpha} + 2\,\beta\ \overline{\beta} + \epsilon\ \mu + \overline{\epsilon}\ \overline{\mu} + \gamma\ \rho + \mu\ \rho + \overline{\gamma}\ \overline{\rho} + \overline{\mu}\ \overline{\rho} + \delta\,\overline{\beta} + \overline{\delta}\,\beta,$$
$$\Phi_{21} + \Psi_3 + 2\,\overline{\beta}\ \overline{\gamma} + \beta\ \lambda + \alpha\ \overline{\mu} + \overline{\epsilon}\ \nu + \overline{\nu}\ \overline{\sigma} + \lambda\ \tau + \gamma\ \overline{\tau} + \Delta\,\alpha + \overline{\delta}\,\overline{\gamma} ==$$

$$
\begin{aligned}
&2\,\overline{\beta}\,\gamma+\overline{\alpha}\,\lambda+\overline{\beta}\,\overline{\mu}+\epsilon\,\nu+\nu\,\rho+\overline{\gamma}\,\overline{\tau}+\overline{\mu}\,\overline{\tau}+\Delta\overline{\beta}+\overline{\delta}\gamma,\\
&\Phi_{21}+2\,\alpha\,\overline{\gamma}+2\,\overline{\beta}\,\overline{\gamma}+\epsilon\,\nu+\overline{\epsilon}\,\nu+\nu\,\rho+\overline{\nu}\,\overline{\sigma}+\overline{\delta}\gamma+\overline{\delta}\,\overline{\gamma}==\\
&\Psi_3+\overline{\alpha}\,\lambda+\beta\,\lambda+\alpha\,\overline{\mu}+\overline{\beta}\,\overline{\mu}+\lambda\,\tau+\gamma\,\overline{\tau}+\overline{\gamma}\,\overline{\tau}+\overline{\mu}\,\overline{\tau}+\Delta\alpha+\Delta\overline{\beta},\\
&\Phi_{10}+\overline{\Psi}_1+2\,\alpha\,\epsilon+\gamma\,\overline{\kappa}+\kappa\,\lambda+\overline{\epsilon}\,\pi+\overline{\beta}\,\rho+\overline{\alpha}\,\overline{\sigma}+\overline{\pi}\,\overline{\sigma}+\mathrm{D}\alpha+\overline{\delta}\,\overline{\epsilon}==\\
&2\,\alpha\,\overline{\epsilon}+\overline{\gamma}\,\overline{\kappa}+\overline{\kappa}\,\overline{\mu}+\epsilon\,\pi+\alpha\,\rho+\pi\,\rho+\beta\,\overline{\sigma}+\mathrm{D}\overline{\beta}+\overline{\delta}\epsilon,\\
&\overline{\Psi}_0+\overline{\kappa}\,\pi+3\,\overline{\epsilon}\,\overline{\sigma}+\rho\,\overline{\sigma}+\overline{\rho}\,\overline{\sigma}+\overline{\delta}\,\overline{\kappa}==\alpha\,\overline{\kappa}+3\,\overline{\beta}\,\overline{\kappa}+\epsilon\,\overline{\sigma}+\overline{\kappa}\,\overline{\tau}+\mathrm{D}\overline{\sigma},\\
&\Phi_{12}+\overline{\Psi}_3+3\,\overline{\beta}\,\overline{\lambda}+\mu\,\overline{\pi}+\overline{\nu}\,\rho+\overline{\delta}\,\overline{\lambda}==\alpha\,\overline{\lambda}+\overline{\alpha}\,\overline{\mu}+\beta\,\overline{\mu}+\overline{\mu}\,\overline{\pi}+\overline{\nu}\,\overline{\rho}+\delta\,\overline{\mu},\\
&\overline{\gamma}\,\lambda+\nu\left(3\,\alpha+\overline{\beta}+\pi\right)+\overline{\delta}\nu==\Psi_4+3\,\gamma\,\lambda+\lambda\,\mu+\lambda\,\overline{\mu}+\nu\,\overline{\tau}+\Delta\lambda,\\
&\Phi_{22}+\alpha\,\overline{\nu}+3\,\overline{\beta}\,\overline{\nu}+\nu\,\overline{\pi}+\overline{\delta}\,\overline{\nu}==\lambda\,\overline{\lambda}+\gamma\,\overline{\mu}+\overline{\gamma}\,\overline{\mu}+\overline{\mu}^{\,2}+\overline{\nu}\,\overline{\tau}+\Delta\overline{\mu},\\
&\Phi_{10}+\overline{\Psi}_1+\overline{\kappa}\,\mu+3\,\overline{\alpha}\,\overline{\sigma}+\rho\,\overline{\tau}+\overline{\delta}\,\overline{\rho}==\overline{\kappa}\,\overline{\mu}+\alpha\,\overline{\rho}+\overline{\beta}\,\overline{\rho}+\beta\,\overline{\sigma}+\overline{\rho}\,\overline{\tau}+\delta\,\overline{\sigma},\\
&2\,\Lambda+\kappa\,\nu+\gamma\,\rho+\overline{\gamma}\,\rho+\overline{\beta}\,\tau+\overline{\delta}\,\tau==\Psi_2+\overline{\mu}\,\rho+\lambda\,\sigma+\alpha\,\tau+\tau\,\overline{\tau}+\Delta\rho,\\
&\Phi_{20}+\overline{\kappa}\,\nu+3\,\overline{\gamma}\,\overline{\sigma}+\alpha\,\overline{\tau}+\overline{\delta}\,\overline{\tau}==\lambda\,\overline{\rho}+\gamma\,\overline{\sigma}+\overline{\mu}\,\overline{\sigma}+\overline{\beta}\,\overline{\tau}+\overline{\tau}^{\,2}+\Delta\overline{\sigma}\Big\}
\end{aligned}
$$

Not all the equations in this bunch are independent. To extract the independent set of equations we just eliminate a certain set of variables. The variables which we are going to eliminate are

In[262]:=
```
Dagger /@
 Flatten@Outer[PDNP[#1]@#2 &, {{1, -NP}, {2, -NP}, {3, -NP}, {4, -NP}}, NPSpinCoefficients, 1]
```

Out[262]=
$$
\begin{aligned}
&\{\mathrm{D}\overline{\alpha},\mathrm{D}\overline{\beta},\mathrm{D}\overline{\gamma},\mathrm{D}\overline{\epsilon},\mathrm{D}\overline{\kappa},\mathrm{D}\overline{\lambda},\mathrm{D}\overline{\mu},\mathrm{D}\overline{\nu},\mathrm{D}\overline{\pi},\mathrm{D}\overline{\rho},\mathrm{D}\overline{\sigma},\mathrm{D}\overline{\tau},\Delta\overline{\alpha},\Delta\overline{\beta},\Delta\overline{\gamma},\Delta\overline{\epsilon},\\
&\Delta\overline{\kappa},\Delta\overline{\lambda},\Delta\overline{\mu},\Delta\overline{\nu},\Delta\overline{\pi},\Delta\overline{\rho},\Delta\overline{\sigma},\Delta\overline{\tau},\overline{\delta}\overline{\alpha},\overline{\delta}\overline{\beta},\overline{\delta}\overline{\gamma},\overline{\delta}\overline{\epsilon},\overline{\delta}\overline{\kappa},\overline{\delta}\overline{\lambda},\overline{\delta}\overline{\mu},\overline{\delta}\overline{\nu},\\
&\overline{\delta}\overline{\pi},\overline{\delta}\overline{\rho},\overline{\delta}\overline{\sigma},\overline{\delta}\overline{\tau},\delta\overline{\alpha},\delta\overline{\beta},\delta\overline{\gamma},\delta\overline{\epsilon},\delta\overline{\kappa},\delta\overline{\lambda},\delta\overline{\mu},\delta\overline{\nu},\delta\overline{\pi},\delta\overline{\rho},\delta\overline{\sigma},\delta\overline{\tau}\}
\end{aligned}
$$

In[263]:=
```
Eliminate[NPRicci, %] // Simplify;
```

In[264]:=
```
List @@ %;
```

In[265]:=
```
Solve[%, Union@Cases[%, PDNP[_]@_, Infinity]] // Flatten // Simplify
```

```
Solve::svars : Equations may not give solutions for all "solve" variables. >>
```

Out[265]=
$$
\begin{aligned}
&\{\mathrm{D}\alpha\to-\Phi_{10}-2\,\alpha\,\epsilon-\overline{\beta}\,\epsilon+\alpha\,\overline{\epsilon}-\gamma\,\overline{\kappa}-\kappa\,\lambda+\epsilon\,\pi+\alpha\,\rho+\pi\,\rho+\beta\,\overline{\sigma}+\overline{\delta}\epsilon,\\
&\mathrm{D}\beta\to\Psi_1-\overline{\alpha}\,\epsilon-\beta\,\overline{\epsilon}-\gamma\,\kappa-\kappa\,\mu+\epsilon\,\overline{\pi}+\beta\,\overline{\rho}+\alpha\,\sigma+\pi\,\sigma+\delta\epsilon,\\
&\mathrm{D}\gamma\to\Lambda-\Phi_{11}+\Psi_2-2\,\gamma\,\epsilon-\overline{\gamma}\,\epsilon-\gamma\,\overline{\epsilon}-\kappa\,\nu+\beta\,\pi+\alpha\,\overline{\pi}+\alpha\,\tau+\pi\,\tau+\beta\,\overline{\tau}+\Delta\epsilon,\\
&\mathrm{D}\lambda\to-\Phi_{20}-3\,\epsilon\,\lambda+\overline{\epsilon}\,\lambda-\overline{\kappa}\,\nu+\alpha\,\pi-\overline{\beta}\,\pi+\pi^2+\lambda\,\rho+\mu\,\overline{\sigma}+\overline{\delta}\pi,\\
&\mathrm{D}\mu\to-2\,\Lambda+\Psi_2-\epsilon\,\mu-\overline{\epsilon}\,\mu-\kappa\,\nu-\overline{\alpha}\,\pi+\beta\,\pi+\pi\,\overline{\pi}+\mu\,\overline{\rho}+\lambda\,\sigma+\delta\pi,\\
&\mathrm{D}\nu\to-\Phi_{21}+\Psi_3-3\,\epsilon\,\nu-\overline{\epsilon}\,\nu+\gamma\,\pi-\overline{\gamma}\,\pi+\mu\,\pi+\lambda\,\overline{\pi}+\lambda\,\tau+\mu\,\overline{\tau}+\Delta\pi,\\
&\mathrm{D}\rho\to-\Phi_{00}-3\,\alpha\,\kappa-\overline{\beta}\,\kappa+\kappa\,\pi+\epsilon\,\rho+\overline{\epsilon}\,\rho+\rho^2+\sigma\,\overline{\sigma}-\overline{\kappa}\,\tau+\overline{\delta}\kappa,\\
&\mathrm{D}\sigma\to\Psi_0-\overline{\alpha}\,\kappa-3\,\beta\,\kappa+\kappa\,\overline{\pi}+3\,\epsilon\,\sigma-\overline{\epsilon}\,\sigma+\rho\,\sigma+\overline{\rho}\,\sigma-\kappa\,\tau+\delta\kappa,\\
&\mathrm{D}\tau\to-\Phi_{01}+\Psi_1-3\,\gamma\,\kappa-\overline{\gamma}\,\kappa+\overline{\pi}\,\rho+\pi\,\sigma+\epsilon\,\tau-\overline{\epsilon}\,\tau+\rho\,\tau+\sigma\,\overline{\tau}+\Delta\kappa,\\
&\Delta\alpha\to-\Psi_3+\overline{\beta}\,\gamma+\alpha\,\overline{\gamma}-\beta\,\lambda-\alpha\,\overline{\mu}+\epsilon\,\nu+\nu\,\rho-\lambda\,\tau-\gamma\,\overline{\tau}+\overline{\delta}\gamma,\\
&\Delta\beta\to\Phi_{12}+\overline{\alpha}\,\gamma+2\,\beta\,\gamma-\beta\,\overline{\gamma}-\alpha\,\overline{\lambda}-\beta\,\mu+\epsilon\,\overline{\nu}+\nu\,\sigma-\gamma\,\tau-\mu\,\tau+\delta\gamma,\\
&\Delta\lambda\to-\Psi_4-3\,\gamma\,\lambda+\overline{\gamma}\,\lambda-\lambda\,\mu-\lambda\,\overline{\mu}+3\,\alpha\,\nu+\overline{\beta}\,\nu+\nu\,\pi-\nu\,\overline{\tau}+\overline{\delta}\nu,\\
&\Delta\mu\to\Phi_{22}-\lambda\,\overline{\lambda}-\gamma\,\mu-\overline{\gamma}\,\mu-\mu^2+\overline{\alpha}\,\nu+3\,\beta\,\nu+\overline{\nu}\,\pi-\nu\,\tau+\delta\nu,\\
&\Delta\rho\to2\,\Lambda-\Psi_2+\kappa\,\nu+\gamma\,\rho+\overline{\gamma}\,\rho-\overline{\mu}\,\rho-\lambda\,\sigma-\alpha\,\tau+\overline{\beta}\,\tau-\tau\,\overline{\tau}+\overline{\delta}\tau,\\
&\Delta\sigma\to\Phi_{02}+\kappa\,\overline{\nu}-\overline{\lambda}\,\rho+3\,\gamma\,\sigma-\overline{\gamma}\,\sigma-\mu\,\sigma+\overline{\alpha}\,\tau-\beta\,\tau-\tau^2+\delta\tau,\\
&\delta\alpha\to-\Lambda-\Phi_{11}-\Psi_2+\alpha\,\overline{\alpha}-2\,\alpha\,\beta+\beta\,\overline{\beta}+\epsilon\,\mu-\epsilon\,\overline{\mu}+\gamma\,\rho+\mu\,\rho-\gamma\,\overline{\rho}-\lambda\,\sigma+\overline{\delta}\beta,\\
&\delta\lambda\to-\Phi_{21}-\Psi_3+\overline{\alpha}\,\lambda-3\,\beta\,\lambda+\alpha\,\mu+\overline{\beta}\,\mu+\mu\,\pi-\overline{\mu}\,\pi+\nu\,\rho-\nu\,\overline{\rho}+\overline{\delta}\mu,\\
&\delta\rho\to-\Phi_{01}-\Psi_1+\kappa\,\mu-\kappa\,\overline{\mu}+\overline{\alpha}\,\rho+\beta\,\rho-3\,\alpha\,\sigma+\overline{\beta}\,\sigma+\rho\,\tau-\overline{\rho}\,\tau+\overline{\delta}\sigma\}
\end{aligned}
$$

In[266]:=
```
NPRicci = (% /. Rule -> Equal);
```

In[267]:= **Map[FullSimplify, NPRicci, {2}]**

Out[267]=
$$\begin{aligned}
&\{D\alpha == -\Phi_{10} - \bar{\beta}\,\epsilon - \gamma\,\bar{\kappa} - \kappa\,\lambda + \pi\,(\epsilon + \rho) + \alpha\,(-2\epsilon + \bar{\epsilon} + \rho) + \beta\,\bar{\sigma} + \bar{\delta}\epsilon,\\
&D\beta == \Psi_1 - \kappa\,(\gamma + \mu) + \epsilon\,(-\bar{\alpha} + \bar{\pi}) + \beta\,(-\bar{\epsilon} + \bar{\rho}) + (\alpha + \pi)\,\sigma + \delta\epsilon,\\
&D\gamma == \Lambda - \Phi_{11} + \Psi_2 - (2\gamma + \bar{\gamma})\,\epsilon - \gamma\,\bar{\epsilon} - \kappa\,\nu + \pi\,(\beta + \tau) + \alpha\,(\bar{\pi} + \tau) + \beta\,\bar{\tau} + \Delta\epsilon,\\
&D\lambda == -\Phi_{20} - \bar{\kappa}\,\nu + \pi\left(\alpha - \bar{\beta} + \pi\right) + \lambda\,(-3\epsilon + \bar{\epsilon} + \rho) + \mu\,\bar{\sigma} + \bar{\delta}\pi,\\
&D\mu == -2\Lambda + \Psi_2 - \kappa\,\nu + \pi\,(-\bar{\alpha} + \beta + \bar{\pi}) - \mu\,(\epsilon + \bar{\epsilon} - \bar{\rho}) + \lambda\,\sigma + \delta\pi,\\
&D\nu == -\Phi_{21} + \Psi_3 - (3\epsilon + \bar{\epsilon})\,\nu + (\gamma - \bar{\gamma} + \mu)\,\pi + \lambda\,(\bar{\pi} + \tau) + \mu\,\bar{\tau} + \Delta\pi,\\
&D\rho == -\Phi_{00} + \kappa\left(-3\alpha - \bar{\beta} + \pi\right) + \rho\,(\epsilon + \bar{\epsilon} + \rho) + \sigma\,\bar{\sigma} - \bar{\kappa}\,\tau + \bar{\delta}\kappa,\\
&D\sigma == \Psi_0 + (3\epsilon - \bar{\epsilon} + \rho + \bar{\rho})\,\sigma - \kappa\,(\bar{\alpha} + 3\beta - \bar{\pi} + \tau) + \delta\kappa,\\
&D\tau == -\Phi_{01} + \Psi_1 - (3\gamma + \bar{\gamma})\,\kappa + \bar{\pi}\,\rho + (\epsilon - \bar{\epsilon} + \rho)\,\tau + \sigma\,(\pi + \bar{\tau}) + \Delta\kappa,\\
&\Delta\alpha == -\Psi_3 + \alpha\,(\bar{\gamma} - \bar{\mu}) + \nu\,(\epsilon + \rho) - \lambda\,(\beta + \tau) + \gamma\left(\bar{\beta} - \bar{\tau}\right) + \bar{\delta}\gamma,\\
&\Delta\beta == \Phi_{12} + \bar{\alpha}\,\gamma + 2\beta\,\gamma - \beta\,\bar{\gamma} - \alpha\,\bar{\lambda} - \beta\,\mu + \epsilon\,\bar{\nu} + \nu\,\sigma - (\gamma + \mu)\,\tau + \delta\gamma,\\
&\Delta\lambda == -\Psi_4 - \lambda\,(3\gamma - \bar{\gamma} + \mu + \bar{\mu}) + \nu\left(3\alpha + \bar{\beta} + \pi - \bar{\tau}\right) + \bar{\delta}\nu,\\
&\Delta\mu == \Phi_{22} - \lambda\,\bar{\lambda} - \mu\,(\gamma + \bar{\gamma} + \mu) + \bar{\nu}\,\pi + \nu\,(\bar{\alpha} + 3\beta - \tau) + \delta\nu,\\
&\Delta\rho == 2\Lambda - \Psi_2 + \kappa\,\nu + (\gamma + \bar{\gamma} - \bar{\mu})\,\rho - \lambda\,\sigma - \tau\left(\alpha - \bar{\beta} + \bar{\tau}\right) + \bar{\delta}\tau,\\
&\Delta\sigma == \Phi_{02} + \kappa\,\bar{\nu} - \bar{\lambda}\,\rho + 3\gamma\,\sigma - (\bar{\gamma} + \mu)\,\sigma + (\bar{\alpha} - \beta - \tau)\,\tau + \delta\tau,\\
&\delta\alpha == -\Lambda - \Phi_{11} - \Psi_2 + \alpha\,(\bar{\alpha} - 2\beta) + \beta\,\bar{\beta} + \epsilon\,(\mu - \bar{\mu}) + (\gamma + \mu)\,\rho - \gamma\,\bar{\rho} - \lambda\,\sigma + \bar{\delta}\beta,\\
&\delta\lambda == -\Phi_{21} - \Psi_3 + (\bar{\alpha} - 3\beta)\,\lambda + \left(\alpha + \bar{\beta}\right)\mu + (\mu - \bar{\mu})\,\pi + \nu\,(\rho - \bar{\rho}) + \bar{\delta}\mu,\\
&\delta\rho == -\Phi_{01} - \Psi_1 + \kappa\,(\mu - \bar{\mu}) + (\bar{\alpha} + \beta)\,\rho + \left(-3\alpha + \bar{\beta}\right)\sigma + (\rho - \bar{\rho})\,\tau + \bar{\delta}\sigma\}
\end{aligned}$$

We may need the complex conjugate set in future computations.

In[268]:= **NPRicci† = Map[Dagger, NPRicci, {2}]**

Out[268]=
$$\begin{aligned}
&\{D\bar{\alpha} == -\Phi_{01} + \bar{\alpha}\,\epsilon - 2\bar{\alpha}\,\bar{\epsilon} - \beta\,\bar{\epsilon} - \bar{\gamma}\,\kappa - \bar{\kappa}\,\bar{\lambda} + \bar{\epsilon}\,\bar{\pi} + \bar{\alpha}\,\bar{\rho} + \bar{\pi}\,\bar{\rho} + \bar{\beta}\,\sigma + \delta\bar{\epsilon},\\
&D\bar{\beta} == \bar{\Psi}_1 - \bar{\beta}\,\epsilon - \alpha\,\bar{\epsilon} - \bar{\gamma}\,\bar{\kappa} - \bar{\kappa}\,\bar{\mu} + \bar{\epsilon}\,\pi + \bar{\beta}\,\rho + \bar{\alpha}\,\bar{\sigma} + \bar{\pi}\,\bar{\sigma} + \bar{\delta}\bar{\epsilon},\\
&D\bar{\gamma} == \Lambda - \Phi_{11} + \bar{\Psi}_2 - \bar{\gamma}\,\epsilon - \gamma\,\bar{\epsilon} - 2\bar{\gamma}\,\bar{\epsilon} - \bar{\kappa}\,\bar{\nu} + \bar{\alpha}\,\pi + \bar{\beta}\,\bar{\pi} + \bar{\beta}\,\tau + \bar{\alpha}\,\bar{\tau} + \bar{\pi}\,\bar{\tau} + \Delta\bar{\epsilon},\\
&D\bar{\lambda} == -\Phi_{02} + \epsilon\,\bar{\lambda} - 3\bar{\epsilon}\,\bar{\lambda} - \kappa\,\bar{\nu} + \bar{\alpha}\,\bar{\pi} - \beta\,\bar{\pi} + \bar{\pi}^2 + \bar{\lambda}\,\bar{\rho} + \bar{\mu}\,\sigma + \delta\bar{\pi},\\
&D\bar{\mu} == -2\Lambda + \bar{\Psi}_2 - \epsilon\,\bar{\mu} - \bar{\epsilon}\,\bar{\mu} - \bar{\kappa}\,\bar{\nu} - \alpha\,\bar{\pi} + \bar{\beta}\,\bar{\pi} + \pi\,\bar{\pi} + \bar{\mu}\,\rho + \bar{\lambda}\,\bar{\sigma} + \bar{\delta}\bar{\pi},\\
&D\bar{\nu} == -\Phi_{12} + \bar{\Psi}_3 - \epsilon\,\bar{\nu} - 3\bar{\epsilon}\,\bar{\nu} + \bar{\lambda}\,\pi - \gamma\,\bar{\pi} + \bar{\gamma}\,\bar{\pi} + \bar{\mu}\,\bar{\pi} + \bar{\mu}\,\tau + \bar{\lambda}\,\bar{\tau} + \Delta\bar{\pi},\\
&D\bar{\rho} == -\Phi_{00} - 3\bar{\alpha}\,\bar{\kappa} - \beta\,\bar{\kappa} + \bar{\kappa}\,\bar{\pi} + \epsilon\,\bar{\rho} + \bar{\epsilon}\,\bar{\rho} + \bar{\rho}^2 + \sigma\,\bar{\sigma} - \kappa\,\bar{\tau} + \delta\bar{\kappa},\\
&D\bar{\sigma} == \bar{\Psi}_0 - \alpha\,\bar{\kappa} - 3\bar{\beta}\,\bar{\kappa} + \bar{\kappa}\,\pi - \epsilon\,\bar{\sigma} + 3\bar{\epsilon}\,\bar{\sigma} + \rho\,\bar{\sigma} + \bar{\rho}\,\bar{\sigma} - \bar{\kappa}\,\bar{\tau} + \bar{\delta}\bar{\kappa},\\
&D\bar{\tau} == -\Phi_{10} + \bar{\Psi}_1 - \gamma\,\bar{\kappa} - 3\bar{\gamma}\,\bar{\kappa} + \pi\,\bar{\rho} + \bar{\pi}\,\bar{\sigma} + \bar{\sigma}\,\tau - \epsilon\,\bar{\tau} + \bar{\epsilon}\,\bar{\tau} + \bar{\rho}\,\bar{\tau} + \Delta\bar{\kappa},\\
&\Delta\bar{\alpha} == -\bar{\Psi}_3 + \bar{\alpha}\,\gamma + \beta\,\bar{\gamma} - \bar{\beta}\,\bar{\lambda} - \bar{\alpha}\,\mu + \bar{\epsilon}\,\bar{\nu} + \bar{\nu}\,\bar{\rho} - \bar{\gamma}\,\tau - \bar{\lambda}\,\bar{\tau} + \delta\bar{\gamma},\\
&\Delta\bar{\beta} == \Phi_{21} - \bar{\beta}\,\gamma + \alpha\,\bar{\gamma} + 2\bar{\beta}\,\bar{\gamma} - \bar{\alpha}\,\lambda - \bar{\beta}\,\bar{\mu} + \bar{\epsilon}\,\nu + \bar{\nu}\,\bar{\sigma} - \bar{\gamma}\,\bar{\tau} - \bar{\mu}\,\bar{\tau} + \bar{\delta}\bar{\gamma},\\
&\Delta\bar{\lambda} == -\bar{\Psi}_4 + \gamma\,\bar{\lambda} - 3\bar{\gamma}\,\bar{\lambda} - \bar{\lambda}\,\mu - \bar{\lambda}\,\bar{\mu} + 3\bar{\alpha}\,\bar{\nu} + \beta\,\bar{\nu} + \bar{\nu}\,\bar{\pi} - \bar{\nu}\,\tau + \delta\bar{\nu},\\
&\Delta\bar{\mu} == \Phi_{22} - \lambda\,\bar{\lambda} - \gamma\,\bar{\mu} - \bar{\gamma}\,\bar{\mu} - \bar{\mu}^2 + \alpha\,\bar{\nu} + 3\bar{\beta}\,\bar{\nu} + \nu\,\bar{\pi} - \bar{\nu}\,\bar{\tau} + \bar{\delta}\bar{\nu},\\
&\Delta\bar{\rho} == 2\Lambda - \bar{\Psi}_2 + \bar{\kappa}\,\bar{\nu} + \gamma\,\bar{\rho} + \bar{\gamma}\,\bar{\rho} - \mu\,\bar{\rho} - \bar{\lambda}\,\bar{\sigma} - \bar{\alpha}\,\bar{\tau} + \beta\,\bar{\tau} - \tau\,\bar{\tau} + \delta\bar{\tau},\\
&\Delta\bar{\sigma} == \Phi_{20} + \bar{\kappa}\,\nu - \lambda\,\bar{\rho} - \gamma\,\bar{\sigma} + 3\bar{\gamma}\,\bar{\sigma} - \bar{\mu}\,\bar{\sigma} + \alpha\,\bar{\tau} - \bar{\beta}\,\bar{\tau} - \bar{\tau}^2 + \bar{\delta}\bar{\tau},\\
&\bar{\delta}\bar{\alpha} == -\Lambda - \Phi_{11} - \bar{\Psi}_2 + \alpha\,\bar{\alpha} - 2\bar{\alpha}\,\bar{\beta} + \beta\,\bar{\beta} - \bar{\epsilon}\,\mu + \bar{\epsilon}\,\bar{\mu} - \bar{\gamma}\,\rho + \bar{\gamma}\,\bar{\rho} + \bar{\mu}\,\bar{\rho} - \bar{\lambda}\,\bar{\sigma} + \delta\bar{\beta},\\
&\bar{\delta}\bar{\lambda} == -\Phi_{12} - \bar{\Psi}_3 + \alpha\,\bar{\lambda} - 3\bar{\beta}\,\bar{\lambda} + \bar{\alpha}\,\bar{\mu} + \bar{\beta}\,\bar{\mu} - \mu\,\bar{\pi} + \bar{\mu}\,\bar{\pi} - \bar{\nu}\,\rho + \bar{\nu}\,\bar{\rho} + \delta\bar{\mu},\\
&\bar{\delta}\bar{\rho} == -\Phi_{10} - \bar{\Psi}_1 - \bar{\kappa}\,\mu + \bar{\kappa}\,\bar{\mu} + \alpha\,\bar{\rho} + \bar{\beta}\,\bar{\rho} - 3\bar{\alpha}\,\bar{\sigma} + \beta\,\bar{\sigma} - \rho\,\bar{\tau} + \bar{\rho}\,\bar{\tau} + \delta\bar{\sigma}\}
\end{aligned}$$

In[269]:= **NPRicci >> "~/calculations/NP5D/NP4D"**

```
OpenWrite::noopen :
 Cannot open /tmp_mnt/netapp/users_home4/garcia/calculations/NP5D/NP4D. >>
```

Out[269]=
$$\{D\alpha == -\Phi_{10} - 2\alpha\epsilon - \overline{\beta}\epsilon + \alpha\overline{\epsilon} - \gamma\overline{\kappa} - \kappa\lambda + \epsilon\pi + \alpha\rho + \pi\rho + \beta\overline{\sigma} + \overline{\delta}\epsilon,$$
$$D\beta == \Psi_1 - \overline{\alpha}\epsilon - \beta\overline{\epsilon} - \gamma\kappa - \kappa\mu + \epsilon\overline{\pi} + \beta\overline{\rho} + \alpha\sigma + \pi\sigma + \delta\epsilon,$$
$$D\gamma == \Lambda - \Phi_{11} + \Psi_2 - 2\gamma\epsilon - \overline{\gamma}\epsilon - \gamma\overline{\epsilon} - \kappa\nu + \beta\pi + \alpha\overline{\pi} + \alpha\tau + \pi\tau + \beta\overline{\tau} + \Delta\epsilon,$$
$$D\lambda == -\Phi_{20} - 3\epsilon\lambda + \overline{\epsilon}\lambda - \overline{\kappa}\nu + \alpha\pi - \overline{\beta}\pi + \pi^2 + \lambda\rho + \mu\overline{\sigma} + \overline{\delta}\pi,$$
$$D\mu == -2\Lambda + \Psi_2 - \epsilon\mu - \overline{\epsilon}\mu - \kappa\nu - \overline{\alpha}\pi + \beta\pi + \pi\overline{\pi} + \mu\overline{\rho} + \lambda\sigma + \delta\pi,$$
$$D\nu == -\Phi_{21} + \Psi_3 - 3\epsilon\nu - \overline{\epsilon}\nu + \gamma\pi - \overline{\gamma}\pi + \mu\pi + \lambda\overline{\pi} + \lambda\tau + \mu\overline{\tau} + \Delta\pi,$$
$$D\rho == -\Phi_{00} - 3\alpha\kappa - \overline{\beta}\kappa + \kappa\pi + \epsilon\rho + \overline{\epsilon}\rho + \rho^2 + \sigma\overline{\sigma} - \overline{\kappa}\tau + \overline{\delta}\kappa,$$
$$D\sigma == \Psi_0 - \overline{\alpha}\kappa - 3\beta\kappa + \kappa\overline{\pi} + 3\epsilon\sigma - \overline{\epsilon}\sigma + \rho\sigma + \overline{\rho}\sigma - \kappa\tau + \delta\kappa,$$
$$D\tau == -\Phi_{01} + \Psi_1 - 3\gamma\kappa - \overline{\gamma}\kappa + \overline{\pi}\rho + \pi\sigma + \epsilon\tau - \overline{\epsilon}\tau + \rho\tau + \sigma\overline{\tau} + \Delta\kappa,$$
$$\Delta\alpha == -\Psi_3 + \overline{\beta}\gamma + \alpha\overline{\gamma} - \beta\lambda - \alpha\overline{\mu} + \epsilon\nu + \nu\rho - \lambda\tau - \gamma\overline{\tau} + \overline{\delta}\gamma,$$
$$\Delta\beta == \Phi_{12} + \overline{\alpha}\gamma + 2\beta\gamma - \beta\overline{\gamma} - \alpha\overline{\lambda} - \beta\mu + \epsilon\overline{\nu} + \nu\sigma - \gamma\tau - \mu\tau + \delta\gamma,$$
$$\Delta\lambda == -\Psi_4 - 3\gamma\lambda + \overline{\gamma}\lambda - \lambda\mu - \lambda\overline{\mu} + 3\alpha\nu + \overline{\beta}\nu + \nu\pi - \nu\overline{\tau} + \overline{\delta}\nu,$$
$$\Delta\mu == \Phi_{22} - \lambda\overline{\lambda} - \gamma\mu - \overline{\gamma}\mu - \mu^2 + \overline{\alpha}\nu + 3\beta\nu + \overline{\nu}\pi - \nu\tau + \delta\nu,$$
$$\Delta\rho == 2\Lambda - \Psi_2 + \kappa\nu + \gamma\rho + \overline{\gamma}\rho - \overline{\mu}\rho - \lambda\sigma - \alpha\tau + \overline{\beta}\tau - \tau\overline{\tau} + \overline{\delta}\tau,$$
$$\Delta\sigma == \Phi_{02} + \kappa\overline{\nu} - \overline{\lambda}\rho + 3\gamma\sigma - \overline{\gamma}\sigma - \mu\sigma + \overline{\alpha}\tau - \beta\tau - \tau^2 + \delta\tau,$$
$$\delta\alpha == -\Lambda - \Phi_{11} - \Psi_2 + \alpha\overline{\alpha} - 2\alpha\beta + \beta\overline{\beta} + \epsilon\mu - \epsilon\overline{\mu} + \gamma\rho + \mu\rho - \gamma\overline{\rho} - \lambda\sigma + \overline{\delta}\beta,$$
$$\delta\lambda == -\Phi_{21} - \Psi_3 + \overline{\alpha}\lambda - 3\beta\lambda + \alpha\mu + \overline{\beta}\mu + \mu\pi - \overline{\mu}\pi + \nu\rho - \nu\overline{\rho} + \overline{\delta}\mu,$$
$$\delta\rho == -\Phi_{01} - \Psi_1 + \kappa\mu - \kappa\overline{\mu} + \overline{\alpha}\rho + \beta\rho - 3\alpha\sigma + \overline{\beta}\sigma + \rho\tau - \overline{\rho}\tau + \overline{\delta}\sigma\} >>$$
`~/calculations/NP5D/NP4D`

■ Bianchi identities

We take next the components of the second Bianchi identity in the null tetrad.

In[270]:= **3 * Antisymmetrize[CD[-a]@RiemannCD[-b, -c, -d, -f], {-a, -b, -c}] == 0**

Out[270]=
$$\frac{1}{2}\left(\nabla_a R[\nabla]_{bcdf} - \nabla_a R[\nabla]_{cbdf} - \nabla_b R[\nabla]_{acdf} + \nabla_b R[\nabla]_{cadf} + \nabla_c R[\nabla]_{abdf} - \nabla_c R[\nabla]_{badf}\right) == 0$$

In[271]:= **ToCanonical /@ %**

Out[271]=
$$\nabla_a R[\nabla]_{bcdf} - \nabla_b R[\nabla]_{acdf} + \nabla_c R[\nabla]_{abdf} == 0$$

In[272]:= **ChangeCovD[#, CD, PDNP] & /@ %**

Out[272]=
$$-\Gamma[\nabla,]^h{}_{cf} R[\nabla]_{abdh} - \Gamma[\nabla,]^h{}_{cd} R[\nabla]_{abhf} + \Gamma[\nabla,]^h{}_{bf} R[\nabla]_{acdh} + \Gamma[\nabla,]^h{}_{bd} R[\nabla]_{achf} + \Gamma[\nabla,]^h{}_{bc} R[\nabla]_{ahdf} -$$
$$\Gamma[\nabla,]^h{}_{cb} R[\nabla]_{ahdf} - \Gamma[\nabla,]^h{}_{af} R[\nabla]_{bcdh} - \Gamma[\nabla,]^h{}_{ad} R[\nabla]_{bchf} - \Gamma[\nabla,]^h{}_{ac} R[\nabla]_{bhdf} - \Gamma[\nabla,]^h{}_{ca} R[\nabla]_{hbdf} -$$
$$\Gamma[\nabla,]^h{}_{ab} R[\nabla]_{hcdf} + \Gamma[\nabla,]^h{}_{ba} R[\nabla]_{hcdf} + {}_a R[\nabla]_{bcdf} - {}_b R[\nabla]_{acdf} + {}_c R[\nabla]_{abdf} == 0$$

In[273]:= **ToCanonical /@ %**

Out[273]=
$$-\Gamma[\nabla,]^h{}_{cf} R[\nabla]_{abdh} + \Gamma[\nabla,]^h{}_{cd} R[\nabla]_{abfh} + \Gamma[\nabla,]^h{}_{bf} R[\nabla]_{acdh} - \Gamma[\nabla,]^h{}_{bd} R[\nabla]_{acfh} + \Gamma[\nabla,]^h{}_{bc} R[\nabla]_{ahdf} -$$
$$\Gamma[\nabla,]^h{}_{cb} R[\nabla]_{ahdf} - \Gamma[\nabla,]^h{}_{af} R[\nabla]_{bcdh} + \Gamma[\nabla,]^h{}_{ad} R[\nabla]_{bcfh} - \Gamma[\nabla,]^h{}_{ac} R[\nabla]_{bhdf} + \Gamma[\nabla,]^h{}_{ca} R[\nabla]_{bhdf} +$$
$$\Gamma[\nabla,]^h{}_{ab} R[\nabla]_{chdf} - \Gamma[\nabla,]^h{}_{ba} R[\nabla]_{chdf} + {}_a R[\nabla]_{bcdf} - {}_b R[\nabla]_{acdf} + {}_c R[\nabla]_{abdf} == 0$$

In[274]:= **% /. ChristoffelCDPDNP -> ChristoffelCDePDNP**

Out[274]=
$$-\Gamma[\nabla,]^h{}_{cf} R[\nabla]_{abdh} + \Gamma[\nabla,]^h{}_{cd} R[\nabla]_{abfh} + \Gamma[\nabla,]^h{}_{bf} R[\nabla]_{acdh} - \Gamma[\nabla,]^h{}_{bd} R[\nabla]_{acfh} + \Gamma[\nabla,]^h{}_{bc} R[\nabla]_{ahdf} -$$
$$\Gamma[\nabla,]^h{}_{cb} R[\nabla]_{ahdf} - \Gamma[\nabla,]^h{}_{af} R[\nabla]_{bcdh} + \Gamma[\nabla,]^h{}_{ad} R[\nabla]_{bcfh} - \Gamma[\nabla,]^h{}_{ac} R[\nabla]_{bhdf} + \Gamma[\nabla,]^h{}_{ca} R[\nabla]_{bhdf} +$$
$$\Gamma[\nabla,]^h{}_{ab} R[\nabla]_{chdf} - \Gamma[\nabla,]^h{}_{ba} R[\nabla]_{chdf} + {}_a R[\nabla]_{bcdf} - {}_b R[\nabla]_{acdf} + {}_c R[\nabla]_{abdf} == 0$$

In[275]:=
```
ToBasis[NP] /@ ToBasis[NP] /@ %
```

Out[275]=
$$-\Gamma[\nabla,\ ]^{h}{}_{cf}\,R[\nabla]_{abdh}+\Gamma[\nabla,\ ]^{h}{}_{cd}\,R[\nabla]_{abfh}+\Gamma[\nabla,\ ]^{h}{}_{bf}\,R[\nabla]_{acdh}-\Gamma[\nabla,\ ]^{h}{}_{bd}\,R[\nabla]_{acfh}+\Gamma[\nabla,\ ]^{h}{}_{bc}\,R[\nabla]_{ahdf}-\Gamma[\nabla,\ ]^{h}{}_{cb}\,R[\nabla]_{ahdf}-\Gamma[\nabla,\ ]^{h}{}_{af}\,R[\nabla]_{bcdh}+\Gamma[\nabla,\ ]^{h}{}_{ad}\,R[\nabla]_{bcfh}-\Gamma[\nabla,\ ]^{h}{}_{ac}\,R[\nabla]_{bhdf}+\Gamma[\nabla,\ ]^{h}{}_{ca}\,R[\nabla]_{bhdf}+\Gamma[\nabla,\ ]^{h}{}_{ab}\,R[\nabla]_{chdf}-\Gamma[\nabla,\ ]^{h}{}_{ba}\,R[\nabla]_{chdf}+{}_{a}\,R[\nabla]_{bcdf}-{}_{b}\,R[\nabla]_{acdf}+{}_{c}\,R[\nabla]_{abdf}==0$$

In[276]:=
```
TraceBasisDummy /@ %;
```

In[277]:=
```
Flatten@ComponentArray@%;
```

In[278]:=
```
NPBianchi = Union[Simplify //@ %];
```

Not all the equations here are independent.

In[279]:=
```
Dagger /@ Flatten@Outer[PDNP[#1]@#2 &,
    {{1, -NP}, {2, -NP}, {3, -NP}, {4, -NP}}, NPCVComponents[[Range[1, 5]]], 1]
```

Out[279]=
$$\{D\,\overline{\Psi}_0,\ D\,\overline{\Psi}_1,\ D\,\overline{\Psi}_2,\ D\,\overline{\Psi}_3,\ D\,\overline{\Psi}_4,\ \Delta\,\overline{\Psi}_0,\ \Delta\,\overline{\Psi}_1,\ \Delta\,\overline{\Psi}_2,\ \Delta\,\overline{\Psi}_3,\ \Delta\,\overline{\Psi}_4,\ \overline{\delta}\,\overline{\Psi}_0,\ \overline{\delta}\,\overline{\Psi}_1,\ \overline{\delta}\,\overline{\Psi}_2,\ \overline{\delta}\,\overline{\Psi}_3,\ \overline{\delta}\,\overline{\Psi}_4,\ \delta\,\overline{\Psi}_0,\ \delta\,\overline{\Psi}_1,\ \delta\,\overline{\Psi}_2,\ \delta\,\overline{\Psi}_3,\ \delta\,\overline{\Psi}_4\}$$

In[280]:=
```
Join[%, {PDNP[{4, -NP}]@LambdaCDe[]}]
```

Out[280]=
$$\{D\,\overline{\Psi}_0,\ D\,\overline{\Psi}_1,\ D\,\overline{\Psi}_2,\ D\,\overline{\Psi}_3,\ D\,\overline{\Psi}_4,\ \Delta\,\overline{\Psi}_0,\ \Delta\,\overline{\Psi}_1,\ \Delta\,\overline{\Psi}_2,\ \Delta\,\overline{\Psi}_3,\ \Delta\,\overline{\Psi}_4,\ \overline{\delta}\,\overline{\Psi}_0,\ \overline{\delta}\,\overline{\Psi}_1,\ \overline{\delta}\,\overline{\Psi}_2,\ \overline{\delta}\,\overline{\Psi}_3,\ \overline{\delta}\,\overline{\Psi}_4,\ \delta\,\overline{\Psi}_0,\ \delta\,\overline{\Psi}_1,\ \delta\,\overline{\Psi}_2,\ \delta\,\overline{\Psi}_3,\ \delta\,\overline{\Psi}_4,\ \overline{\delta}\,\Lambda\}$$

In[281]:=
```
List @@ Eliminate[NPBianchi, %];
```

The final result for the Bianchi identities is

In[282]:=
```
NPBianchi = Map[FullSimplify, %, {2}]
```

Out[282]=
$$\begin{aligned}
&\{D\Phi_{01} == 2\Psi_1\epsilon - 2\Phi_{11}\kappa - 3\Psi_2\kappa - \Phi_{02}\overline{\kappa} + \Psi_0(-4\alpha + \pi) + \\
&\quad \Phi_{00}(-2(\overline{\alpha} + \beta) + \overline{\pi}) + 4\Psi_1\rho + 2\Phi_{01}(\epsilon + \overline{\rho}) + 2\Phi_{10}\sigma - D\Psi_1 + \delta\Phi_{00} + \overline{\delta}\Psi_0, \\
&D\Phi_{02} == -2\Phi_{01}\beta + 2\Psi_1\beta - 2\Phi_{12}\kappa - \Phi_{00}\overline{\lambda} + \Psi_0(-4\gamma + \mu) + 2\Phi_{01}\overline{\pi} + \\
&\quad \Phi_{02}(2\epsilon - 2\overline{\epsilon} + \overline{\rho}) + 2\Phi_{11}\sigma - 3\Psi_2\sigma + 4\Psi_1\tau + \Delta\Psi_0 + \delta\Phi_{01} - \delta\Psi_1, \\
&D\Phi_{11} == -(\Phi_{21} + 2\Psi_3)\kappa - \Phi_{12}\overline{\kappa} - \Psi_0\lambda - \Phi_{00}\mu + \Phi_{01}\pi + 2\Psi_1(-\alpha + \pi) + \\
&\quad \Phi_{10}(-2\overline{\alpha} + \overline{\pi}) + 3\Psi_2\rho + 2\Phi_{11}\overline{\rho} + \Phi_{20}\sigma - D\Lambda - D\Psi_2 + \delta\Phi_{10} + \overline{\delta}\Psi_1, \\
&D\Phi_{22} == -\Phi_{20}\overline{\lambda} - 2\Phi_{11}\mu + 3\Psi_2\mu - 2\Psi_1\nu + 2\Phi_{12}\pi + 2\Phi_{21}(\beta + \overline{\pi}) + \\
&\quad \Phi_{22}(-2(\epsilon + \overline{\epsilon}) + \overline{\rho}) - \Psi_4\sigma + 2\Psi_3(-\beta + \tau) - 2(\Delta\Lambda) + \Delta\Psi_2 + \delta\Phi_{21} - \delta\Psi_3, \\
&D\Psi_4 == -2\Phi_{21}\alpha + 2\Psi_3\alpha - 4\Psi_4\epsilon + 2\Phi_{11}\lambda - 3\Psi_2\lambda + \Phi_{20}(2\gamma - 2\overline{\gamma} + \overline{\mu}) - \\
&\quad 2\Phi_{10}\nu + 4\Psi_3\pi + \Psi_4\rho - \Phi_{22}\overline{\sigma} + 2\Phi_{21}\overline{\tau} + \Delta\Phi_{20} - \overline{\delta}\Phi_{21} + \overline{\delta}\Psi_3, \\
&\Delta\Phi_{00} == 2\Phi_{00}(\gamma + \overline{\gamma}) + 2\Psi_3\kappa + \Psi_0\lambda - \Phi_{00}\overline{\mu} + 2\Psi_1(\alpha - \pi) + 2\Phi_{11}\rho - \\
&\quad 3\Psi_2\rho + \Phi_{02}\overline{\sigma} - 2\Phi_{10}\tau - 2\Phi_{01}(\alpha + \overline{\tau}) - 2(D\Lambda) + D\Psi_2 + \overline{\delta}\Phi_{01} - \overline{\delta}\Psi_1, \\
&\Delta\Phi_{11} == 2\Psi_3\beta + 2\Phi_{12}\overline{\beta} - \Phi_{02}\lambda - 3\Psi_2\mu - 2\Phi_{11}\overline{\mu} + \Phi_{01}\nu + 2\Psi_1\nu + \Phi_{10}\overline{\nu} + \\
&\quad \Phi_{22}\rho + \Psi_4\sigma - \Phi_{21}\tau - 2\Psi_3\tau - \Phi_{12}\overline{\tau} - \Delta\Lambda - \Delta\Psi_2 + \delta\Psi_3 + \overline{\delta}\Phi_{12}, \\
&\Delta\Phi_{21} == 2\Phi_{22}(\alpha + \overline{\beta}) - 2\Phi_{12}\lambda - 2(\Psi_3(\gamma + 2\mu) + \Phi_{21}(\gamma + \overline{\mu})) + 2\Phi_{11}\nu + \\
&\quad 3\Psi_2\nu + \Phi_{20}\overline{\nu} + \Psi_4(4\beta - \tau) - \Phi_{22}\overline{\tau} - \Delta\Psi_3 + \delta\Psi_4 + \overline{\delta}\Phi_{22}, \\
&\delta\Phi_{11} == 2\Psi_1\gamma + 2\Phi_{12}\overline{\epsilon} + \Phi_{22}\kappa + \Phi_{10}\overline{\lambda} + \Phi_{01}\mu - 2\Psi_1\mu + \Psi_0\nu - \Phi_{02}\pi - \\
&\quad 2\Phi_{11}\overline{\pi} - \Phi_{12}\overline{\rho} - \Phi_{21}\sigma + 2\Psi_3\sigma - 3\Psi_2\tau + D\Phi_{12} - \Delta\Psi_1 + \delta\Lambda + \delta\Psi_2, \\
&\overline{\delta}\Phi_{02} == -2\Phi_{01}\gamma - 2\Psi_1\gamma + 2\Psi_1\mu + 2\Phi_{01}\overline{\mu} - \Psi_0\nu - \Phi_{00}\overline{\nu} - 2\Phi_{12}\rho - 2\Psi_3\sigma + \\
&\quad 2\Phi_{11}\tau + 3\Psi_2\tau + \Phi_{02}(2\alpha - 2\overline{\beta} + \overline{\tau}) + \Delta\Phi_{01} + \Delta\Psi_1 + 2(\delta\Lambda) - \delta\Psi_2, \\
&2(\overline{\delta}\Phi_{11}) == -6\Psi_3\epsilon - 3\Psi_4\kappa - \Phi_{22}\overline{\kappa} + 2\Phi_{01}\lambda - 6\Psi_1\lambda - 2\Phi_{10}(2\overline{\gamma} + \mu - \overline{\mu}) - \\
&\quad 2\Phi_{00}\nu + 2\Phi_{11}\pi + 9\Psi_2\pi + 6\Psi_3\rho - 2\Phi_{21}(\epsilon + \rho - \overline{\rho}) - 2\Phi_{12}\overline{\sigma} + \\
&\quad \Phi_{20}(-2\overline{\alpha} + 2\beta + \overline{\pi} + 2\tau) + 4\Phi_{11}\overline{\tau} - D\Phi_{21} - 3(D\Psi_3) + 2(\Delta\Phi_{10}) + \delta\Phi_{20} + 3(\overline{\delta}\Psi_2)\}
\end{aligned}$$

■ 3. The GHP formalism

The Geroch–Held–Penrose formalism (known as GHP formalism) is a particularisation of the Newman–Penrose formalism in which the spin coefficients are separated according to their *boost* and *spin weight.* Also the direcctional derivatives associated to the null tetrad are replaced by differential operators which behave well under boost and spin transformations and therefore we can speak of the boost and spin weight of these operators. This enables us to re–write the basic equations used in the Newman–Penrose formalism in terms of boost–weighted quantities only. In this section we show how this is accomplished in the frame–work of `Spinors`` and `xCoba``.

■ Boost and spin weights

The GHP formalism only admits *weighted quantities*. In this subsection we introduce a command, called **`GHPWeightOf`**, which computes the boost and spin weight of any quantity. These are integers which are kept in a list. The command **`GHPWeightOf`** also checks that the GHP weight of the terms of any sum all agree, throwing an error otherwise.

In[283]:=
```
GHPWeightOf::usage =
  "GHPWeightOf[expr] returns the GHP boost and spin weight of any expression.
    If the expression is not weighted, then an error message is returned";
```

Weights of integers, rationals and complex numbers.

In[284]:=
```
GHPWeightOf[num_?IntegerQ] := {0, 0};
GHPWeightOf[_Rational] := {0, 0};
GHPWeightOf[_Complex] := {0, 0};
```

Weights of products.

In[287]:=
```
GHPWeightOf[expr_Times] := Apply[Plus, GHPWeightOf /@ Level[expr, {1}]];
```

Weights of sums

In[288]:=
```
GHPWeightOf@expr_Plus := If[Length@Union[GHPWeightOf /@ Level[expr, 1]] > 1,
    Throw@Message[GHPWeightOf::Error], First@Union[GHPWeightOf /@ Level[expr, 1]]];
GHPWeightOf::Error = "Inconsistent weights on Plus expression";
```

Weights of powers

In[290]:=
```
GHPWeightOf[expr_^(n_?IntegerQ)] := n GHPWeightOf@expr;
```

■ Weights of the spin coefficients

Some of the spin coefficients are weighted quantities. These are in the following list.

In[291]:=
```
GHPSpinCoefficients = NPSpinCoefficients[[#]] & /@ Range[5, 12]
```

Out[291]=
$$\{\kappa\ ,\ \lambda\ ,\ \mu\ ,\ \nu\ ,\ \pi\ ,\ \rho\ ,\ \sigma\ ,\ \tau\ \}$$

These are the spin and boost weights of these quantities :

In[292]:=
```
{GHPWeightOf@κ Dyad[] ^= {3, 1}, GHPWeightOf@λ Dyad[] ^= {-3, 1},
  GHPWeightOf@μ Dyad[] ^= {-1, -1}, GHPWeightOf@ν Dyad[] ^= {-3, -1},
  GHPWeightOf@π Dyad[] ^= {-1, 1}, GHPWeightOf@ρ Dyad[] ^= {1, 1},
  GHPWeightOf@σ Dyad[] ^= {3, -1}, GHPWeightOf@τ Dyad[] ^= {1, -1}};
```

From here we compute the weights of the complex conjugate of these quantities

In[293]:=
```
Dagger /@ GHPSpinCoefficients
```

Out[293]=
$$\{\overline{\kappa}\ ,\ \overline{\lambda}\ ,\ \overline{\mu}\ ,\ \overline{\nu}\ ,\ \overline{\pi}\ ,\ \overline{\rho}\ ,\ \overline{\sigma}\ ,\ \overline{\tau}\ \}$$

In[294]:=
```
UpSet[GHPWeightOf@#, Reverse@GHPWeightOf@Dagger@#] & /@ %
```

Out[294]=
```
{{1, 3}, {1, -3}, {-1, -1}, {-1, -3}, {1, -1}, {1, 1}, {-1, 3}, {-1, 1}}
```

■ Weights of the components of the curvature spinors

It is easy to deduce the weights of the components of the curvature spinors. These are their values.

In[295]:=
```
{GHPWeightOf@PsiCDe0Dyad[] ^= {4, 0},
  GHPWeightOf@PsiCDe1Dyad[] ^= {2, 0}, GHPWeightOf@PsiCDe2Dyad[] ^= {0, 0},
  GHPWeightOf@PsiCDe3Dyad[] ^= {-2, 0}, GHPWeightOf@PsiCDe4Dyad[] ^= {-4, 0}};
```

In[296]:=
```
{GHPWeightOf@PsiCDe0Dyad†[] ^= Reverse@{4, 0}, GHPWeightOf@PsiCDe1Dyad†[] ^= Reverse@{2, 0},
  GHPWeightOf@PsiCDe2Dyad†[] ^= Reverse@{0, 0}, GHPWeightOf@PsiCDe3Dyad†[] ^=
   Reverse@{-2, 0}, GHPWeightOf@PsiCDe4Dyad†[] ^= Reverse@{-4, 0}};
```

In[297]:=
```
{GHPWeightOf@PhiCDe00Dyad[] ^= {2, 2},
  GHPWeightOf@PhiCDe10Dyad[] ^= {0, 2}, GHPWeightOf@PhiCDe01Dyad[] ^= {2, 0},
  GHPWeightOf@PhiCDe11Dyad[] ^= {0, 0}, GHPWeightOf@PhiCDe20Dyad[] ^= {-2, 2},
  GHPWeightOf@PhiCDe21Dyad[] ^= {-2, 0}, GHPWeightOf@PhiCDe02Dyad[] ^= {2, -2},
  GHPWeightOf@PhiCDe12Dyad[] ^= {0, -2}, GHPWeightOf@PhiCDe22Dyad[] ^= {-2, -2}};
```

In[298]:=
```
GHPWeightOf@LambdaCDe[] ^= {0, 0};
```

GHP differential operators

The GHP differential operators are constructed by adding some extra terms to the four NP direccional derivatives. The result is a set of four differential operators having good transformation properties under spin and boost.

The GHP differential operators can be regarded as the "components" of a certain differential operator which we define next. This operator is not a true covariant derivative but just a differential operator acting on scalars, so we need to use some primitive xAct` instructions in order to create it. We use the notation Θ_a as a representation for it.

In[299]:=
```
CovDQ[Θ] ^= True;
xAct`xTensor`Private`MakeLinearDerivative[{Θ[A_], Θ[A]}, True];
SymmetryGroupOfCovD[Θ] ^= StrongGenSet[{}, GenSet[]];
ManifoldOfCovD[Θ] ^= ManifoldOfCovD[CDe];
VBundlesOfCovD[Θ] ^= {};
Dagger[Θ[A_][expr_]] ^:= Θ[DaggerIndex[A]][Dagger[expr]];
SymbolOfCovD[Θ] ^= {"Θ"};
```

Formatting of the GHP operators with their standard symbols.

In[306]:=
```
xTensorFormStop[CovD]
FormatBasis[Θ[{1, -NP}], "þ"];
FormatBasis[Θ[{2, -NP}], "þ'"];
FormatBasis[Θ[{3, -NP}], "ð"];
FormatBasis[Θ[{4, -NP}], "ð'"];
xTensorFormStart[CovD]
```

The following constants represent respectively generic boost and spin weights.

In[312]:=
```
DefConstantSymbol[w]
```

```
** DefConstantSymbol: Defining constant symbol w.
```

In[313]:=
```
DefConstantSymbol[z]
```

```
** DefConstantSymbol: Defining constant symbol z.
```

We introduce a complex scalar of weight (p,q).

In[314]:=
```
DefSpinor[η[LI@{p, q}], M4]
```

```
** DefTensor: Defining tensor η[LI[{p, q}]].

** DefTensor: Defining tensor η ł[LI[{p, q}]].
```

We specify the weight of the scalar quantity $\eta^{\{w,\ z\}}$ (note that the integers w and z are zero weighted).

In[315]:=
```
GHPWeightOf@η[LI@{w, z}] ^= {w, z};
GHPWeightOf@w ^= {0, 0}; GHPWeightOf@z ^= {0, 0};
```

The action of the differential operator Θ_a on a weighted scalar $\eta^{\{w,\,z\}}$ is, by definition

In[317]:=
```
{Θ[{1, -NP}][η[LI[{w, z}]]] ->
   (-(w * ϵ Dyad[]) - z * ϵ Dyad†[]) * η[LI[{w, z}]] + PDNP[{1, -NP}][η[LI[{w, z}]]],
 Θ[{2, -NP}][η[LI[{w, z}]]] -> (-(w * γ Dyad[]) - z * γ Dyad†[]) * η[LI[{w, z}]] +
   PDNP[{2, -NP}][η[LI[{w, z}]]],
 Θ[{3, -NP}][η[LI[{w, z}]]] -> (-(z * α Dyad†[]) - w * β Dyad[]) * η[LI[{w, z}]] +
   PDNP[{3, -NP}][η[LI[{w, z}]]],
 Θ[{4, -NP}][η[LI[{w, z}]]] -> (-(w * α Dyad[]) - z * β Dyad†[]) * η[LI[{w, z}]] +
   PDNP[{4, -NP}][η[LI[{w, z}]]]}
```

Out[317]=
$$\{þ\,\eta^{\{w,\,z\}} \to (-w\,\epsilon\ -z\,\overline{\epsilon}\ )\,\eta^{\{w,\,z\}} + D\,\eta^{\{w,\,z\}},\ þ'\,\eta^{\{w,\,z\}} \to (-w\,\gamma\ -z\,\overline{\gamma}\ )\,\eta^{\{w,\,z\}} + \Delta\,\eta^{\{w,\,z\}},$$
$$ð\,\eta^{\{w,\,z\}} \to (-z\,\overline{\alpha}\ -w\,\beta\ )\,\eta^{\{w,\,z\}} + \delta\,\eta^{\{w,\,z\}},\ ð'\,\eta^{\{w,\,z\}} \to \left(-w\,\alpha\ -z\,\overline{\beta}\ \right)\eta^{\{w,\,z\}} + \overline{\delta}\,\eta^{\{w,\,z\}}\}$$

In[318]:=
```
GHPDerivativesToFrame = %;
```

In[319]:=
```
% /. Rule → Equal;
```

In[320]:=
```
Solve[%, Cases[%, PDNP[_]@_, Infinity]] // Flatten
```

Out[320]=
$$\{D\,\eta^{\{w,\,z\}} \to w\,\epsilon\ \ \eta^{\{w,\,z\}} + z\,\overline{\epsilon}\ \ \eta^{\{w,\,z\}} + þ\,\eta^{\{w,\,z\}},\ \Delta\,\eta^{\{w,\,z\}} \to w\,\gamma\ \ \eta^{\{w,\,z\}} + z\,\overline{\gamma}\ \ \eta^{\{w,\,z\}} + þ'\,\eta^{\{w,\,z\}},$$
$$\delta\,\eta^{\{w,\,z\}} \to z\,\overline{\alpha}\ \ \eta^{\{w,\,z\}} + w\,\beta\ \ \eta^{\{w,\,z\}} + ð\,\eta^{\{w,\,z\}},\ \overline{\delta}\,\eta^{\{w,\,z\}} \to w\,\alpha\ \ \eta^{\{w,\,z\}} + z\,\overline{\beta}\ \ \eta^{\{w,\,z\}} + ð'\,\eta^{\{w,\,z\}}\}$$

In[321]:=
```
FrameToGHPDerivatives = %;
```

We transform previous rules in delayed rules with patterns which can be used for any weighted quantity

In[322]:=
```
GHPDerivativesToFrame /. Rule -> RuleDelayed /. η[LI@{w, z}] -> η
```

Out[322]=
$$\{þ\,\eta :\to (-w\,\epsilon\ -z\,\overline{\epsilon}\ )\,\eta + D\,\eta,\ þ'\,\eta :\to (-w\,\gamma\ -z\,\overline{\gamma}\ )\,\eta + \Delta\,\eta,$$
$$ð\,\eta :\to (-z\,\overline{\alpha}\ -w\,\beta\ )\,\eta + \delta\,\eta,\ ð'\,\eta :\to \left(-w\,\alpha\ -z\,\overline{\beta}\ \right)\eta + \overline{\delta}\,\eta\}$$

In[323]:=
```
% /. Θ[comp__]@η :> Θ[comp]@η _
```

Out[323]=
$$\{þ\,\eta\_ :\to (-w\,\epsilon\ -z\,\overline{\epsilon}\ )\,\eta + D\,\eta,\ þ'\,\eta\_ :\to (-w\,\gamma\ -z\,\overline{\gamma}\ )\,\eta + \Delta\,\eta,$$
$$ð\,\eta\_ :\to (-z\,\overline{\alpha}\ -w\,\beta\ )\,\eta + \delta\,\eta,\ ð'\,\eta\_ :\to \left(-w\,\alpha\ -z\,\overline{\beta}\ \right)\eta + \overline{\delta}\,\eta\}$$

In[324]:=
```
GHPToNP = (% /. w :> First@GHPWeightOf@η /. z :> Last@GHPWeightOf@η)
```

Out[324]=
$$\{þ\,\eta\_ :\to (-\text{First}[\text{GHPWeightOf}[\eta]]\ \epsilon\ -\text{Last}[\text{GHPWeightOf}[\eta]]\ \overline{\epsilon}\ )\,\eta + D\,\eta,$$
$$þ'\,\eta\_ :\to (-\text{First}[\text{GHPWeightOf}[\eta]]\ \gamma\ -\text{Last}[\text{GHPWeightOf}[\eta]]\ \overline{\gamma}\ )\,\eta + \Delta\,\eta,$$
$$ð\,\eta\_ :\to (-\text{Last}[\text{GHPWeightOf}[\eta]]\ \overline{\alpha}\ -\text{First}[\text{GHPWeightOf}[\eta]]\ \beta\ )\,\eta + \delta\,\eta,$$
$$ð'\,\eta\_ :\to \left(-\text{First}[\text{GHPWeightOf}[\eta]]\ \alpha\ -\text{Last}[\text{GHPWeightOf}[\eta]]\ \overline{\beta}\ \right)\eta + \overline{\delta}\,\eta\}$$

In[325]:=
```
FrameToGHPDerivatives /. Rule -> RuleDelayed /. η[LI@{w, z}] -> η
```

Out[325]=
$$\{D\,\eta :\to w\,\epsilon\ \ \eta + z\,\overline{\epsilon}\ \ \eta + þ\,\eta,\ \Delta\,\eta :\to w\,\gamma\ \ \eta + z\,\overline{\gamma}\ \ \eta + þ'\,\eta,\ \delta\,\eta :\to z\,\overline{\alpha}\ \ \eta + w\,\beta\ \ \eta + ð\,\eta,\ \overline{\delta}\,\eta :\to w\,\alpha\ \ \eta + z\,\overline{\beta}\ \ \eta + ð'\,\eta\}$$

In[326]:=
```
% /. PDNP[comp__]@η :> PDNP[comp]@η _
```

Out[326]=
$$\{D\,\eta\_ :\to w\,\epsilon\ \ \eta + z\,\overline{\epsilon}\ \ \eta + þ\,\eta,\ \Delta\,\eta\_ :\to w\,\gamma\ \ \eta + z\,\overline{\gamma}\ \ \eta + þ'\,\eta,$$
$$\delta\,\eta\_ :\to z\,\overline{\alpha}\ \ \eta + w\,\beta\ \ \eta + ð\,\eta,\ \overline{\delta}\,\eta\_ :\to w\,\alpha\ \ \eta + z\,\overline{\beta}\ \ \eta + ð'\,\eta\}$$

In[327]:=
```
NPToGHP = (% /. w :> First@GHPWeightOf@η /. z :> Last@GHPWeightOf@η)
```

Out[327]=
$$\{\text{D}\,\eta\_ :\to \text{First}[\text{GHPWeightOf}[\eta]]\,\epsilon\ \eta + \text{Last}[\text{GHPWeightOf}[\eta]]\,\overline{\epsilon}\ \eta + \text{þ}\,\eta,\ \Delta\,\eta\_ :\to \text{First}[\text{GHPWeightOf}[\eta]]\,\gamma\ \eta + \text{Last}[\text{GHPWeightOf}[\eta]]\,\overline{\gamma}\ \eta + \text{þ}'\,\eta,\ \delta\,\eta\_ :\to \text{Last}[\text{GHPWeightOf}[\eta]]\,\overline{\alpha}\ \eta + \text{First}[\text{GHPWeightOf}[\eta]]\,\beta\ \eta + \text{ð}\,\eta,\ \overline{\delta}\,\eta\_ :\to \text{First}[\text{GHPWeightOf}[\eta]]\,\alpha\ \eta + \text{Last}[\text{GHPWeightOf}[\eta]]\,\overline{\beta}\ \eta + \text{ð}'\,\eta\}$$

Each of the quantities $\text{þ}\,\eta^{\{w,\,z\}}$, $\text{þ}'\,\eta^{\{w,\,z\}}$, $\text{ð}\,\eta^{\{w,\,z\}}$, $\text{ð}'\,\eta^{\{w,\,z\}}$ is a spin weighted quantity. These are the specific values for the weights of each of them.

In[328]:=
```
GHPWeightOf@@0[{1, -NP}] ^= {1, 1};
GHPWeightOf@@0[{2, -NP}] ^= {-1, -1};
GHPWeightOf@@0[{3, -NP}] ^= {1, -1};
GHPWeightOf@@0[{4, -NP}] ^= {-1, 1};
```

We also need to teach GHPWeightOf how to behave when acting on differential expressions.

In[332]:=
```
GHPWeightOf[Θ[ind_?CIndexQ]@sc_] := {Part[GHPWeightOf@@Θ[ind], 1] + Part[GHPWeightOf@sc, 1],
    Part[GHPWeightOf@@Θ[ind], 2] + Part[GHPWeightOf@sc, 2]};
```

■ Commutation of the GHP differential operators

In this subsection we work out the commutation relations of each of the GHP operators. The commutator of two GHP operators is a new differential operator which acts on a generic spin weighted scalar $\eta^{\{w,\,z\}}$.

This is the commutation between two generic GHP operators.

In[333]:=
```
Θ[-{a, NP}]@Θ[-{b, NP}]@η[LI@{w, z}] - Θ[-{b, NP}]@Θ[-{a, NP}]@η[LI@{w, z}]
```

Out[333]=
$$\Theta_a\,\Theta_b\,\eta^{\{w,\,z\}} - \Theta_b\,\Theta_a\,\eta^{\{w,\,z\}}$$

We expand this into components in order to obtain the set of commutation relations.

In[334]:=
```
Delete[Union@Flatten@ComponentArray@%, 1]
```

Out[334]=
$$\{\text{þ}\,\text{þ}'\,\eta^{\{w,\,z\}} - \text{þ}'\,\text{þ}\,\eta^{\{w,\,z\}},\ -\left(\text{þ}\,\text{þ}'\,\eta^{\{w,\,z\}}\right) + \text{þ}'\,\text{þ}\,\eta^{\{w,\,z\}},\ \text{þ}\,\text{ð}\,\eta^{\{w,\,z\}} - \text{ð}\,\text{þ}\,\eta^{\{w,\,z\}},\ -\left(\text{þ}\,\text{ð}\,\eta^{\{w,\,z\}}\right) + \text{ð}\,\text{þ}\,\eta^{\{w,\,z\}},\ \text{þ}'\,\text{ð}\,\eta^{\{w,\,z\}} - \text{ð}\,\text{þ}'\,\eta^{\{w,\,z\}},\ -\left(\text{þ}'\,\text{ð}\,\eta^{\{w,\,z\}}\right) + \text{ð}\,\text{þ}'\,\eta^{\{w,\,z\}},\ \text{þ}\,\text{ð}'\,\eta^{\{w,\,z\}} - \text{ð}'\,\text{þ}\,\eta^{\{w,\,z\}},\ -\left(\text{þ}\,\text{ð}'\,\eta^{\{w,\,z\}}\right) + \text{ð}'\,\text{þ}\,\eta^{\{w,\,z\}},\ \text{þ}'\,\text{ð}'\,\eta^{\{w,\,z\}} - \text{ð}'\,\text{þ}'\,\eta^{\{w,\,z\}},\ -\left(\text{þ}'\,\text{ð}'\,\eta^{\{w,\,z\}}\right) + \text{ð}'\,\text{þ}'\,\eta^{\{w,\,z\}},\ \text{ð}\,\text{ð}'\,\eta^{\{w,\,z\}} - \text{ð}'\,\text{ð}\,\eta^{\{w,\,z\}},\ -\left(\text{ð}\,\text{ð}'\,\eta^{\{w,\,z\}}\right) + \text{ð}'\,\text{ð}\,\eta^{\{w,\,z\}}\}$$

In[335]:=
```
Union[%, SameTest → (-#1 === #2 &)]
```

Out[335]=
$$\{\text{þ}\,\text{þ}'\,\eta^{\{w,\,z\}} - \text{þ}'\,\text{þ}\,\eta^{\{w,\,z\}},\ \text{þ}\,\text{ð}\,\eta^{\{w,\,z\}} - \text{ð}\,\text{þ}\,\eta^{\{w,\,z\}},\ \text{þ}'\,\text{ð}\,\eta^{\{w,\,z\}} - \text{ð}\,\text{þ}'\,\eta^{\{w,\,z\}},\ \text{þ}\,\text{ð}'\,\eta^{\{w,\,z\}} - \text{ð}'\,\text{þ}\,\eta^{\{w,\,z\}},\ \text{þ}'\,\text{ð}'\,\eta^{\{w,\,z\}} - \text{ð}'\,\text{þ}'\,\eta^{\{w,\,z\}},\ \text{ð}\,\text{ð}'\,\eta^{\{w,\,z\}} - \text{ð}'\,\text{ð}\,\eta^{\{w,\,z\}}\}$$

In[336]:=
```
Union[%, SameTest → (Dagger@#1 === #2 &)]
```

Out[336]=
$$\{\text{þ}\,\text{þ}'\,\eta^{\{w,\,z\}} - \text{þ}'\,\text{þ}\,\eta^{\{w,\,z\}},\ \text{þ}\,\text{ð}\,\eta^{\{w,\,z\}} - \text{ð}\,\text{þ}\,\eta^{\{w,\,z\}},\ \text{þ}'\,\text{ð}\,\eta^{\{w,\,z\}} - \text{ð}\,\text{þ}'\,\eta^{\{w,\,z\}},\ \text{þ}\,\text{ð}'\,\eta^{\{w,\,z\}} - \text{ð}'\,\text{þ}\,\eta^{\{w,\,z\}},\ \text{þ}'\,\text{ð}'\,\eta^{\{w,\,z\}} - \text{ð}'\,\text{þ}'\,\eta^{\{w,\,z\}},\ \text{ð}\,\text{ð}'\,\eta^{\{w,\,z\}} - \text{ð}'\,\text{ð}\,\eta^{\{w,\,z\}}\}$$

In[337]:= **Equal[#, # //. GHPToNP] & /@ %**

Out[337]=
$$\{ þ\,þ'\,\eta^{\{w,\,z\}} - þ'\,þ\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\,(-w\,(D\,\gamma) - z\,(D\,\overline{\gamma})) + (-w\,\gamma - z\,\overline{\gamma})\,\left(D\,\eta^{\{w,\,z\}}\right) -$$
$$(-(1+w)\,\gamma - (1+z)\,\overline{\gamma})\,\left((-w\,\epsilon - z\,\overline{\epsilon})\,\eta^{\{w,\,z\}} + D\,\eta^{\{w,\,z\}}\right) + D\,\Delta\,\eta^{\{w,\,z\}} - \eta^{\{w,\,z\}}\,(-w\,(\Delta\,\epsilon) - z\,(\Delta\,\overline{\epsilon})) -$$
$$(-w\,\epsilon - z\,\overline{\epsilon})\,\left(\Delta\,\eta^{\{w,\,z\}}\right) + (-(-1+w)\,\epsilon - (-1+z)\,\overline{\epsilon})\,\left((-w\,\gamma - z\,\overline{\gamma})\,\eta^{\{w,\,z\}} + \Delta\,\eta^{\{w,\,z\}}\right) - \Delta\,D\,\eta^{\{w,\,z\}},$$
$$þ\,ð\,\eta^{\{w,\,z\}} - ð\,þ\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\,(-z\,(D\,\overline{\alpha}) - w\,(D\,\beta)) + (-z\,\overline{\alpha} - w\,\beta)\,\left(D\,\eta^{\{w,\,z\}}\right) -$$
$$(-(1+z)\,\overline{\alpha} - (1+w)\,\beta)\,\left((-w\,\epsilon - z\,\overline{\epsilon})\,\eta^{\{w,\,z\}} + D\,\eta^{\{w,\,z\}}\right) + D\,\delta\,\eta^{\{w,\,z\}} - \eta^{\{w,\,z\}}\,(-w\,(\delta\,\epsilon) - z\,(\delta\,\overline{\epsilon})) -$$
$$(-w\,\epsilon - z\,\overline{\epsilon})\,\left(\delta\,\eta^{\{w,\,z\}}\right) + (-(1+w)\,\epsilon - (-1+z)\,\overline{\epsilon})\,\left((-z\,\overline{\alpha} - w\,\beta)\,\eta^{\{w,\,z\}} + \delta\,\eta^{\{w,\,z\}}\right) - \delta\,D\,\eta^{\{w,\,z\}},$$
$$þ'\,ð\,\eta^{\{w,\,z\}} - ð\,þ'\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\,(-z\,(\Delta\,\overline{\alpha}) - w\,(\Delta\,\beta)) + (-z\,\overline{\alpha} - w\,\beta)\,\left(\Delta\,\eta^{\{w,\,z\}}\right) -$$
$$(-(-1+z)\,\overline{\alpha} - (-1+w)\,\beta)\,\left((-w\,\gamma - z\,\overline{\gamma})\,\eta^{\{w,\,z\}} + \Delta\,\eta^{\{w,\,z\}}\right) +$$
$$\Delta\,\delta\,\eta^{\{w,\,z\}} - \eta^{\{w,\,z\}}\,(-w\,(\delta\,\gamma) - z\,(\delta\,\overline{\gamma})) - (-w\,\gamma - z\,\overline{\gamma})\,\left(\delta\,\eta^{\{w,\,z\}}\right) +$$
$$(-(-1+w)\,\gamma - (-1+z)\,\overline{\gamma})\,\left((-z\,\overline{\alpha} - w\,\beta)\,\eta^{\{w,\,z\}} + \delta\,\eta^{\{w,\,z\}}\right) - \delta\,\Delta\,\eta^{\{w,\,z\}},$$
$$þ\,ð'\,\eta^{\{w,\,z\}} - ð'\,þ\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\,\left(-w\,(D\,\alpha) - z\,\left(D\,\overline{\beta}\right)\right) + \left(-w\,\alpha - z\,\overline{\beta}\right)\,\left(D\,\eta^{\{w,\,z\}}\right) -$$
$$\left(-(1+w)\,\alpha - (1+z)\,\overline{\beta}\right)\,\left((-w\,\epsilon - z\,\overline{\epsilon})\,\eta^{\{w,\,z\}} + D\,\eta^{\{w,\,z\}}\right) + D\,\overline{\delta}\,\eta^{\{w,\,z\}} - \eta^{\{w,\,z\}}\,\left(-w\,\left(\overline{\delta}\,\epsilon\right) - z\,\left(\overline{\delta}\,\overline{\epsilon}\right)\right) -$$
$$(-w\,\epsilon - z\,\overline{\epsilon})\,\left(\overline{\delta}\,\eta^{\{w,\,z\}}\right) + (-(-1+w)\,\epsilon - (1+z)\,\overline{\epsilon})\,\left(\left(-w\,\alpha - z\,\overline{\beta}\right)\,\eta^{\{w,\,z\}} + \overline{\delta}\,\eta^{\{w,\,z\}}\right) - \overline{\delta}\,D\,\eta^{\{w,\,z\}},$$
$$þ'\,ð'\,\eta^{\{w,\,z\}} - ð'\,þ'\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\,\left(-w\,(\Delta\,\alpha) - z\,\left(\Delta\,\overline{\beta}\right)\right) + \left(-w\,\alpha - z\,\overline{\beta}\right)\,\left(\Delta\,\eta^{\{w,\,z\}}\right) -$$
$$\left(-(-1+w)\,\alpha - (-1+z)\,\overline{\beta}\right)\,\left((-w\,\gamma - z\,\overline{\gamma})\,\eta^{\{w,\,z\}} + \Delta\,\eta^{\{w,\,z\}}\right) +$$
$$\Delta\,\overline{\delta}\,\eta^{\{w,\,z\}} - \eta^{\{w,\,z\}}\,\left(-w\,\left(\overline{\delta}\,\gamma\right) - z\,\left(\overline{\delta}\,\overline{\gamma}\right)\right) - (-w\,\gamma - z\,\overline{\gamma})\,\left(\overline{\delta}\,\eta^{\{w,\,z\}}\right) +$$
$$(-(-1+w)\,\gamma - (1+z)\,\overline{\gamma})\,\left(\left(-w\,\alpha - z\,\overline{\beta}\right)\,\eta^{\{w,\,z\}} + \overline{\delta}\,\eta^{\{w,\,z\}}\right) - \overline{\delta}\,\Delta\,\eta^{\{w,\,z\}},$$
$$ð\,ð'\,\eta^{\{w,\,z\}} - ð'\,ð\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\,\left(-w\,(\delta\,\alpha) - z\,\left(\delta\,\overline{\beta}\right)\right) + \left(-w\,\alpha - z\,\overline{\beta}\right)\,\left(\delta\,\eta^{\{w,\,z\}}\right) -$$
$$\left(-(1+w)\,\alpha - (-1+z)\,\overline{\beta}\right)\,\left((-z\,\overline{\alpha} - w\,\beta)\,\eta^{\{w,\,z\}} + \delta\,\eta^{\{w,\,z\}}\right) +$$
$$\delta\,\overline{\delta}\,\eta^{\{w,\,z\}} - \eta^{\{w,\,z\}}\,\left(-z\,\left(\overline{\delta}\,\overline{\alpha}\right) - w\,\left(\overline{\delta}\,\beta\right)\right) - (-z\,\overline{\alpha} - w\,\beta)\,\left(\overline{\delta}\,\eta^{\{w,\,z\}}\right) +$$
$$(-(1+z)\,\overline{\alpha} - (-1+w)\,\beta)\,\left(\left(-w\,\alpha - z\,\overline{\beta}\right)\,\eta^{\{w,\,z\}} + \overline{\delta}\,\eta^{\{w,\,z\}}\right) - \overline{\delta}\,\delta\,\eta^{\{w,\,z\}}\}$$

In[338]:= **% /. (NPCommutators /. ψ[] -> η[LI@{w, z}] /. Equal -> Rule);**

In[339]:= **% /. (Map[Dagger, NPCommutators, {2}] /. ψ[] -> η[LI@{w, z}] /. Equal -> Rule);**

In[340]:= **Map[FullSimplify, %, {2}]**

Out[340]=
$$\{ þ\,þ'\,\eta^{\{w,\,z\}} - þ'\,þ\,\eta^{\{w,\,z\}} == -\eta^{\{w,\,z\}}$$
$$(2\,w\,\gamma\,\epsilon + w\,\overline{\gamma}\,\epsilon + z\,\overline{\gamma}\,\epsilon + w\,\gamma\,\overline{\epsilon} + z\,\gamma\,\overline{\epsilon} + 2\,z\,\overline{\gamma}\,\overline{\epsilon} + w\,(D\,\gamma) + z\,(D\,\overline{\gamma}) - w\,(\Delta\,\epsilon) - z\,(\Delta\,\overline{\epsilon})) +$$
$$(\pi + \overline{\tau})\,\left(\delta\,\eta^{\{w,\,z\}}\right) + (\overline{\pi} + \tau)\,\left(\overline{\delta}\,\eta^{\{w,\,z\}}\right),\ þ\,ð\,\eta^{\{w,\,z\}} - ð\,þ\,\eta^{\{w,\,z\}} == \overline{\pi}\,\left(D\,\eta^{\{w,\,z\}}\right) -$$
$$\kappa\,\left(\Delta\,\eta^{\{w,\,z\}}\right) - \eta^{\{w,\,z\}}\,((w-z)\,\overline{\alpha}\,\epsilon + (2\,z\,\overline{\alpha} + (w+z)\,\beta)\,\overline{\epsilon} + z\,(D\,\overline{\alpha}) + w\,(D\,\beta - \delta\,\epsilon) - z\,(\delta\,\overline{\epsilon})) +$$
$$\overline{\rho}\,\left(\delta\,\eta^{\{w,\,z\}}\right) + \sigma\,\left(\overline{\delta}\,\eta^{\{w,\,z\}}\right),\ þ'\,ð\,\eta^{\{w,\,z\}} - ð\,þ'\,\eta^{\{w,\,z\}} == \overline{\nu}\,\left(D\,\eta^{\{w,\,z\}}\right) - \tau\,\left(\Delta\,\eta^{\{w,\,z\}}\right) +$$
$$\eta^{\{w,\,z\}}\,(w\,\overline{\alpha}\,\gamma + z\,\overline{\alpha}\,\gamma + 2\,w\,\beta\,\gamma - w\,\beta\,\overline{\gamma} + z\,\beta\,\overline{\gamma} - z\,(\Delta\,\overline{\alpha}) - w\,(\Delta\,\beta) + w\,(\delta\,\gamma) + z\,(\delta\,\overline{\gamma})) -$$
$$\mu\,\left(\delta\,\eta^{\{w,\,z\}}\right) - \overline{\lambda}\,\left(\overline{\delta}\,\eta^{\{w,\,z\}}\right),$$
$$þ\,ð'\,\eta^{\{w,\,z\}} - ð'\,þ\,\eta^{\{w,\,z\}} == \pi\,\left(D\,\eta^{\{w,\,z\}}\right) - \overline{\kappa}\,\left(\Delta\,\eta^{\{w,\,z\}}\right) + \overline{\sigma}\,\left(\delta\,\eta^{\{w,\,z\}}\right) - \eta^{\{w,\,z\}}$$
$$\left(2\,w\,\alpha\,\epsilon + w\,\overline{\beta}\,\epsilon + z\,\overline{\beta}\,\epsilon - w\,\alpha\,\overline{\epsilon} + z\,\alpha\,\overline{\epsilon} + w\,(D\,\alpha) + z\,\left(D\,\overline{\beta}\right) - w\,\left(\overline{\delta}\,\epsilon\right) - z\,\left(\overline{\delta}\,\overline{\epsilon}\right)\right) + \rho\,\left(\overline{\delta}\,\eta^{\{w,\,z\}}\right),$$
$$þ'\,ð'\,\eta^{\{w,\,z\}} - ð'\,þ'\,\eta^{\{w,\,z\}} == \nu\,\left(D\,\eta^{\{w,\,z\}}\right) - \overline{\tau}\,\left(\Delta\,\eta^{\{w,\,z\}}\right) - \lambda\,\left(\delta\,\eta^{\{w,\,z\}}\right) +$$
$$\eta^{\{w,\,z\}}\,\left((w-z)\,\overline{\beta}\,\gamma + \left((w+z)\,\alpha + 2\,z\,\overline{\beta}\right)\,\overline{\gamma} - w\,(\Delta\,\alpha) + w\,\left(\overline{\delta}\,\gamma\right) + z\,\left(-\left(\Delta\,\overline{\beta}\right) + \overline{\delta}\,\overline{\gamma}\right)\right) - \overline{\mu}\,\left(\overline{\delta}\,\eta^{\{w,\,z\}}\right),$$
$$ð\,ð'\,\eta^{\{w,\,z\}} - ð'\,ð\,\eta^{\{w,\,z\}} == (\mu - \overline{\mu})\,\left(D\,\eta^{\{w,\,z\}}\right) + (\rho - \overline{\rho})\,\left(\Delta\,\eta^{\{w,\,z\}}\right) +$$
$$\eta^{\{w,\,z\}}\,\left(w\,\left(\alpha\,(\overline{\alpha} - 2\,\beta) + \beta\,\overline{\beta}\right) - z\,\left(\alpha\,\overline{\alpha} + (-2\,\overline{\alpha} + \beta)\,\overline{\beta}\right) + z\,\left(-\left(\delta\,\overline{\beta}\right) + \overline{\delta}\,\overline{\alpha}\right) + w\,\left(-(\delta\,\alpha) + \overline{\delta}\,\beta\right)\right)\}$$

We use the Newman–Penrose equations on this relation.

In[341]:= **% /. (NPRicci /. Equal → Rule);**

In[342]:= **% /. (NPRicci† /. Equal → Rule);**

In[343]:=
```
Map[Simplify, %, {2}]
```

Out[343]=
$$\{\text{þ}\,\text{þ}'\,\eta^{\{w,\,z\}} - \text{þ}'\,\text{þ}\,\eta^{\{w,\,z\}} == -\eta^{\{w,\,z\}}\,\big(w\,(\Lambda - \Psi_{11} + \Psi_2 - \kappa\,\nu + \beta\,\pi + \alpha\,\overline{\pi} + \alpha\,\tau + \pi\,\tau + \beta\,\overline{\tau}) + z\,\big(\Lambda - \Psi_{11} + \overline{\Psi}_2 - \overline{\kappa}\,\overline{\nu} + \overline{\alpha}\,\pi + \overline{\beta}\,\overline{\pi} + \overline{\beta}\,\tau + \overline{\alpha}\,\overline{\tau} + \overline{\pi}\,\overline{\tau}\big)\big) + (\pi + \overline{\tau})\,\big(\delta\,\eta^{\{w,\,z\}}\big) + (\overline{\pi} + \tau)\,\big(\overline{\delta}\,\eta^{\{w,\,z\}}\big),$$

$$\text{þ}\,\text{ð}\,\eta^{\{w,\,z\}} - \text{ð}\,\text{þ}\,\eta^{\{w,\,z\}} == -\eta^{\{w,\,z\}}\,\big(z\,\big(-\Psi_{01} - \overline{\gamma}\,\kappa - \overline{\kappa}\,\overline{\lambda} + \overline{\epsilon}\,\overline{\pi} + \overline{\alpha}\,\overline{\rho} + \overline{\pi}\,\overline{\rho} + \overline{\beta}\,\sigma\big) + w\,(\Psi_1 - \gamma\,\kappa - \kappa\,\mu + \epsilon\,\overline{\pi} + \beta\,\overline{\rho} + \alpha\,\sigma + \pi\,\sigma)\big) + \overline{\pi}\,\big(D\,\eta^{\{w,\,z\}}\big) - \kappa\,\big(\Delta\,\eta^{\{w,\,z\}}\big) + \overline{\rho}\,\big(\delta\,\eta^{\{w,\,z\}}\big) + \sigma\,\big(\overline{\delta}\,\eta^{\{w,\,z\}}\big),$$

$$\text{þ}'\,\text{ð}\,\eta^{\{w,\,z\}} - \text{ð}\,\text{þ}'\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\,\big(w\,\big(-\Psi_{12} + \alpha\,\overline{\lambda} + \beta\,\mu - \epsilon\,\overline{\nu} - \nu\,\sigma + \gamma\,\tau + \mu\,\tau\big) + z\,\big(\overline{\Psi}_3 + \overline{\beta}\,\overline{\lambda} + \overline{\alpha}\,\mu - \overline{\epsilon}\,\overline{\nu} - \overline{\nu}\,\overline{\rho} + \overline{\gamma}\,\tau + \overline{\lambda}\,\overline{\tau}\big)\big) + \overline{\nu}\,\big(D\,\eta^{\{w,\,z\}}\big) - \tau\,\big(\Delta\,\eta^{\{w,\,z\}}\big) - \mu\,\big(\delta\,\eta^{\{w,\,z\}}\big) - \overline{\lambda}\,\big(\overline{\delta}\,\eta^{\{w,\,z\}}\big),$$

$$\text{þ}\,\text{ð}'\,\eta^{\{w,\,z\}} - \text{ð}'\,\text{þ}\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\,\big(w\,(\Psi_{10} + \gamma\,\overline{\kappa} + \kappa\,\lambda - \epsilon\,\pi - \alpha\,\rho - \pi\,\rho - \beta\,\overline{\sigma}) - z\,\big(\overline{\Psi}_1 - \overline{\gamma}\,\overline{\kappa} - \overline{\kappa}\,\overline{\mu} + \overline{\epsilon}\,\pi + \overline{\beta}\,\rho + \overline{\alpha}\,\overline{\sigma} + \overline{\pi}\,\overline{\sigma}\big)\big) + \pi\,\big(D\,\eta^{\{w,\,z\}}\big) - \overline{\kappa}\,\big(\Delta\,\eta^{\{w,\,z\}}\big) + \overline{\sigma}\,\big(\delta\,\eta^{\{w,\,z\}}\big) + \rho\,\big(\overline{\delta}\,\eta^{\{w,\,z\}}\big),$$

$$\text{þ}'\,\text{ð}'\,\eta^{\{w,\,z\}} - \text{ð}'\,\text{þ}'\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\,\big(w\,(\Psi_3 + \beta\,\lambda + \alpha\,\overline{\mu} - \epsilon\,\nu - \nu\,\rho + \lambda\,\tau + \gamma\,\overline{\tau}) + z\,\big(-\Psi_{21} + \overline{\alpha}\,\lambda + \overline{\beta}\,\overline{\mu} - \overline{\epsilon}\,\nu - \overline{\nu}\,\overline{\sigma} + \overline{\gamma}\,\overline{\tau} + \overline{\mu}\,\overline{\tau}\big)\big) + \nu\,\big(D\,\eta^{\{w,\,z\}}\big) - \overline{\tau}\,\big(\Delta\,\eta^{\{w,\,z\}}\big) - \lambda\,\big(\delta\,\eta^{\{w,\,z\}}\big) - \overline{\mu}\,\big(\overline{\delta}\,\eta^{\{w,\,z\}}\big),$$

$$\text{ð}\,\text{ð}'\,\eta^{\{w,\,z\}} - \text{ð}'\,\text{ð}\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\,\big(w\,(\Lambda + \Psi_{11} + \Psi_2 - \epsilon\,\mu + \epsilon\,\overline{\mu} - \gamma\,\rho - \mu\,\rho + \gamma\,\overline{\rho} + \lambda\,\sigma) - z\,\big(\Lambda + \Psi_{11} + \overline{\Psi}_2 + \overline{\epsilon}\,\mu - \overline{\epsilon}\,\overline{\mu} + \overline{\gamma}\,\rho - \overline{\gamma}\,\overline{\rho} - \overline{\mu}\,\overline{\rho} + \overline{\lambda}\,\overline{\sigma}\big)\big) + (\mu - \overline{\mu})\,\big(D\,\eta^{\{w,\,z\}}\big) + (\rho - \overline{\rho})\,\big(\Delta\,\eta^{\{w,\,z\}}\big)\}$$

Finally we write $\delta\,\eta^{\{w,\,z\}}$, $\overline{\delta}\,\eta^{\{w,\,z\}}$ in terms of $\text{ð}\,\eta^{\{w,\,z\}}$, $\text{ð}'\,\eta^{\{w,\,z\}}$ and simplify the resulting expression.

In[344]:
```
% /. NPToGHP;
```

In[345]:
```
Map[FullSimplify, %, {2}]
```

Out[345]=
$$\{\text{þ}\,\text{þ}'\,\eta^{\{w,\,z\}} - \text{þ}'\,\text{þ}\,\eta^{\{w,\,z\}} == -\eta^{\{w,\,z\}}\,\big(w\,(\Lambda - \Psi_{11} + \Psi_2 - \kappa\,\nu + \pi\,\tau) + z\,\big(\Lambda - \Psi_{11} + \overline{\Psi}_2 - \overline{\kappa}\,\overline{\nu} + \overline{\pi}\,\overline{\tau}\big) + (\pi + \overline{\tau})\,\big(\text{ð}\,\eta^{\{w,\,z\}}\big) + (\overline{\pi} + \tau)\,\big(\text{ð}'\,\eta^{\{w,\,z\}}\big),$$

$$\text{þ}\,\text{ð}\,\eta^{\{w,\,z\}} - \text{ð}\,\text{þ}\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\,\big(z\,\big(\Psi_{01} + \overline{\kappa}\,\overline{\lambda} - \overline{\pi}\,\overline{\rho}\big) - w\,(\Psi_1 - \kappa\,\mu + \pi\,\sigma)\big) + \overline{\pi}\,\big(\text{þ}\,\eta^{\{w,\,z\}}\big) - \kappa\,\big(\text{þ}'\,\eta^{\{w,\,z\}}\big) + \overline{\rho}\,\big(\text{ð}\,\eta^{\{w,\,z\}}\big) + \sigma\,\big(\text{ð}'\,\eta^{\{w,\,z\}}\big),$$

$$\text{þ}'\,\text{ð}\,\eta^{\{w,\,z\}} - \text{ð}\,\text{þ}'\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\,\big(-w\,(\Psi_{12} + \nu\,\sigma - \mu\,\tau) + z\,\big(\overline{\Psi}_3 - \overline{\nu}\,\overline{\rho} + \overline{\lambda}\,\overline{\tau}\big)\big) + \overline{\nu}\,\big(\text{þ}\,\eta^{\{w,\,z\}}\big) - \tau\,\big(\text{þ}'\,\eta^{\{w,\,z\}}\big) - \mu\,\big(\text{ð}\,\eta^{\{w,\,z\}}\big) - \overline{\lambda}\,\big(\text{ð}'\,\eta^{\{w,\,z\}}\big),$$

$$\text{þ}\,\text{ð}'\,\eta^{\{w,\,z\}} - \text{ð}'\,\text{þ}\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\,\big(w\,(\Psi_{10} + \kappa\,\lambda - \pi\,\rho) - z\,\big(\overline{\Psi}_1 - \overline{\kappa}\,\overline{\mu} + \overline{\pi}\,\overline{\sigma}\big)\big) + \pi\,\big(\text{þ}\,\eta^{\{w,\,z\}}\big) - \overline{\kappa}\,\big(\text{þ}'\,\eta^{\{w,\,z\}}\big) + \overline{\sigma}\,\big(\text{ð}\,\eta^{\{w,\,z\}}\big) + \rho\,\big(\text{ð}'\,\eta^{\{w,\,z\}}\big),$$

$$\text{þ}'\,\text{ð}'\,\eta^{\{w,\,z\}} - \text{ð}'\,\text{þ}'\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\,(w\,(\Psi_3 - \nu\,\rho + \lambda\,\tau) - z\,(\Phi_{21} + \overline{\nu}\,\overline{\sigma} - \overline{\mu}\,\overline{\tau})) + \nu\,\big(\text{þ}\,\eta^{\{w,\,z\}}\big) - \overline{\tau}\,\big(\text{þ}'\,\eta^{\{w,\,z\}}\big) - \lambda\,\big(\text{ð}\,\eta^{\{w,\,z\}}\big) - \overline{\mu}\,\big(\text{ð}'\,\eta^{\{w,\,z\}}\big),$$

$$\text{ð}\,\text{ð}'\,\eta^{\{w,\,z\}} - \text{ð}'\,\text{ð}\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\,\big(w\,(\Lambda + \Psi_{11} + \Psi_2 - \mu\,\rho + \lambda\,\sigma) - z\,\big(\Lambda + \Psi_{11} + \overline{\Psi}_2 - \overline{\mu}\,\overline{\rho} + \overline{\lambda}\,\overline{\sigma}\big)\big) + (\mu - \overline{\mu})\,\big(\text{þ}\,\eta^{\{w,\,z\}}\big) + (\rho - \overline{\rho})\,\big(\text{þ}'\,\eta^{\{w,\,z\}}\big)\}$$

In[346]:
```
GHPCommutators = %;
```

We check that everything is OK

In[347]:
```
Map[GHPWeightOf, GHPCommutators, {2}]
```

Out[347]=
```
{True, True, True, True, True, True}
```

■ Ricci identities in the GHP formalism

In this subsection we use the Newman–Penrose Ricci identities (Newman–Penrose equations) to find a set of equations which only contain the GHP differential operators and spin weighted quantities. This set of equations is known as GHP equations or compacted spin coefficient equations.

These are the weighted spin coefficients.

In[348]:=
```
GHPSpinCoefficients
```

Out[348]=
```
{κ , λ , μ , ν , π , ρ , σ , τ }
```

These are the NP differentials of these spin coefficients.

In[349]:=
```
Flatten@Outer[PDNP[#1][#2] &, {{1, -NP}, {2, -NP}, {3, -NP}, {4, -NP}}, GHPSpinCoefficients, 1]
```

Out[349]=
```
{Dκ , Dλ , Dμ , Dν , Dπ , Dρ , Dσ , Dτ , Δκ , Δλ , Δμ , Δν , Δπ , Δρ , Δσ , Δτ ,
 δκ , δλ , δμ , δν , δπ , δρ , δσ , δτ , δ̄κ , δ̄λ , δ̄μ , δ̄ν , δ̄π , δ̄ρ , δ̄σ , δ̄τ }
```

We compute the GHP differentials of the weighted Newman–Penrose spin coefficients.

In[350]:=
```
Rule[#, # /. NPToGHP] & /@ %
```

Out[350]=
```
{Dκ → 3ϵ κ + ϵ̄ κ + þκ , Dλ → -3ϵ λ + ϵ̄ λ + þλ ,
 Dμ → -ϵ μ - ϵ̄ μ + þμ , Dν → -3ϵ ν - ϵ̄ ν + þν , Dπ → -ϵ π + ϵ̄ π + þπ ,
 Dρ → ϵ ρ + ϵ̄ ρ + þρ , Dσ → 3ϵ σ - ϵ̄ σ + þσ , Dτ → ϵ τ - ϵ̄ τ + þτ ,
 Δκ → 3γ κ + γ̄ κ + þ′κ , Δλ → -3γ λ + γ̄ λ + þ′λ , Δμ → -γ μ - γ̄ μ + þ′μ ,
 Δν → -3γ ν - γ̄ ν + þ′ν , Δπ → -γ π + γ̄ π + þ′π , Δρ → γ ρ + γ̄ ρ + þ′ρ ,
 Δσ → 3γ σ - γ̄ σ + þ′σ , Δτ → γ τ - γ̄ τ + þ′τ , δκ → ᾱ κ + 3β κ + ðκ ,
 δλ → ᾱ λ - 3β λ + ðλ , δμ → -ᾱ μ - β μ + ðμ , δν → -ᾱ ν - 3β ν + ðν ,
 δπ → ᾱ π - β π + ðπ , δρ → ᾱ ρ + β ρ + ðρ , δσ → -ᾱ σ + 3β σ + ðσ ,
 δτ → -ᾱ τ + β τ + ðτ , δ̄κ → 3α κ + β̄ κ + ð′κ , δ̄λ → -3α λ + β̄ λ + ð′λ ,
 δ̄μ → -α μ - β̄ μ + ð′μ , δ̄ν → -3α ν - β̄ ν + ð′ν , δ̄π → -α π + β̄ π + ð′π ,
 δ̄ρ → α ρ + β̄ ρ + ð′ρ , δ̄σ → 3α σ - β̄ σ + ð′σ , δ̄τ → α τ - β̄ τ + ð′τ }
```

With this information we can render the Newman–Penrose equations in terms of the GHP differential operators.

In[351]:=
```
NPRicci /. %
```

Out[351]=
$$\begin{aligned}
&\{\text{D}\alpha == -\Phi_{10} - 2\alpha\epsilon - \overline{\beta}\epsilon + \alpha\overline{\epsilon} - \gamma\overline{\kappa} - \kappa\lambda + \epsilon\pi + \alpha\rho + \pi\rho + \beta\overline{\sigma} + \overline{\delta}\epsilon,\\
&\text{D}\beta == \Psi_1 - \overline{\alpha}\epsilon - \beta\overline{\epsilon} - \gamma\kappa - \kappa\mu + \epsilon\overline{\pi} + \beta\overline{\rho} + \alpha\sigma + \pi\sigma + \delta\epsilon,\\
&\text{D}\gamma == \Lambda - \Phi_{11} + \Psi_2 - 2\gamma\epsilon - \overline{\gamma}\epsilon - \gamma\overline{\epsilon} - \kappa\nu + \beta\pi + \alpha\overline{\pi} + \alpha\tau + \pi\tau + \beta\overline{\tau} + \Delta\epsilon,\\
&-3\epsilon\lambda + \overline{\epsilon}\lambda + \text{Þ}\lambda == -\Phi_{20} - 3\epsilon\lambda + \overline{\epsilon}\lambda - \overline{\kappa}\nu + \pi^2 + \lambda\rho + \mu\overline{\sigma} + \text{ð}'\pi,\\
&-\epsilon\mu - \overline{\epsilon}\mu + \text{Þ}\mu == -2\Lambda + \Psi_2 - \epsilon\mu - \overline{\epsilon}\mu - \kappa\nu + \pi\overline{\pi} + \mu\overline{\rho} + \lambda\sigma + \text{ð}\pi,\\
&-3\epsilon\nu - \overline{\epsilon}\nu + \text{Þ}\nu == -\Phi_{21} + \Psi_3 - 3\epsilon\nu - \overline{\epsilon}\nu + \mu\pi + \lambda\overline{\pi} + \lambda\tau + \mu\overline{\tau} + \text{Þ}'\pi,\\
&\epsilon\rho + \overline{\epsilon}\rho + \text{Þ}\rho == -\Phi_{00} + \kappa\pi + \epsilon\rho + \overline{\epsilon}\rho + \rho^2 + \sigma\overline{\sigma} - \overline{\kappa}\tau + \text{ð}'\kappa,\\
&3\epsilon\sigma - \overline{\epsilon}\sigma + \text{Þ}\sigma == \Psi_0 + \kappa\overline{\pi} + 3\epsilon\sigma - \overline{\epsilon}\sigma + \rho\sigma + \overline{\rho}\sigma - \kappa\tau + \text{ð}\kappa,\\
&\epsilon\tau - \overline{\epsilon}\tau + \text{Þ}\tau == -\Phi_{01} + \Psi_1 + \overline{\pi}\rho + \pi\sigma + \epsilon\tau - \overline{\epsilon}\tau + \rho\tau + \sigma\overline{\tau} + \text{Þ}'\kappa,\\
&\Delta\alpha == -\Psi_3 + \overline{\beta}\gamma + \alpha\overline{\gamma} - \beta\lambda - \alpha\overline{\mu} + \epsilon\nu + \nu\rho - \lambda\tau - \gamma\overline{\tau} + \overline{\delta}\gamma,\\
&\Delta\beta == \Phi_{12} + \overline{\alpha}\gamma + 2\beta\gamma - \beta\overline{\gamma} - \alpha\overline{\lambda} - \beta\mu + \epsilon\overline{\nu} + \nu\sigma - \gamma\tau - \mu\tau + \delta\gamma,\\
&-3\gamma\lambda + \overline{\gamma}\lambda + \text{Þ}'\lambda == -\Psi_4 - 3\gamma\lambda + \overline{\gamma}\lambda - \lambda\mu - \lambda\overline{\mu} + \nu\pi - \nu\overline{\tau} + \text{ð}'\nu,\\
&-\gamma\mu - \overline{\gamma}\mu + \text{Þ}'\mu == \Phi_{22} - \lambda\overline{\lambda} - \gamma\mu - \overline{\gamma}\mu - \mu^2 + \overline{\nu}\pi - \nu\tau + \text{ð}\nu,\\
&\gamma\rho + \overline{\gamma}\rho + \text{Þ}'\rho == 2\Lambda - \Psi_2 + \kappa\nu + \gamma\rho + \overline{\gamma}\rho - \overline{\mu}\rho - \lambda\sigma - \tau\overline{\tau} + \text{ð}'\tau,\\
&3\gamma\sigma - \overline{\gamma}\sigma + \text{Þ}'\sigma == \Phi_{02} + \kappa\overline{\nu} - \overline{\lambda}\rho + 3\gamma\sigma - \overline{\gamma}\sigma - \mu\sigma - \tau^2 + \text{ð}\tau,\\
&\delta\alpha == -\Lambda - \Phi_{11} - \Psi_2 + \alpha\overline{\alpha} - 2\alpha\beta + \beta\overline{\beta} + \epsilon\mu - \epsilon\overline{\mu} + \gamma\rho + \mu\rho - \gamma\overline{\rho} - \lambda\sigma + \overline{\delta}\beta,\\
&\overline{\alpha}\lambda - 3\beta\lambda + \text{ð}\lambda == -\Phi_{21} - \Psi_3 + \overline{\alpha}\lambda - 3\beta\lambda + \mu\pi - \overline{\mu}\pi + \nu\rho - \nu\overline{\rho} + \text{ð}'\mu,\\
&\overline{\alpha}\rho + \beta\rho + \text{ð}\rho == -\Phi_{01} - \Psi_1 + \kappa\mu - \kappa\overline{\mu} + \overline{\alpha}\rho + \beta\rho + \rho\tau - \overline{\rho}\tau + \text{ð}'\sigma\}
\end{aligned}$$

In[352]:=
```
Simplify //@ %
```

Out[352]=
$$\begin{aligned}
&\{\Phi_{10} + 2\alpha\epsilon + \overline{\beta}\epsilon + \gamma\overline{\kappa} + \kappa\lambda + \text{D}\alpha == \epsilon\pi + \pi\rho + \alpha(\overline{\epsilon} + \rho) + \beta\overline{\sigma} + \overline{\delta}\epsilon,\\
&\Psi_1 + \epsilon\overline{\pi} + \beta\overline{\rho} + \alpha\sigma + \pi\sigma + \delta\epsilon == \overline{\alpha}\epsilon + \beta\overline{\epsilon} + \gamma\kappa + \kappa\mu + \text{D}\beta,\\
&\Lambda + \Psi_2 + \beta\pi + \alpha\overline{\pi} + \alpha\tau + \pi\tau + \beta\overline{\tau} + \Delta\epsilon == \Phi_{11} + 2\gamma\epsilon + \overline{\gamma}\epsilon + \gamma\overline{\epsilon} + \kappa\nu + \text{D}\gamma,\\
&\Phi_{20} + \overline{\kappa}\nu + \text{Þ}\lambda == \pi^2 + \lambda\rho + \mu\overline{\sigma} + \text{ð}'\pi,\ 2\Lambda + \kappa\nu + \text{Þ}\mu == \Psi_2 + \pi\overline{\pi} + \mu\overline{\rho} + \lambda\sigma + \text{ð}\pi,\\
&\Phi_{21} + \text{Þ}\nu == \Psi_3 + \mu\pi + \lambda\overline{\pi} + \lambda\tau + \mu\overline{\tau} + \text{Þ}'\pi,\ \Phi_{00} + \overline{\kappa}\tau + \text{Þ}\rho == \kappa\pi + \rho^2 + \sigma\overline{\sigma} + \text{ð}'\kappa,\\
&\Psi_0 + \kappa\overline{\pi} + \rho\sigma + \overline{\rho}\sigma + \text{ð}\kappa == \kappa\tau + \text{Þ}\sigma,\ \Phi_{01} + \text{Þ}\tau == \Psi_1 + \overline{\pi}\rho + \pi\sigma + \rho\tau + \sigma\overline{\tau} + \text{Þ}'\kappa,\\
&\Psi_3 + \beta\lambda + \alpha\overline{\mu} + \lambda\tau + \gamma\overline{\tau} + \Delta\alpha == \overline{\beta}\gamma + \alpha\overline{\gamma} + \epsilon\nu + \nu\rho + \overline{\delta}\gamma,\\
&\alpha\overline{\lambda} + \beta(\overline{\gamma} + \mu) + \gamma\tau + \mu\tau + \Delta\beta == \Phi_{12} + \overline{\alpha}\gamma + 2\beta\gamma + \epsilon\overline{\nu} + \nu\sigma + \delta\gamma,\\
&\Psi_4 + \lambda\mu + \lambda\overline{\mu} + \nu\overline{\tau} + \text{Þ}'\lambda == \nu\pi + \text{ð}'\nu,\ \Phi_{22} + \overline{\nu}\pi + \text{ð}\nu == \lambda\overline{\lambda} + \mu^2 + \nu\tau + \text{Þ}'\mu,\\
&2\Lambda + \kappa\nu + \text{ð}'\tau == \Psi_2 + \overline{\mu}\rho + \lambda\sigma + \tau\overline{\tau} + \text{Þ}'\rho,\ \Phi_{02} + \kappa\overline{\nu} + \text{ð}\tau == \overline{\lambda}\rho + \mu\sigma + \tau^2 + \text{Þ}'\sigma,\\
&\Lambda + \Phi_{11} + \Psi_2 + 2\alpha\beta + \epsilon\overline{\mu} + \gamma\overline{\rho} + \lambda\sigma + \delta\alpha == \alpha\overline{\alpha} + \beta\overline{\beta} + \epsilon\mu + \gamma\rho + \mu\rho + \overline{\delta}\beta,\\
&\Phi_{21} + \Psi_3 + \overline{\mu}\pi + \nu\overline{\rho} + \text{ð}\lambda == \mu\pi + \nu\rho + \text{ð}'\mu,\ \Phi_{01} + \Psi_1 + \kappa\overline{\mu} + \overline{\rho}\tau + \text{ð}\rho == \kappa\mu + \rho\tau + \text{ð}'\sigma\}
\end{aligned}$$

We are interested in these equations which only contain weighted spin coefficients and GHP operators. These are

In[353]:=
```
Part[%, Union[First /@ Position[%, Θ[_]@_]]]
```

Out[353]=
$$\begin{aligned}
&\{\Phi_{20} + \overline{\kappa}\nu + \text{Þ}\lambda == \pi^2 + \lambda\rho + \mu\overline{\sigma} + \text{ð}'\pi,\ 2\Lambda + \kappa\nu + \text{Þ}\mu == \Psi_2 + \pi\overline{\pi} + \mu\overline{\rho} + \lambda\sigma + \text{ð}\pi,\\
&\Phi_{21} + \text{Þ}\nu == \Psi_3 + \mu\pi + \lambda\overline{\pi} + \lambda\tau + \mu\overline{\tau} + \text{Þ}'\pi,\ \Phi_{00} + \overline{\kappa}\tau + \text{Þ}\rho == \kappa\pi + \rho^2 + \sigma\overline{\sigma} + \text{ð}'\kappa,\\
&\Psi_0 + \kappa\overline{\pi} + \rho\sigma + \overline{\rho}\sigma + \text{ð}\kappa == \kappa\tau + \text{Þ}\sigma,\ \Phi_{01} + \text{Þ}\tau == \Psi_1 + \overline{\pi}\rho + \pi\sigma + \rho\tau + \sigma\overline{\tau} + \text{Þ}'\kappa,\\
&\Psi_4 + \lambda\mu + \lambda\overline{\mu} + \nu\overline{\tau} + \text{Þ}'\lambda == \nu\pi + \text{ð}'\nu,\ \Phi_{22} + \overline{\nu}\pi + \text{ð}\nu == \lambda\overline{\lambda} + \mu^2 + \nu\tau + \text{Þ}'\mu,\\
&2\Lambda + \kappa\nu + \text{ð}'\tau == \Psi_2 + \overline{\mu}\rho + \lambda\sigma + \tau\overline{\tau} + \text{Þ}'\rho,\ \Phi_{02} + \kappa\overline{\nu} + \text{ð}\tau == \overline{\lambda}\rho + \mu\sigma + \tau^2 + \text{Þ}'\sigma,\\
&\Phi_{21} + \Psi_3 + \overline{\mu}\pi + \nu\overline{\rho} + \text{ð}\lambda == \mu\pi + \nu\rho + \text{ð}'\mu,\ \Phi_{01} + \Psi_1 + \kappa\overline{\mu} + \overline{\rho}\tau + \text{ð}\rho == \kappa\mu + \rho\tau + \text{ð}'\sigma\}
\end{aligned}$$

We may write these equations in an alternative way.

In[354]:=
```
Solve[%, Union@Cases[%, 0[__]@_, Infinity]] // Flatten
```

```
Solve::svars : Equations may not give solutions for all "solve" variables. >>
```

Out[354]=
$$\{\text{þ}\lambda \to -\Phi_{20} - \overline{\kappa}\,\nu + \pi^2 + \lambda\,\rho + \mu\,\overline{\sigma} + \text{ð}'\pi,\ \text{þ}\mu \to -2\Lambda + \Psi_2 - \kappa\,\nu + \pi\,\overline{\pi} + \mu\,\overline{\rho} + \lambda\,\sigma + \text{ð}\pi,$$
$$\text{þ}\nu \to -\Phi_{21} + \Psi_3 + \mu\,\pi + \lambda\,\overline{\pi} + \lambda\,\tau + \mu\,\overline{\tau} + \text{þ}'\pi,$$
$$\text{þ}\rho \to -\Phi_{00} + \kappa\,\pi + \rho^2 + \sigma\,\overline{\sigma} - \overline{\kappa}\,\tau + \text{ð}'\kappa,\ \text{þ}\sigma \to \Psi_0 + \kappa\,\overline{\pi} + \rho\,\sigma + \overline{\rho}\,\sigma - \kappa\,\tau + \text{ð}\kappa,$$
$$\text{þ}\tau \to -\Phi_{01} + \Psi_1 + \overline{\pi}\,\rho + \pi\,\sigma + \rho\,\tau + \sigma\,\overline{\tau} + \text{þ}'\kappa,\ \text{þ}'\lambda \to -\Psi_4 - \lambda\,\mu - \lambda\,\overline{\mu} + \nu\,\pi - \nu\,\overline{\tau} + \text{ð}'\nu,$$
$$\text{þ}'\mu \to \Phi_{22} - \lambda\,\overline{\lambda} - \mu^2 + \overline{\nu}\,\pi - \nu\,\tau + \text{ð}\nu,\ \text{þ}'\rho \to 2\Lambda - \Psi_2 + \kappa\,\nu - \overline{\mu}\,\rho - \lambda\,\sigma - \tau\,\overline{\tau} + \text{ð}'\tau,$$
$$\text{þ}'\sigma \to \Phi_{02} + \kappa\,\overline{\nu} - \overline{\lambda}\,\rho - \mu\,\sigma - \tau^2 + \text{ð}\tau,\ \text{ð}\lambda \to -\Phi_{21} - \Psi_3 + \mu\,\pi - \overline{\mu}\,\pi + \nu\,\rho - \nu\,\overline{\rho} + \text{ð}'\mu,$$
$$\text{ð}\rho \to -\Phi_{01} - \Psi_1 + \kappa\,\mu - \kappa\,\overline{\mu} + \rho\,\tau - \overline{\rho}\,\tau + \text{ð}'\sigma\}$$

In[355]:=
```
FullSimplify //@ %
```

Out[355]=
$$\{\text{þ}\lambda \to -\Phi_{20} - \overline{\kappa}\,\nu + \pi^2 + \lambda\,\rho + \mu\,\overline{\sigma} + \text{ð}'\pi,\ \text{þ}\mu \to -2\Lambda + \Psi_2 - \kappa\,\nu + \pi\,\overline{\pi} + \mu\,\overline{\rho} + \lambda\,\sigma + \text{ð}\pi,$$
$$\text{þ}\nu \to -\Phi_{21} + \Psi_3 + \lambda\,(\overline{\pi} + \tau) + \mu\,(\pi + \overline{\tau}) + \text{þ}'\pi,\ \text{þ}\rho \to -\Phi_{00} + \kappa\,\pi + \rho^2 + \sigma\,\overline{\sigma} - \overline{\kappa}\,\tau + \text{ð}'\kappa,$$
$$\text{þ}\sigma \to \Psi_0 + (\rho + \overline{\rho})\,\sigma + \kappa\,(\overline{\pi} - \tau) + \text{ð}\kappa,\ \text{þ}\tau \to -\Phi_{01} + \Psi_1 + \rho\,(\overline{\pi} + \tau) + \sigma\,(\pi + \overline{\tau}) + \text{þ}'\kappa,$$
$$\text{þ}'\lambda \to -\Psi_4 - \lambda\,(\mu + \overline{\mu}) + \nu\,(\pi - \overline{\tau}) + \text{ð}'\nu,\ \text{þ}'\mu \to \Phi_{22} - \lambda\,\overline{\lambda} - \mu^2 + \overline{\nu}\,\pi - \nu\,\tau + \text{ð}\nu,$$
$$\text{þ}'\rho \to 2\Lambda - \Psi_2 + \kappa\,\nu - \overline{\mu}\,\rho - \lambda\,\sigma - \tau\,\overline{\tau} + \text{ð}'\tau,\ \text{þ}'\sigma \to \Phi_{02} + \kappa\,\overline{\nu} - \overline{\lambda}\,\rho - \mu\,\sigma - \tau^2 + \text{ð}\tau,$$
$$\text{ð}\lambda \to -\Phi_{21} - \Psi_3 + \mu\,\pi - \overline{\mu}\,\pi + \nu\,\rho - \nu\,\overline{\rho} + \text{ð}'\mu,$$
$$\text{ð}\rho \to -\Phi_{01} - \Psi_1 + \kappa\,\mu - \kappa\,\overline{\mu} + \rho\,\tau - \overline{\rho}\,\tau + \text{ð}'\sigma\}$$

In[356]:=
```
% /. Rule -> Equal
```

Out[356]=
$$\{\text{þ}\lambda == -\Phi_{20} - \overline{\kappa}\,\nu + \pi^2 + \lambda\,\rho + \mu\,\overline{\sigma} + \text{ð}'\pi,\ \text{þ}\mu == -2\Lambda + \Psi_2 - \kappa\,\nu + \pi\,\overline{\pi} + \mu\,\overline{\rho} + \lambda\,\sigma + \text{ð}\pi,$$
$$\text{þ}\nu == -\Phi_{21} + \Psi_3 + \lambda\,(\overline{\pi} + \tau) + \mu\,(\pi + \overline{\tau}) + \text{þ}'\pi,\ \text{þ}\rho == -\Phi_{00} + \kappa\,\pi + \rho^2 + \sigma\,\overline{\sigma} - \overline{\kappa}\,\tau + \text{ð}'\kappa,$$
$$\text{þ}\sigma == \Psi_0 + (\rho + \overline{\rho})\,\sigma + \kappa\,(\overline{\pi} - \tau) + \text{ð}\kappa,\ \text{þ}\tau == -\Phi_{01} + \Psi_1 + \rho\,(\overline{\pi} + \tau) + \sigma\,(\pi + \overline{\tau}) + \text{þ}'\kappa,$$
$$\text{þ}'\lambda == -\Psi_4 - \lambda\,(\mu + \overline{\mu}) + \nu\,(\pi - \overline{\tau}) + \text{ð}'\nu,\ \text{þ}'\mu == \Phi_{22} - \lambda\,\overline{\lambda} - \mu^2 + \overline{\nu}\,\pi - \nu\,\tau + \text{ð}\nu,$$
$$\text{þ}'\rho == 2\Lambda - \Psi_2 + \kappa\,\nu - \overline{\mu}\,\rho - \lambda\,\sigma - \tau\,\overline{\tau} + \text{ð}'\tau,\ \text{þ}'\sigma == \Phi_{02} + \kappa\,\overline{\nu} - \overline{\lambda}\,\rho - \mu\,\sigma - \tau^2 + \text{ð}\tau,$$
$$\text{ð}\lambda == -\Phi_{21} - \Psi_3 + \mu\,\pi - \overline{\mu}\,\pi + \nu\,\rho - \nu\,\overline{\rho} + \text{ð}'\mu,$$
$$\text{ð}\rho == -\Phi_{01} - \Psi_1 + \kappa\,\mu - \kappa\,\overline{\mu} + \rho\,\tau - \overline{\rho}\,\tau + \text{ð}'\sigma\}$$

Previous set of equations is known as GHP equations or compacted spin coefficient equations.

In[357]:=
```
Length[GHPRicci = %]
```

Out[357]= 12

■ Bianchi identities in the GHP formalism

Each of the components of the curvature spinors is a weighted scalar. Using this property it is possible to rewrite the Newman–Penrose form of the Bianchi identities in terms of spin weighted quantities. This is accomplished in this subsection.

We write the differentials of the curvature components which appear in the NP Bianchi identities in terms of differentials written in ters of the GHP operators.

In[358]:=
```
NPBianchi /. NPToGHP
```

Out[358]=
$$\{2\,\Phi_{01}\,\epsilon + \text{þ}\,\Phi_{01} == 4\,\Psi_0\,\alpha + 2\,\Phi_{00}\,\overline{\alpha} + 2\,\Phi_{00}\,\beta - 2\,\Phi_{11}\,\kappa - 3\,\Psi_2\,\kappa - \Phi_{02}\,\overline{\kappa} + \Psi_0\,(-4\,\alpha + \pi) + \Phi_{00}\,(-2\,(\overline{\alpha} + \beta) + \overline{\pi}) + 4\,\Psi_1\,\rho + 2\,\Phi_{01}\,(\epsilon + \overline{\rho}) + 2\,\Phi_{10}\,\sigma - \text{þ}\,\Psi_1 - \text{ð}\,\Phi_{00} + \text{ð}'\,\Psi_0,$$
$$2\,\Phi_{02}\,\epsilon - 2\,\Phi_{02}\,\overline{\epsilon} + \text{þ}\,\Phi_{02} == 4\,\Psi_0\,\gamma - 2\,\Phi_{12}\,\kappa - \Phi_{00}\,\overline{\lambda} + \Psi_0\,(-4\,\gamma + \mu) + 2\,\Phi_{01}\,\overline{\pi} + \Phi_{02}\,(2\,\epsilon - 2\,\overline{\epsilon} + \overline{\rho}) + 2\,\Phi_{11}\,\sigma - 3\,\Psi_2\,\sigma + 4\,\Psi_1\,\tau + \text{þ}'\,\Psi_0 + \text{ð}\,\Phi_{01} - \text{ð}\,\Psi_1,$$
$$\text{þ}\,\Phi_{11} == 2\,\Psi_1\,\alpha + 2\,\Phi_{10}\,\overline{\alpha} - (\Phi_{21} + 2\,\Psi_3)\,\kappa - \Phi_{12}\,\overline{\kappa} - \Psi_0\,\lambda - \Phi_{00}\,\mu + \Phi_{01}\,\pi + 2\,\Psi_1\,(-\alpha + \pi) + \Phi_{10}\,(-2\,\overline{\alpha} + \overline{\pi}) + 3\,\Psi_2\,\rho + 2\,\Phi_{11}\,\overline{\rho} + \Phi_{20}\,\sigma - \text{þ}\,\Lambda - \text{þ}\,\Psi_2 + \text{ð}\,\Phi_{10} + \text{ð}'\,\Psi_1,$$
$$-2\,\Phi_{22}\,\epsilon - 2\,\Phi_{22}\,\overline{\epsilon} + \text{þ}\,\Phi_{22} == -2\,\Phi_{21}\,\beta + 2\,\Psi_3\,\beta - \Phi_{20}\,\overline{\lambda} - 2\,\Phi_{11}\,\mu + 3\,\Psi_2\,\mu - 2\,\Psi_1\,\nu + 2\,\Phi_{12}\,\pi + 2\,\Phi_{21}\,(\beta + \overline{\pi}) + \Phi_{22}\,(-2\,(\epsilon + \overline{\epsilon}) + \overline{\rho}) - \Psi_4\,\sigma + 2\,\Psi_3\,(-\beta + \tau) - 2\,(\text{þ}'\,\Lambda) + \text{þ}'\,\Psi_2 + \text{ð}\,\Phi_{21} - \text{ð}\,\Psi_3,$$
$$-4\,\Psi_4\,\epsilon + \text{þ}\,\Psi_4 == -2\,\Phi_{20}\,\gamma + 2\,\Phi_{20}\,\overline{\gamma} - 4\,\Psi_4\,\epsilon + 2\,\Phi_{11}\,\lambda - 3\,\Psi_2\,\lambda + \Phi_{20}\,(2\,\gamma - 2\,\overline{\gamma} + \overline{\mu}) - 2\,\Phi_{10}\,\nu + 4\,\Psi_3\,\pi + \Psi_4\,\rho - \Phi_{22}\,\overline{\sigma} + 2\,\Phi_{21}\,\overline{\tau} + \text{þ}'\,\Phi_{20} - \text{ð}'\,\Phi_{21} + \text{ð}'\,\Psi_3,\ 2\,\Phi_{00}\,\gamma + 2\,\Phi_{00}\,\overline{\gamma} + \text{þ}'\,\Phi_{00} == 2\,\Phi_{01}\,\alpha - 2\,\Psi_1\,\alpha + 2\,\Phi_{00}\,(\gamma + \overline{\gamma}) + 2\,\Psi_3\,\kappa + \Psi_0\,\lambda - \Phi_{00}\,\overline{\mu} + 2\,\Psi_1\,(\alpha - \pi) + 2\,\Phi_{11}\,\rho - 3\,\Psi_2\,\rho + \Phi_{02}\,\overline{\sigma} - 2\,\Phi_{10}\,\tau - 2\,\Phi_{01}\,(\alpha + \overline{\tau}) - 2\,(\text{þ}\,\Lambda) + \text{þ}\,\Psi_2 + \text{ð}'\,\Phi_{01} - \text{ð}'\,\Psi_1,$$
$$\text{þ}'\,\Phi_{11} == -\Phi_{02}\,\lambda - 3\,\Psi_2\,\mu - 2\,\Phi_{11}\,\overline{\mu} + \Phi_{01}\,\nu + 2\,\Psi_1\,\nu + \Phi_{10}\,\overline{\nu} + \Phi_{22}\,\rho + \Psi_4\,\sigma - \Phi_{21}\,\tau - 2\,\Psi_3\,\tau - \Phi_{12}\,\overline{\tau} - \text{þ}'\,\Lambda - \text{þ}'\,\Psi_2 + \text{ð}\,\Psi_3 + \text{ð}'\,\Phi_{12},\ -2\,\Phi_{21}\,\gamma + \text{þ}'\,\Phi_{21} == -2\,\Phi_{22}\,\alpha - 4\,\Psi_4\,\beta - 2\,\Phi_{22}\,\overline{\beta} + 2\,\Phi_{22}\,\left(\alpha + \overline{\beta}\right) + 2\,\Psi_3\,\gamma - 2\,\Phi_{12}\,\lambda - 2\,(\Psi_3\,(\gamma + 2\,\mu) + \Phi_{21}\,(\gamma + \overline{\mu})) + 2\,\Phi_{11}\,\nu + 3\,\Psi_2\,\nu + \Phi_{20}\,\overline{\nu} + \Psi_4\,(4\,\beta - \tau) - \Phi_{22}\,\overline{\tau} - \text{þ}'\,\Psi_3 + \text{ð}\,\Psi_4 + \text{ð}'\,\Phi_{22},$$
$$\text{ð}\,\Phi_{11} == \Phi_{22}\,\kappa + \Phi_{10}\,\overline{\lambda} + \Phi_{01}\,\mu - 2\,\Psi_1\,\mu + \Psi_0\,\nu - \Phi_{02}\,\pi - 2\,\Phi_{11}\,\overline{\pi} - \Phi_{12}\,\overline{\rho} - \Phi_{21}\,\sigma + 2\,\Psi_3\,\sigma - 3\,\Psi_2\,\tau + \text{þ}\,\Phi_{12} - \text{þ}'\,\Psi_1 + \text{ð}\,\Lambda + \text{ð}\,\Psi_2,$$
$$2\,\Phi_{02}\,\alpha - 2\,\Phi_{02}\,\overline{\beta} + \text{ð}'\,\Phi_{02} == 2\,\Psi_1\,\mu + 2\,\Phi_{01}\,\overline{\mu} - \Psi_0\,\nu - \Phi_{00}\,\overline{\nu} - 2\,\Phi_{12}\,\rho - 2\,\Psi_3\,\sigma + 2\,\Phi_{11}\,\tau + 3\,\Psi_2\,\tau + \Phi_{02}\,\left(2\,\alpha - 2\,\overline{\beta} + \overline{\tau}\right) + \text{þ}'\,\Phi_{01} + \text{þ}'\,\Psi_1 + 2\,(\text{ð}\,\Lambda) - \text{ð}\,\Psi_2,\ 2\,(\text{ð}'\,\Phi_{11}) == 2\,\Phi_{20}\,\overline{\alpha} - 2\,\Phi_{20}\,\beta + 2\,\Phi_{21}\,\epsilon - 6\,\Psi_3\,\epsilon - 3\,\Psi_4\,\kappa - \Phi_{22}\,\overline{\kappa} + 2\,\Phi_{01}\,\lambda - 6\,\Psi_1\,\lambda - 2\,\Phi_{10}\,(2\,\overline{\gamma} + \mu - \overline{\mu}) - 2\,\Phi_{00}\,\nu + 2\,\Phi_{11}\,\pi + 9\,\Psi_2\,\pi + 6\,\Psi_3\,\rho - 2\,\Phi_{21}\,(\epsilon + \rho - \overline{\rho}) - 2\,\Phi_{12}\,\overline{\sigma} + \Phi_{20}\,(-2\,\overline{\alpha} + 2\,\beta + \overline{\pi} + 2\,\tau) + 4\,\Phi_{11}\,\overline{\tau} - \text{þ}\,\Phi_{21} - 3\,(-2\,\Psi_3\,\epsilon + \text{þ}\,\Psi_3) + 2\,(2\,\Phi_{10}\,\overline{\gamma} + \text{þ}'\,\Phi_{10}) + \text{ð}\,\Phi_{20} + 3\,(\text{ð}'\,\Psi_2)\}$$

In[359]:=
```
Simplify //@ %
```

Out[359]=
$$\{2\,\Phi_{11}\,\kappa + 3\,\Psi_2\,\kappa + \Phi_{02}\,\overline{\kappa} + \text{þ}\,\Phi_{01} + \text{þ}\,\Psi_1 == \Psi_0\,\pi + \Phi_{00}\,\overline{\pi} + 4\,\Psi_1\,\rho + 2\,\Phi_{01}\,\overline{\rho} + 2\,\Phi_{10}\,\sigma + \text{ð}\,\Phi_{00} + \text{ð}'\,\Psi_0,$$
$$2\,\Phi_{12}\,\kappa + \Phi_{00}\,\overline{\lambda} + 3\,\Psi_2\,\sigma + \text{þ}\,\Phi_{02} + \text{ð}\,\Psi_1 == \Psi_0\,\mu + 2\,\Phi_{01}\,\overline{\pi} + \Phi_{02}\,\overline{\rho} + 2\,\Phi_{11}\,\sigma + 4\,\Psi_1\,\tau + \text{þ}'\,\Psi_0 + \text{ð}\,\Phi_{01},$$
$$\Phi_{21}\,\kappa + 2\,\Psi_3\,\kappa + \Phi_{12}\,\overline{\kappa} + \Psi_0\,\lambda + \Phi_{00}\,\mu + \text{þ}\,\Lambda + \text{þ}\,\Phi_{11} + \text{þ}\,\Psi_2 == \Phi_{01}\,\pi + 2\,\Psi_1\,\pi + \Phi_{10}\,\overline{\pi} + 3\,\Psi_2\,\rho + 2\,\Phi_{11}\,\overline{\rho} + \Phi_{20}\,\sigma + \text{ð}\,\Phi_{10} + \text{ð}'\,\Psi_1,$$
$$\Phi_{20}\,\overline{\lambda} + 2\,\Phi_{11}\,\mu + 2\,\Psi_1\,\nu + \Psi_4\,\sigma + \text{þ}\,\Phi_{22} + 2\,(\text{þ}'\,\Lambda) + \text{ð}\,\Psi_3 == 3\,\Psi_2\,\mu + 2\,\Phi_{12}\,\pi + 2\,\Phi_{21}\,\overline{\pi} + \Phi_{22}\,\overline{\rho} + 2\,\Psi_3\,\tau + \text{þ}'\,\Psi_2 + \text{ð}\,\Phi_{21},$$
$$2\,\Phi_{11}\,\lambda + \Phi_{20}\,\overline{\mu} + 4\,\Psi_3\,\pi + \Psi_4\,\rho + 2\,\Phi_{21}\,\overline{\tau} + \text{þ}'\,\Phi_{20} + \text{ð}'\,\Psi_3 == 3\,\Psi_2\,\lambda + 2\,\Phi_{10}\,\nu + \Phi_{22}\,\overline{\sigma} + \text{þ}\,\Psi_4 + \text{ð}'\,\Phi_{21},\ 2\,\Psi_3\,\kappa + \Psi_0\,\lambda + 2\,\Phi_{11}\,\rho + \Phi_{02}\,\overline{\sigma} + \text{þ}\,\Psi_2 + \text{ð}'\,\Phi_{01} == \Phi_{00}\,\overline{\mu} + 2\,\Psi_1\,\pi + 3\,\Psi_2\,\rho + 2\,\Phi_{10}\,\tau + 2\,\Phi_{01}\,\overline{\tau} + 2\,(\text{þ}\,\Lambda) + \text{þ}'\,\Phi_{00} + \text{ð}'\,\Psi_1,$$
$$\Phi_{02}\,\lambda + 3\,\Psi_2\,\mu + 2\,\Phi_{11}\,\overline{\mu} + \Phi_{21}\,\tau + 2\,\Psi_3\,\tau + \Phi_{12}\,\overline{\tau} + \text{þ}'\,\Lambda + \text{þ}'\,\Phi_{11} + \text{þ}'\,\Psi_2 == \Phi_{01}\,\nu + 2\,\Psi_1\,\nu + \Phi_{10}\,\overline{\nu} + \Phi_{22}\,\rho + \Psi_4\,\sigma + \text{ð}\,\Psi_3 + \text{ð}'\,\Phi_{12},$$
$$2\,\Phi_{12}\,\lambda + 4\,\Psi_3\,\mu + 2\,\Phi_{21}\,\overline{\mu} + \Psi_4\,\tau + \Phi_{22}\,\overline{\tau} + \text{þ}'\,\Phi_{21} + \text{þ}'\,\Psi_3 == 2\,\Phi_{11}\,\nu + 3\,\Psi_2\,\nu + \Phi_{20}\,\overline{\nu} + \text{ð}\,\Psi_4 + \text{ð}'\,\Phi_{22},$$
$$\Phi_{22}\,\kappa + \Phi_{10}\,\overline{\lambda} + \Phi_{01}\,\mu + \Psi_0\,\nu + 2\,\Psi_3\,\sigma + \text{þ}\,\Phi_{12} + \text{ð}\,\Lambda + \text{ð}\,\Psi_2 == 2\,\Psi_1\,\mu + \Phi_{02}\,\pi + 2\,\Phi_{11}\,\overline{\pi} + \Phi_{12}\,\overline{\rho} + \Phi_{21}\,\sigma + 3\,\Psi_2\,\tau + \text{þ}'\,\Psi_1 + \text{ð}\,\Phi_{11},$$
$$2\,\Psi_1\,\mu + 2\,\Phi_{01}\,\overline{\mu} + 2\,\Phi_{11}\,\tau + 3\,\Psi_2\,\tau + \Phi_{02}\,\overline{\tau} + \text{þ}'\,\Phi_{01} + \text{þ}'\,\Psi_1 + 2\,(\text{ð}\,\Lambda) == \Psi_0\,\nu + \Phi_{00}\,\overline{\nu} + 2\,\Phi_{12}\,\rho + 2\,\Psi_3\,\sigma + \text{ð}\,\Psi_2 + \text{ð}'\,\Phi_{02},$$
$$3\,\Psi_4\,\kappa + \Phi_{22}\,\overline{\kappa} + 6\,\Psi_1\,\lambda + 2\,\Phi_{10}\,\mu + 2\,\Phi_{00}\,\nu + 2\,\Phi_{21}\,\rho + 2\,\Phi_{12}\,\overline{\sigma} + \text{þ}\,\Phi_{21} + 3\,(\text{þ}\,\Psi_3) + 2\,(\text{ð}'\,\Phi_{11}) == 2\,\Phi_{01}\,\lambda + 2\,\Phi_{10}\,\overline{\mu} + 2\,\Phi_{11}\,\pi + 9\,\Psi_2\,\pi + \Phi_{20}\,\overline{\pi} + 6\,\Psi_3\,\rho + 2\,\Phi_{21}\,\overline{\rho} + 2\,\Phi_{20}\,\tau + 4\,\Phi_{11}\,\overline{\tau} + 2\,(\text{þ}'\,\Phi_{10}) + \text{ð}\,\Phi_{20} + 3\,(\text{ð}'\,\Psi_2)\}$$

We may write these equations in an alternative way.

In[360]:=
```
Solve[%, Union@Cases[%, θ[__]@_, Infinity]] // Flatten
```

```
Solve::svars : Equations may not give solutions for all "solve" variables. >>
```

Out[360]=

$$\{ \text{þ}\Lambda \to \frac{1}{2}(2\Psi_3\ \kappa + \Psi_0\ \lambda - \Phi_{00}\ \bar{\mu} - 2\Psi_1\ \pi + 2\Phi_{11}\ \rho - 3\Psi_2\ \rho + \Phi_{02}\ \bar{\sigma} - 2\Phi_{10}\ \tau - 2\Phi_{01}\ \bar{\tau}) +$$
$$\frac{1}{2}(\text{þ}\Psi_2) - \frac{1}{2}(\text{þ}'\Phi_{00}) + \frac{1}{2}(\text{ð}'\Phi_{01}) - \frac{1}{2}(\text{ð}'\Psi_1),\ \text{þ}\Phi_{01} \to -2\Phi_{11}\ \kappa - 3\Psi_2\ \kappa -$$
$$\Phi_{02}\ \bar{\kappa} + \Psi_0\ \pi + \Phi_{00}\ \bar{\pi} + 4\Psi_1\ \rho + 2\Phi_{01}\ \bar{\rho} + 2\Phi_{10}\ \sigma - \text{þ}\Psi_1 + \text{ð}\Phi_{00} + \text{ð}'\Psi_0,\ \text{þ}\Phi_{02} \to$$
$$-2\Phi_{12}\ \kappa - \Phi_{00}\ \bar{\lambda} + \Psi_0\ \mu + 2\Phi_{01}\ \bar{\pi} + \Phi_{02}\ \bar{\rho} + 2\Phi_{11}\ \sigma - 3\Psi_2\ \sigma + 4\Psi_1\ \tau + \text{þ}'\Psi_0 + \text{ð}\Phi_{01} - \text{ð}\Psi_1,$$
$$\text{þ}\Phi_{11} \to \frac{1}{2}(-2\Phi_{21}\ \kappa - 6\Psi_3\ \kappa - 2\Phi_{12}\ \bar{\kappa} - 3\Psi_0\ \lambda - 2\Phi_{00}\ \mu + \Phi_{00}\ \bar{\mu} + 2\Phi_{01}\ \pi + 6\Psi_1\ \pi +$$
$$2\Phi_{10}\ \bar{\pi} - 2\Phi_{11}\ \rho + 9\Psi_2\ \rho + 4\Phi_{11}\ \bar{\rho} + 2\Phi_{20}\ \sigma - \Phi_{02}\ \bar{\sigma} + 2\Phi_{10}\ \tau + 2\Phi_{01}\ \bar{\tau}) -$$
$$\frac{3}{2}(\text{þ}\Psi_2) + \frac{1}{2}(\text{þ}'\Phi_{00}) + \text{ð}\Phi_{10} - \frac{1}{2}(\text{ð}'\Phi_{01}) + \frac{3}{2}(\text{ð}'\Psi_1),$$
$$\text{þ}\Phi_{12} \to -\Phi_{22}\ \kappa - \Phi_{10}\ \bar{\lambda} - \Phi_{01}\ \mu + 2\Psi_1\ \mu - \Psi_0\ \nu + \Phi_{02}\ \pi + 2\Phi_{11}\ \bar{\pi} +$$
$$\Phi_{12}\ \bar{\rho} + \Phi_{21}\ \sigma - 2\Psi_3\ \sigma + 3\Psi_2\ \tau + \text{þ}'\Psi_1 - \text{ð}\Lambda + \text{ð}\Phi_{11} - \text{ð}\Psi_2,$$
$$\text{þ}\Phi_{21} \to -3\Psi_4\ \kappa - \Phi_{22}\ \bar{\kappa} + 2\Phi_{01}\ \lambda - 6\Psi_1\ \lambda - 2\Phi_{10}\ \mu + 2\Phi_{10}\ \bar{\mu} - 2\Phi_{00}\ \nu +$$
$$2\Phi_{11}\ \pi + 9\Psi_2\ \pi + \Phi_{20}\ \bar{\pi} - 2\Phi_{21}\ \rho + 6\Psi_3\ \rho + 2\Phi_{21}\ \bar{\rho} - 2\Phi_{12}\ \bar{\sigma} +$$
$$2\Phi_{20}\ \tau + 4\Phi_{11}\ \bar{\tau} - 3(\text{þ}\Psi_3) + 2(\text{þ}'\Phi_{10}) + \text{ð}\Phi_{20} - 2(\text{ð}'\Phi_{11}) + 3(\text{ð}'\Psi_2),$$
$$\text{þ}\Phi_{22} \to 2\Phi_{02}\ \lambda - \Phi_{20}\ \bar{\lambda} - 2\Phi_{11}\ \mu + 9\Psi_2\ \mu + 4\Phi_{11}\ \bar{\mu} - 2\Phi_{01}\ \nu - 6\Psi_1\ \nu -$$
$$2\Phi_{10}\ \bar{\nu} + 2\Phi_{12}\ \pi + 2\Phi_{21}\ \bar{\pi} - 2\Phi_{22}\ \rho + \Phi_{22}\ \bar{\rho} - 3\Psi_4\ \sigma + 2\Phi_{21}\ \tau + 6\Psi_3\ \tau +$$
$$2\Phi_{12}\ \bar{\tau} + 2(\text{þ}'\Phi_{11}) + 3(\text{þ}'\Psi_2) + \text{ð}\Phi_{21} - 3(\text{ð}\Psi_3) - 2(\text{ð}'\Phi_{12}),\ \text{þ}\Psi_4 \to$$
$$2\Phi_{11}\ \lambda - 3\Psi_2\ \lambda + \Phi_{20}\ \bar{\mu} - 2\Phi_{10}\ \nu + 4\Psi_3\ \pi + \Psi_4\ \rho - \Phi_{22}\ \bar{\sigma} + 2\Phi_{21}\ \bar{\tau} + \text{þ}'\Phi_{20} - \text{ð}'\Phi_{21} + \text{ð}'\Psi_3,$$
$$\text{þ}'\Lambda \to -\Phi_{02}\ \lambda - 3\Psi_2\ \mu - 2\Phi_{11}\ \bar{\mu} + \Phi_{01}\ \nu + 2\Psi_1\ \nu + \Phi_{10}\ \bar{\nu} + \Phi_{22}\ \rho +$$
$$\Psi_4\ \sigma - \Phi_{21}\ \tau - 2\Psi_3\ \tau - \Phi_{12}\ \bar{\tau} - \text{þ}'\Phi_{11} - \text{þ}'\Psi_2 + \text{ð}\Psi_3 + \text{ð}'\Phi_{12},$$
$$\text{þ}'\Phi_{01} \to -2\Psi_1\ \mu - 2\Phi_{01}\ \bar{\mu} + \Psi_0\ \nu + \Phi_{00}\ \bar{\nu} + 2\Phi_{12}\ \rho + 2\Psi_3\ \sigma - 2\Phi_{11}\ \tau -$$
$$3\Psi_2\ \tau - \Phi_{02}\ \bar{\tau} - \text{þ}'\Psi_1 - 2(\text{ð}\Lambda) + \text{ð}\Psi_2 + \text{ð}'\Phi_{02},\ \text{þ}'\Phi_{21} \to$$
$$-2\Phi_{12}\ \lambda - 4\Psi_3\ \mu - 2\Phi_{21}\ \bar{\mu} + 2\Phi_{11}\ \nu + 3\Psi_2\ \nu + \Phi_{20}\ \bar{\nu} - \Psi_4\ \tau - \Phi_{22}\ \bar{\tau} - \text{þ}'\Psi_3 + \text{ð}\Psi_4 + \text{ð}'\Phi_{22}\ \}$$

In[361]:= **FullSimplify //@ %**

Out[361]=
$$\{\text{þ}\,\Lambda \to \frac{1}{2}(2\,\Psi_3\,\kappa + \Psi_0\,\lambda - \Phi_{00}\,\overline{\mu} - 2\,\Psi_1\,\pi + 2\,\Phi_{11}\,\rho - 3\,\Psi_2\,\rho + \Phi_{02}\,\overline{\sigma} - 2\,\Phi_{10}\,\tau - 2\,\Phi_{01}\,\overline{\tau} + \text{þ}\,\Psi_2 - \text{þ}'\,\Phi_{00} + \text{ð}'\,\Phi_{01} - \text{ð}'\,\Psi_1),\ \text{þ}\,\Phi_{01} \to -2\,\Phi_{11}\,\kappa - 3\,\Psi_2\,\kappa - \Phi_{02}\,\overline{\kappa} + \Psi_0\,\pi + \Phi_{00}\,\overline{\pi} + 4\,\Psi_1\,\rho + 2\,\Phi_{01}\,\overline{\rho} + 2\,\Phi_{10}\,\sigma - \text{þ}\,\Psi_1 + \text{ð}\,\Phi_{00} + \text{ð}'\,\Psi_0,$$
$$\text{þ}\,\Phi_{02} \to -2\,\Phi_{12}\,\kappa - \Phi_{00}\,\overline{\lambda} + \Psi_0\,\mu + 2\,\Phi_{01}\,\overline{\pi} + \Phi_{02}\,\overline{\rho} + 2\,\Phi_{11}\,\sigma - 3\,\Psi_2\,\sigma + 4\,\Psi_1\,\tau + \text{þ}'\,\Psi_0 + \text{ð}\,\Phi_{01} - \text{ð}\,\Psi_1,$$
$$\text{þ}\,\Phi_{11} \to \frac{1}{2}(-2\,(\Phi_{21} + 3\,\Psi_3)\,\kappa - 2\,\Phi_{12}\,\overline{\kappa} - 3\,\Psi_0\,\lambda + \Phi_{00}\,(-2\,\mu + \overline{\mu}) + 6\,\Psi_1\,\pi - 2\,\Phi_{11}\,\rho + 9\,\Psi_2\,\rho + 4\,\Phi_{11}\,\overline{\rho} + 2\,\Phi_{20}\,\sigma - \Phi_{02}\,\overline{\sigma} + 2\,\Phi_{10}\,(\overline{\pi} + \tau) + 2\,\Phi_{01}\,(\pi + \overline{\tau}) - 3\,(\text{þ}\,\Psi_2) + \text{þ}'\,\Phi_{00} + 2\,(\text{ð}\,\Phi_{10}) - \text{ð}'\,\Phi_{01} + 3\,(\text{ð}'\,\Psi_1)),$$
$$\text{þ}\,\Phi_{12} \to -\Phi_{22}\,\kappa - \Phi_{10}\,\overline{\lambda} - \Phi_{01}\,\mu + 2\,\Psi_1\,\mu - \Psi_0\,\nu + \Phi_{02}\,\pi + 2\,\Phi_{11}\,\overline{\pi} + \Phi_{12}\,\overline{\rho} + \Phi_{21}\,\sigma - 2\,\Psi_3\,\sigma + 3\,\Psi_2\,\tau + \text{þ}'\,\Psi_1 - \text{ð}\,\Lambda + \text{ð}\,\Phi_{11} - \text{ð}\,\Psi_2,$$
$$\text{þ}\,\Phi_{21} \to -3\,\Psi_4\,\kappa - \Phi_{22}\,\overline{\kappa} + 2\,\Phi_{01}\,\lambda - 6\,\Psi_1\,\lambda - 2\,\Phi_{10}\,\mu + 2\,\Phi_{10}\,\overline{\mu} - 2\,\Phi_{00}\,\nu + 2\,\Phi_{11}\,\pi + 9\,\Psi_2\,\pi + \Phi_{20}\,\overline{\pi} - 2\,\Phi_{21}\,\rho + 6\,\Psi_3\,\rho + 2\,\Phi_{21}\,\overline{\rho} - 2\,\Phi_{12}\,\overline{\sigma} + 2\,\Phi_{20}\,\tau + 4\,\Phi_{11}\,\overline{\tau} - 3\,(\text{þ}\,\Psi_3) + 2\,(\text{þ}'\,\Phi_{10}) + \text{ð}\,\Phi_{20} - 2\,(\text{ð}'\,\Phi_{11}) + 3\,(\text{ð}'\,\Psi_2),$$
$$\text{þ}\,\Phi_{22} \to 2\,\Phi_{02}\,\lambda - \Phi_{20}\,\overline{\lambda} - 2\,\Phi_{11}\,\mu + 9\,\Psi_2\,\mu + 4\,\Phi_{11}\,\overline{\mu} - 2\,(\Phi_{01} + 3\,\Psi_1)\,\nu - 2\,\Phi_{10}\,\overline{\nu} - 2\,\Phi_{22}\,\rho + \Phi_{22}\,\overline{\rho} - 3\,\Psi_4\,\sigma + 6\,\Psi_3\,\tau + 2\,\Phi_{21}\,(\overline{\pi} + \tau) + 2\,\Phi_{12}\,(\pi + \overline{\tau}) + 2\,(\text{þ}'\,\Phi_{11}) + 3\,(\text{þ}'\,\Psi_2) + \text{ð}\,\Phi_{21} - 3\,(\text{ð}\,\Psi_3) - 2\,(\text{ð}'\,\Phi_{12}),\ \text{þ}\,\Psi_4 \to 2\,\Phi_{11}\,\lambda - 3\,\Psi_2\,\lambda + \Phi_{20}\,\overline{\mu} - 2\,\Phi_{10}\,\nu + 4\,\Psi_3\,\pi + \Psi_4\,\rho - \Phi_{22}\,\overline{\sigma} + 2\,\Phi_{21}\,\overline{\tau} + \text{þ}'\,\Phi_{20} - \text{ð}'\,\Phi_{21} + \text{ð}'\,\Psi_3,$$
$$\text{þ}'\,\Lambda \to -\Phi_{02}\,\lambda - 3\,\Psi_2\,\mu - 2\,\Phi_{11}\,\overline{\mu} + (\Phi_{01} + 2\,\Psi_1)\,\nu + \Phi_{10}\,\overline{\nu} + \Phi_{22}\,\rho + \Psi_4\,\sigma - (\Phi_{21} + 2\,\Psi_3)\,\tau - \Phi_{12}\,\overline{\tau} - \text{þ}'\,\Phi_{11} - \text{þ}'\,\Psi_2 + \text{ð}\,\Psi_3 + \text{ð}'\,\Phi_{12},$$
$$\text{þ}'\,\Phi_{01} \to -2\,\Psi_1\,\mu - 2\,\Phi_{01}\,\overline{\mu} + \Psi_0\,\nu + \Phi_{00}\,\overline{\nu} + 2\,\Phi_{12}\,\rho + 2\,\Psi_3\,\sigma - 2\,\Phi_{11}\,\tau - 3\,\Psi_2\,\tau - \Phi_{02}\,\overline{\tau} - \text{þ}'\,\Psi_1 - 2\,(\text{ð}\,\Lambda) + \text{ð}\,\Psi_2 + \text{ð}'\,\Phi_{02},\ \text{þ}'\,\Phi_{21} \to -2\,\Phi_{12}\,\lambda - 4\,\Psi_3\,\mu - 2\,\Phi_{21}\,\overline{\mu} + 2\,\Phi_{11}\,\nu + 3\,\Psi_2\,\nu + \Phi_{20}\,\overline{\nu} - \Psi_4\,\tau - \Phi_{22}\,\overline{\tau} - \text{þ}'\,\Psi_3 + \text{ð}\,\Psi_4 + \text{ð}'\,\Phi_{22}\}$$

In[362]:= **% /. Rule -> Equal**

Out[362]=
$$\{\text{þ}\,\Lambda == \frac{1}{2}(2\,\Psi_3\,\kappa + \Psi_0\,\lambda - \Phi_{00}\,\overline{\mu} - 2\,\Psi_1\,\pi + 2\,\Phi_{11}\,\rho - 3\,\Psi_2\,\rho + \Phi_{02}\,\overline{\sigma} - 2\,\Phi_{10}\,\tau - 2\,\Phi_{01}\,\overline{\tau} + \text{þ}\,\Psi_2 - \text{þ}'\,\Phi_{00} + \text{ð}'\,\Phi_{01} - \text{ð}'\,\Psi_1),\ \text{þ}\,\Phi_{01} == -2\,\Phi_{11}\,\kappa - 3\,\Psi_2\,\kappa - \Phi_{02}\,\overline{\kappa} + \Psi_0\,\pi + \Phi_{00}\,\overline{\pi} + 4\,\Psi_1\,\rho + 2\,\Phi_{01}\,\overline{\rho} + 2\,\Phi_{10}\,\sigma - \text{þ}\,\Psi_1 + \text{ð}\,\Phi_{00} + \text{ð}'\,\Psi_0,$$
$$\text{þ}\,\Phi_{02} == -2\,\Phi_{12}\,\kappa - \Phi_{00}\,\overline{\lambda} + \Psi_0\,\mu + 2\,\Phi_{01}\,\overline{\pi} + \Phi_{02}\,\overline{\rho} + 2\,\Phi_{11}\,\sigma - 3\,\Psi_2\,\sigma + 4\,\Psi_1\,\tau + \text{þ}'\,\Psi_0 + \text{ð}\,\Phi_{01} - \text{ð}\,\Psi_1,$$
$$\text{þ}\,\Phi_{11} == \frac{1}{2}(-2\,(\Phi_{21} + 3\,\Psi_3)\,\kappa - 2\,\Phi_{12}\,\overline{\kappa} - 3\,\Psi_0\,\lambda + \Phi_{00}\,(-2\,\mu + \overline{\mu}) + 6\,\Psi_1\,\pi - 2\,\Phi_{11}\,\rho + 9\,\Psi_2\,\rho + 4\,\Phi_{11}\,\overline{\rho} + 2\,\Phi_{20}\,\sigma - \Phi_{02}\,\overline{\sigma} + 2\,\Phi_{10}\,(\overline{\pi} + \tau) + 2\,\Phi_{01}\,(\pi + \overline{\tau}) - 3\,(\text{þ}\,\Psi_2) + \text{þ}'\,\Phi_{00} + 2\,(\text{ð}\,\Phi_{10}) - \text{ð}'\,\Phi_{01} + 3\,(\text{ð}'\,\Psi_1)),$$
$$\text{þ}\,\Phi_{12} == -\Phi_{22}\,\kappa - \Phi_{10}\,\overline{\lambda} - \Phi_{01}\,\mu + 2\,\Psi_1\,\mu - \Psi_0\,\nu + \Phi_{02}\,\pi + 2\,\Phi_{11}\,\overline{\pi} + \Phi_{12}\,\overline{\rho} + \Phi_{21}\,\sigma - 2\,\Psi_3\,\sigma + 3\,\Psi_2\,\tau + \text{þ}'\,\Psi_1 - \text{ð}\,\Lambda + \text{ð}\,\Phi_{11} - \text{ð}\,\Psi_2,$$
$$\text{þ}\,\Phi_{21} == -3\,\Psi_4\,\kappa - \Phi_{22}\,\overline{\kappa} + 2\,\Phi_{01}\,\lambda - 6\,\Psi_1\,\lambda - 2\,\Phi_{10}\,\mu + 2\,\Phi_{10}\,\overline{\mu} - 2\,\Phi_{00}\,\nu + 2\,\Phi_{11}\,\pi + 9\,\Psi_2\,\pi + \Phi_{20}\,\overline{\pi} - 2\,\Phi_{21}\,\rho + 6\,\Psi_3\,\rho + 2\,\Phi_{21}\,\overline{\rho} - 2\,\Phi_{12}\,\overline{\sigma} + 2\,\Phi_{20}\,\tau + 4\,\Phi_{11}\,\overline{\tau} - 3\,(\text{þ}\,\Psi_3) + 2\,(\text{þ}'\,\Phi_{10}) + \text{ð}\,\Phi_{20} - 2\,(\text{ð}'\,\Phi_{11}) + 3\,(\text{ð}'\,\Psi_2),$$
$$\text{þ}\,\Phi_{22} == 2\,\Phi_{02}\,\lambda - \Phi_{20}\,\overline{\lambda} - 2\,\Phi_{11}\,\mu + 9\,\Psi_2\,\mu + 4\,\Phi_{11}\,\overline{\mu} - 2\,(\Phi_{01} + 3\,\Psi_1)\,\nu - 2\,\Phi_{10}\,\overline{\nu} - 2\,\Phi_{22}\,\rho + \Phi_{22}\,\overline{\rho} - 3\,\Psi_4\,\sigma + 6\,\Psi_3\,\tau + 2\,\Phi_{21}\,(\overline{\pi} + \tau) + 2\,\Phi_{12}\,(\pi + \overline{\tau}) + 2\,(\text{þ}'\,\Phi_{11}) + 3\,(\text{þ}'\,\Psi_2) + \text{ð}\,\Phi_{21} - 3\,(\text{ð}\,\Psi_3) - 2\,(\text{ð}'\,\Phi_{12}),\ \text{þ}\,\Psi_4 == 2\,\Phi_{11}\,\lambda - 3\,\Psi_2\,\lambda + \Phi_{20}\,\overline{\mu} - 2\,\Phi_{10}\,\nu + 4\,\Psi_3\,\pi + \Psi_4\,\rho - \Phi_{22}\,\overline{\sigma} + 2\,\Phi_{21}\,\overline{\tau} + \text{þ}'\,\Phi_{20} - \text{ð}'\,\Phi_{21} + \text{ð}'\,\Psi_3,$$
$$\text{þ}'\,\Lambda == -\Phi_{02}\,\lambda - 3\,\Psi_2\,\mu - 2\,\Phi_{11}\,\overline{\mu} + (\Phi_{01} + 2\,\Psi_1)\,\nu + \Phi_{10}\,\overline{\nu} + \Phi_{22}\,\rho + \Psi_4\,\sigma - (\Phi_{21} + 2\,\Psi_3)\,\tau - \Phi_{12}\,\overline{\tau} - \text{þ}'\,\Phi_{11} - \text{þ}'\,\Psi_2 + \text{ð}\,\Psi_3 + \text{ð}'\,\Phi_{12},$$
$$\text{þ}'\,\Phi_{01} == -2\,\Psi_1\,\mu - 2\,\Phi_{01}\,\overline{\mu} + \Psi_0\,\nu + \Phi_{00}\,\overline{\nu} + 2\,\Phi_{12}\,\rho + 2\,\Psi_3\,\sigma - 2\,\Phi_{11}\,\tau - 3\,\Psi_2\,\tau - \Phi_{02}\,\overline{\tau} - \text{þ}'\,\Psi_1 - 2\,(\text{ð}\,\Lambda) + \text{ð}\,\Psi_2 + \text{ð}'\,\Phi_{02},\ \text{þ}'\,\Phi_{21} == -2\,\Phi_{12}\,\lambda - 4\,\Psi_3\,\mu - 2\,\Phi_{21}\,\overline{\mu} + 2\,\Phi_{11}\,\nu + 3\,\Psi_2\,\nu + \Phi_{20}\,\overline{\nu} - \Psi_4\,\tau - \Phi_{22}\,\overline{\tau} - \text{þ}'\,\Psi_3 + \text{ð}\,\Psi_4 + \text{ð}'\,\Phi_{22}\}$$

Previous set of equations is known as GHP Bianchi identities or compacted Bianchi Identities.

In[363]:=
```
Length[GHPBianchi = %]
```

Out[363]= 11

■ 4. Quick start

In this section we show how to use in practical applications the NP and GHP equations computed in previous sections. This is intended to be a short tutorial which enables the user to start working as quick as possible with the equations of the NP and GHP formalisms, so one does not need to read the contents of previous sections to use it. However, to take full advantage of the flexibility offered by our implementation and the remaining features of `xAct`, the complete knowledge of sections 1,2 and 3 is needed

The basic variables of the NP and GHP formalisms are the spin coefficients and the components of the curvature tensor. These are stored in lists.

In[364]:=
```
NPSpinCoefficients
```

Out[364]= $\{\alpha, \beta, \gamma, \epsilon, \kappa, \lambda, \mu, \nu, \pi, \rho, \sigma, \tau\}$

In[365]:=
```
NPCVComponents
```

Out[365]= $\{\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Phi_{00}, \Phi_{01}, \Phi_{02}, \Phi_{10}, \Phi_{11}, \Phi_{12}, \Phi_{20}, \Phi_{21}, \Phi_{22}, \Lambda\}$

To type any of these quantities you need to know the imput form of them. This is achieved with the *Mathematica* com–mand `InputForm`. For example, let us display the input form of the spin coefficient γ :

In[366]:=
```
NPSpinCoefficients[[3]] // InputForm
```

Out[366]//InputForm=
```
γDyad[]
```

Recall that Greek letters are typed by using the key "esc" followed by the Latin name of the letter followed by "esc". Hence

In[367]:=
```
γ Dyad[]
```

Out[367]= γ

Other essential ingredient of the NP and GHP formalisms are the *frame differentiations.* In the NP formalism these are just the directional derivatives associated to the four elements of the null tetrad. To see how they are typed, we consider their action on the spin coefficient γ .

In[368]:=
```
{PDNP[{1, -NP}]@γ Dyad[], PDNP[{2, -NP}]@γ Dyad[],
 PDNP[{3, -NP}]@γ Dyad[], PDNP[{4, -NP}]@γ Dyad[]}
```

Out[368]= $\{D\gamma, \Delta\gamma, \delta\gamma, \overline{\delta}\gamma\}$

The head `PDNP` denotes a generic NP operator associated to the null tetrad, whereas the arguments `{1,-NP}`, ... `{4,-NP}` are used to refer to each of the directional derivatives of the null tetrad elements. In this way `{1,-NP}` means the first element of the null tetrad, `{2,-NP}` the second element and so on. The elements of the null tetrad are numbered according to the usual convention. In the case of the GHP formalism, we have to take into account that only weighted quantities appear in the formalism. All the components of the Riemann tensor are weighted, but not all the spin coefficients are so. The weighted spin coefficients are stored in the following list.

In[369]:=
```
GHPSpinCoefficients
```

Out[369]= $\{\kappa, \lambda, \mu, \nu, \pi, \rho, \sigma, \tau\}$

Let us take κ : the action of the four GHP operators on it is

```
In[370]:= {Θ[{1, -NP}]@κ Dyad[], Θ[{2, -NP}]@κ Dyad[], Θ[{3, -NP}]@κ Dyad[], Θ[{4, -NP}]@κ Dyad[]}
Out[370]= {þ κ , þ′ κ , ð κ , ð′ κ }
```

In this case, the head Θ (typed esc–Q–esc) is the notation for a generic GHP differential operator while

{1,-NP},..,{4,-NP} are, as before, the labels for each of the elements of the null tetrad. We can transform the GHP differentials into NP differentials as follows.

```
In[371]:= % /. GHPToNP
Out[371]= {(-3 ϵ - ϵ̄ ) κ + D κ , (-3 γ - γ̄ ) κ + Δ κ , (-ᾱ - 3 β ) κ + δ κ , (-3 α - β̄ ) κ + δ̄ κ }
```

NP differentials can be also transformed back into GHP differentials.

```
In[372]:= % /. NPToGHP
Out[372]= {3 ϵ κ + (-3 ϵ - ϵ̄ ) κ + ϵ̄ κ + þ κ , 3 γ κ + (-3 γ - γ̄ ) κ + γ̄ κ + þ′ κ ,
           ᾱ κ + (-ᾱ - 3 β ) κ + 3 β κ + ð κ , 3 α κ + (-3 α - β̄ ) κ + β̄ κ + ð′ κ }

In[373]:= % // Simplify
Out[373]= {þ κ , þ′ κ , ð κ , ð′ κ }
```

Previous operations can be performed on any weighted quantity and the appropriate weights will be automatically inserted. For example:

```
In[374]:= {Θ[{1, -NP}]@PsiCDe4Dyad[], Θ[{2, -NP}]@PsiCDe4Dyad[],
           Θ[{3, -NP}]@PsiCDe4Dyad[], Θ[{4, -NP}]@PsiCDe4Dyad[]}
Out[374]= {þ Ψ4 , þ′ Ψ4 , ð Ψ4 , ð′ Ψ4 }

In[375]:= % //. GHPToNP
Out[375]= {4 Ψ4 ϵ + D Ψ4 , 4 Ψ4 γ + Δ Ψ4 , 4 Ψ4 β + δ Ψ4 , 4 Ψ4 α + δ̄ Ψ4 }
```

The basic equations of each formalism are stored in lists. These are the names of the lists and the correspondences:

Newman–Penrose commutation relations:

```
In[376]:= NPCommutators
Out[376]= {D Δ ψ == -(γ + γ̄ ) (D ψ ) - (ϵ + ϵ̄ ) (Δ ψ ) + Δ D ψ + (π + τ̄ ) (δ ψ ) + (π̄ + τ ) (δ̄ ψ ),
           D δ ψ == -(ᾱ + β - π̄ ) (D ψ ) - κ (Δ ψ ) + (ϵ - ϵ̄ + ρ̄ ) (δ ψ ) + δ D ψ + σ (δ̄ ψ ),
           Δ δ ψ == ν̄ (D ψ ) + (ᾱ + β - τ ) (Δ ψ ) + (γ - γ̄ - μ ) (δ ψ ) + δ Δ ψ - λ̄ (δ̄ ψ ),
           δ δ̄ ψ == (μ - μ̄ ) (D ψ ) + (ρ - ρ̄ ) (Δ ψ ) + (-α + β̄ ) (δ ψ ) + (ᾱ - β ) (δ̄ ψ ) + δ̄ δ ψ }
```

Newman–Penrose Ricci identities:

In[377]:= **NPRicci**

Out[377]=
$$\begin{aligned}
&\{D\alpha == -\Phi_{10} - 2\alpha\,\epsilon - \overline{\beta}\,\epsilon + \alpha\,\overline{\epsilon} - \gamma\,\overline{\kappa} - \kappa\,\lambda + \epsilon\,\pi + \alpha\,\rho + \pi\,\rho + \beta\,\overline{\sigma} + \overline{\delta}\epsilon,\\
&D\beta == \Psi_1 - \overline{\alpha}\,\epsilon - \beta\,\overline{\epsilon} - \gamma\,\kappa - \kappa\,\mu + \epsilon\,\overline{\pi} + \beta\,\overline{\rho} + \alpha\,\sigma + \pi\,\sigma + \delta\epsilon,\\
&D\gamma == \Lambda - \Phi_{11} + \Psi_2 - 2\gamma\,\epsilon - \overline{\gamma}\,\epsilon - \gamma\,\overline{\epsilon} - \kappa\,\nu + \beta\,\pi + \alpha\,\overline{\pi} + \alpha\,\tau + \pi\,\tau + \beta\,\overline{\tau} + \Delta\epsilon,\\
&D\lambda == -\Phi_{20} - 3\epsilon\,\lambda + \overline{\epsilon}\,\lambda - \overline{\kappa}\,\nu + \alpha\,\pi - \overline{\beta}\,\pi + \pi^2 + \lambda\,\rho + \mu\,\overline{\sigma} + \overline{\delta}\pi,\\
&D\mu == -2\Lambda + \Psi_2 - \epsilon\,\mu - \overline{\epsilon}\,\mu - \kappa\,\nu - \overline{\alpha}\,\pi + \beta\,\pi + \pi\,\overline{\pi} + \mu\,\overline{\rho} + \lambda\,\sigma + \delta\pi,\\
&D\nu == -\Phi_{21} + \Psi_3 - 3\epsilon\,\nu - \overline{\epsilon}\,\nu + \gamma\,\pi - \overline{\gamma}\,\pi + \mu\,\pi + \lambda\,\overline{\pi} + \lambda\,\tau + \mu\,\overline{\tau} + \Delta\pi,\\
&D\rho == -\Phi_{00} - 3\alpha\,\kappa - \overline{\beta}\,\kappa + \kappa\,\pi + \epsilon\,\rho + \overline{\epsilon}\,\rho + \rho^2 + \sigma\,\overline{\sigma} - \overline{\kappa}\,\tau + \overline{\delta}\kappa,\\
&D\sigma == \Psi_0 - \overline{\alpha}\,\kappa - 3\beta\,\kappa + \kappa\,\overline{\pi} + 3\epsilon\,\sigma - \overline{\epsilon}\,\sigma + \rho\,\sigma + \overline{\rho}\,\sigma - \kappa\,\tau + \delta\kappa,\\
&D\tau == -\Phi_{01} + \Psi_1 - 3\gamma\,\kappa - \overline{\gamma}\,\kappa + \overline{\pi}\,\rho + \pi\,\sigma + \epsilon\,\tau - \overline{\epsilon}\,\tau + \rho\,\tau + \sigma\,\overline{\tau} + \Delta\kappa,\\
&\Delta\alpha == -\Psi_3 + \overline{\beta}\,\gamma + \alpha\,\overline{\gamma} - \beta\,\lambda - \alpha\,\overline{\mu} + \epsilon\,\nu + \nu\,\rho - \lambda\,\tau - \gamma\,\overline{\tau} + \overline{\delta}\gamma,\\
&\Delta\beta == \Phi_{12} + \overline{\alpha}\,\gamma + 2\beta\,\gamma - \beta\,\overline{\gamma} - \alpha\,\overline{\lambda} - \beta\,\mu + \epsilon\,\overline{\nu} + \nu\,\sigma - \gamma\,\tau - \mu\,\tau + \delta\gamma,\\
&\Delta\lambda == -\Psi_4 - 3\gamma\,\lambda + \overline{\gamma}\,\lambda - \lambda\,\mu - \lambda\,\overline{\mu} + 3\alpha\,\nu + \overline{\beta}\,\nu + \nu\,\pi - \nu\,\overline{\tau} + \overline{\delta}\nu,\\
&\Delta\mu == \Phi_{22} - \lambda\,\overline{\lambda} - \gamma\,\mu - \overline{\gamma}\,\mu - \mu^2 + \overline{\alpha}\,\nu + 3\beta\,\nu + \overline{\nu}\,\pi - \nu\,\tau + \delta\nu,\\
&\Delta\rho == 2\Lambda - \Psi_2 + \kappa\,\nu + \gamma\,\rho + \overline{\gamma}\,\rho - \overline{\mu}\,\rho - \lambda\,\sigma - \alpha\,\tau + \overline{\beta}\,\tau - \tau\,\overline{\tau} + \overline{\delta}\tau,\\
&\Delta\sigma == \Phi_{02} + \kappa\,\overline{\nu} - \overline{\lambda}\,\rho + 3\gamma\,\sigma - \overline{\gamma}\,\sigma - \mu\,\sigma + \overline{\alpha}\,\tau - \beta\,\tau - \tau^2 + \delta\tau,\\
&\delta\alpha == -\Lambda - \Phi_{11} - \Psi_2 + \alpha\,\overline{\alpha} - 2\alpha\,\beta + \beta\,\overline{\beta} + \epsilon\,\mu - \epsilon\,\overline{\mu} + \gamma\,\rho + \mu\,\rho - \gamma\,\overline{\rho} - \lambda\,\sigma + \overline{\delta}\beta,\\
&\delta\lambda == -\Phi_{21} - \Psi_3 + \overline{\alpha}\,\lambda - 3\beta\,\lambda + \alpha\,\mu + \overline{\beta}\,\mu + \mu\,\pi - \overline{\mu}\,\pi + \nu\,\rho - \nu\,\overline{\rho} + \overline{\delta}\mu,\\
&\delta\rho == -\Phi_{01} - \Psi_1 + \kappa\,\mu - \kappa\,\overline{\mu} + \overline{\alpha}\,\rho + \beta\,\rho - 3\alpha\,\sigma + \overline{\beta}\,\sigma + \rho\,\tau - \overline{\rho}\,\tau + \overline{\delta}\sigma\}
\end{aligned}$$

Newman–Penrose Bianchi identities:

In[378]:= **NPBianchi**

Out[378]=
$$\begin{aligned}
&\{D\Phi_{01} == 2\Psi_1\,\epsilon - 2\Phi_{11}\,\kappa - 3\Psi_2\,\kappa - \Phi_{02}\,\overline{\kappa} + \Psi_0\,(-4\alpha + \pi) +\\
&\quad \Phi_{00}\,(-2(\overline{\alpha} + \beta) + \overline{\pi}) + 4\Psi_1\,\rho + 2\Phi_{01}\,(\epsilon + \overline{\rho}) + 2\Phi_{10}\,\sigma - D\Psi_1 + \delta\Phi_{00} + \overline{\delta}\Psi_0,\\
&D\Phi_{02} == -2\Phi_{01}\,\beta + 2\Psi_1\,\beta - 2\Phi_{12}\,\kappa - \Phi_{00}\,\overline{\lambda} + \Psi_0\,(-4\gamma + \mu) + 2\Phi_{01}\,\overline{\pi} +\\
&\quad \Phi_{02}\,(2\epsilon - 2\overline{\epsilon} + \overline{\rho}) + 2\Phi_{11}\,\sigma - 3\Psi_2\,\sigma + 4\Psi_1\,\tau + \Delta\Psi_0 + \delta\Phi_{01} - \delta\Psi_1,\\
&D\Phi_{11} == -(\Phi_{21} + 2\Psi_3)\,\kappa - \Phi_{12}\,\overline{\kappa} - \Psi_0\,\lambda - \Phi_{00}\,\mu + \Phi_{01}\,\pi + 2\Psi_1\,(-\alpha + \pi) +\\
&\quad \Phi_{10}\,(-2\overline{\alpha} + \overline{\pi}) + 3\Psi_2\,\rho + 2\Phi_{11}\,\overline{\rho} + \Phi_{20}\,\sigma - D\Lambda - D\Psi_2 + \delta\Phi_{10} + \overline{\delta}\Psi_1,\\
&D\Phi_{22} == -\Phi_{20}\,\overline{\lambda} - 2\Phi_{11}\,\mu + 3\Psi_2\,\mu - 2\Psi_1\,\nu + 2\Phi_{12}\,\pi + 2\Phi_{21}\,(\beta + \overline{\pi}) +\\
&\quad \Phi_{22}\,(-2(\epsilon + \overline{\epsilon}) + \overline{\rho}) - \Psi_4\,\sigma + 2\Psi_3\,(-\beta + \tau) - 2(\Delta\Lambda) + \Delta\Psi_2 + \delta\Phi_{21} - \delta\Psi_3,\\
&D\Psi_4 == -2\Phi_{21}\,\alpha + 2\Psi_3\,\alpha - 4\Psi_4\,\epsilon + 2\Phi_{11}\,\lambda - 3\Psi_2\,\lambda + \Phi_{20}\,(2\gamma - 2\overline{\gamma} + \overline{\mu}) -\\
&\quad 2\Phi_{10}\,\nu + 4\Psi_3\,\pi + \Psi_4\,\rho - \Phi_{22}\,\overline{\sigma} + 2\Phi_{21}\,\overline{\tau} + \Delta\Phi_{20} - \overline{\delta}\Phi_{21} + \overline{\delta}\Psi_3,\\
&\Delta\Phi_{00} == 2\Phi_{00}\,(\gamma + \overline{\gamma}) + 2\Psi_3\,\kappa + \Psi_0\,\lambda - \Phi_{00}\,\overline{\mu} + 2\Psi_1\,(\alpha - \pi) + 2\Phi_{11}\,\rho -\\
&\quad 3\Psi_2\,\rho + \Phi_{02}\,\overline{\sigma} - 2\Phi_{10}\,\tau - 2\Phi_{01}\,(\alpha + \overline{\tau}) - 2(D\Lambda) + D\Psi_2 + \overline{\delta}\Phi_{01} - \overline{\delta}\Psi_1,\\
&\Delta\Phi_{11} == 2\Psi_3\,\beta + 2\Phi_{12}\,\overline{\beta} - \Phi_{02}\,\lambda - 3\Psi_2\,\mu - 2\Phi_{11}\,\overline{\mu} + \Phi_{01}\,\nu + 2\Psi_1\,\nu + \Phi_{10}\,\overline{\nu} +\\
&\quad \Phi_{22}\,\rho + \Psi_4\,\sigma - \Phi_{21}\,\tau - 2\Psi_3\,\tau - \Phi_{12}\,\overline{\tau} - \Delta\Lambda - \Delta\Psi_2 + \delta\Psi_3 + \overline{\delta}\Phi_{12},\\
&\Delta\Phi_{21} == 2\Phi_{22}\,\left(\alpha + \overline{\beta}\right) - 2\Phi_{12}\,\lambda - 2(\Psi_3\,(\gamma + 2\mu) + \Phi_{21}\,(\gamma + \overline{\mu})) + 2\Phi_{11}\,\nu +\\
&\quad 3\Psi_2\,\nu + \Phi_{20}\,\overline{\nu} + \Psi_4\,(4\beta - \tau) - \Phi_{22}\,\overline{\tau} - \Delta\Psi_3 + \delta\Psi_4 + \overline{\delta}\Phi_{22},\\
&\delta\Phi_{11} == 2\Psi_1\,\gamma + 2\Phi_{12}\,\overline{\epsilon} + \Phi_{22}\,\kappa + \Phi_{10}\,\overline{\lambda} + \Phi_{01}\,\mu - 2\Psi_1\,\mu + \Psi_0\,\nu - \Phi_{02}\,\pi -\\
&\quad 2\Phi_{11}\,\overline{\pi} - \Phi_{12}\,\overline{\rho} - \Phi_{21}\,\sigma + 2\Psi_3\,\sigma - 3\Psi_2\,\tau + D\Phi_{12} - \Delta\Psi_1 + \delta\Lambda + \delta\Psi_2,\\
&\overline{\delta}\Phi_{02} == -2\Phi_{01}\,\gamma - 2\Psi_1\,\gamma + 2\Psi_1\,\mu + 2\Phi_{01}\,\overline{\mu} - \Psi_0\,\nu - \Phi_{00}\,\overline{\nu} - 2\Phi_{12}\,\rho - 2\Psi_3\,\sigma +\\
&\quad 2\Phi_{11}\,\tau + 3\Psi_2\,\tau + \Phi_{02}\,\left(2\alpha - 2\overline{\beta} + \overline{\tau}\right) + \Delta\Phi_{01} + \Delta\Psi_1 + 2(\delta\Lambda) - \delta\Psi_2,\\
&2\left(\overline{\delta}\Phi_{11}\right) == -6\Psi_3\,\epsilon - 3\Psi_4\,\kappa - \Phi_{22}\,\overline{\kappa} + 2\Phi_{01}\,\lambda - 6\Psi_1\,\lambda - 2\Phi_{10}\,(2\overline{\gamma} + \mu - \overline{\mu}) -\\
&\quad 2\Phi_{00}\,\nu + 2\Phi_{11}\,\pi + 9\Psi_2\,\pi + 6\Psi_3\,\rho - 2\Phi_{21}\,(\epsilon + \rho - \overline{\rho}) - 2\Phi_{12}\,\overline{\sigma} +\\
&\quad \Phi_{20}\,(-2\overline{\alpha} + 2\beta + \overline{\pi} + 2\tau) + 4\Phi_{11}\,\overline{\tau} - D\Phi_{21} - 3(D\Psi_3) + 2(\Delta\Phi_{10}) + \delta\Phi_{20} + 3\left(\overline{\delta}\Psi_2\right)\}
\end{aligned}$$

GHP commutators acting on a generic quantity $\eta^{\{w,\,z\}}$ of boost weight w and spin weight z.

In[379]:= **GHPCommutators**

Out[379]=
$$\{ þ\,þ'\,\eta^{\{w,\,z\}} - þ'\,þ\,\eta^{\{w,\,z\}} == -\eta^{\{w,\,z\}} \left(w\,(\Lambda - \Phi_{11} + \Psi_2 - \kappa\,\nu + \pi\,\tau) + z\left(\Lambda - \Phi_{11} + \overline{\Psi}_2 - \overline{\kappa}\,\overline{\nu} + \overline{\pi}\,\overline{\tau}\right)\right) + (\pi + \overline{\tau})\left(ð\,\eta^{\{w,\,z\}}\right) + (\overline{\pi} + \tau)\left(ð'\,\eta^{\{w,\,z\}}\right),$$
$$þ\,ð\,\eta^{\{w,\,z\}} - ð\,þ\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\left(z\left(\Phi_{01} + \overline{\kappa}\,\overline{\lambda} - \overline{\pi}\,\overline{\rho}\right) - w\,(\Psi_1 - \kappa\,\mu + \pi\,\sigma)\right) + \overline{\pi}\left(þ\,\eta^{\{w,\,z\}}\right) - \kappa\left(þ'\,\eta^{\{w,\,z\}}\right) + \overline{\rho}\left(ð\,\eta^{\{w,\,z\}}\right) + \sigma\left(ð'\,\eta^{\{w,\,z\}}\right),$$
$$þ'\,ð\,\eta^{\{w,\,z\}} - ð\,þ'\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\left(-w\,(\Phi_{12} + \nu\,\sigma - \mu\,\tau) + z\left(\overline{\Psi}_3 - \overline{\nu}\,\overline{\rho} + \overline{\lambda}\,\overline{\tau}\right)\right) + \overline{\nu}\left(þ\,\eta^{\{w,\,z\}}\right) - \tau\left(þ'\,\eta^{\{w,\,z\}}\right) - \mu\left(ð\,\eta^{\{w,\,z\}}\right) - \overline{\lambda}\left(ð'\,\eta^{\{w,\,z\}}\right),$$
$$þ\,ð'\,\eta^{\{w,\,z\}} - ð'\,þ\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\left(w\,(\Phi_{10} + \kappa\,\lambda - \pi\,\rho) - z\left(\overline{\Psi}_1 - \overline{\kappa}\,\overline{\mu} + \overline{\pi}\,\overline{\sigma}\right)\right) + \pi\left(þ\,\eta^{\{w,\,z\}}\right) - \overline{\kappa}\left(þ'\,\eta^{\{w,\,z\}}\right) + \overline{\sigma}\left(ð\,\eta^{\{w,\,z\}}\right) + \rho\left(ð'\,\eta^{\{w,\,z\}}\right),$$
$$þ'\,ð'\,\eta^{\{w,\,z\}} - ð'\,þ'\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\,(w\,(\Psi_3 - \nu\,\rho + \lambda\,\tau) - z\,(\Phi_{21} + \overline{\nu}\,\overline{\sigma} - \overline{\mu}\,\overline{\tau})) + \nu\left(þ\,\eta^{\{w,\,z\}}\right) - \overline{\tau}\left(þ'\,\eta^{\{w,\,z\}}\right) - \lambda\left(ð\,\eta^{\{w,\,z\}}\right) - \overline{\mu}\left(ð'\,\eta^{\{w,\,z\}}\right),$$
$$ð\,ð'\,\eta^{\{w,\,z\}} - ð'\,ð\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\left(w\,(\Lambda + \Phi_{11} + \Psi_2 - \mu\,\rho + \lambda\,\sigma) - z\left(\Lambda + \Phi_{11} + \overline{\Psi}_2 - \overline{\mu}\,\overline{\rho} + \overline{\lambda}\,\overline{\sigma}\right)\right) + (\mu - \overline{\mu})\left(þ\,\eta^{\{w,\,z\}}\right) + (\rho - \overline{\rho})\left(þ'\,\eta^{\{w,\,z\}}\right)\}$$

GHP Ricci identities:

In[380]:= **GHPRicci**

Out[380]=
$$\{ þ\,\lambda == -\Phi_{20} - \overline{\kappa}\,\nu + \pi^2 + \lambda\,\rho + \mu\,\overline{\sigma} + ð'\,\pi,\; þ\,\mu == -2\,\Lambda + \Psi_2 - \kappa\,\nu + \pi\,\overline{\pi} + \mu\,\overline{\rho} + \lambda\,\sigma + ð\,\pi,$$
$$þ\,\nu == -\Phi_{21} + \Psi_3 + \lambda\,(\overline{\pi} + \tau) + \mu\,(\pi + \overline{\tau}) + þ'\,\pi,\; þ\,\rho == -\Phi_{00} + \kappa\,\pi + \rho^2 + \sigma\,\overline{\sigma} - \overline{\kappa}\,\tau + ð'\,\kappa,$$
$$þ\,\sigma == \Psi_0 + (\rho + \overline{\rho})\,\sigma + \kappa\,(\overline{\pi} - \tau) + ð\,\kappa,\; þ\,\tau == -\Phi_{01} + \Psi_1 + \rho\,(\overline{\pi} + \tau) + \sigma\,(\pi + \overline{\tau}) + þ'\,\kappa,$$
$$þ'\,\lambda == -\Psi_4 - \lambda\,(\mu + \overline{\mu}) + \nu\,(\pi - \overline{\tau}) + ð'\,\nu,\; þ'\,\mu == \Phi_{22} - \lambda\,\overline{\lambda} - \mu^2 + \overline{\nu}\,\pi - \nu\,\tau + ð\,\nu,$$
$$þ'\,\rho == 2\,\Lambda - \Psi_2 + \kappa\,\nu - \overline{\mu}\,\rho - \lambda\,\sigma - \tau\,\overline{\tau} + ð'\,\tau,\; þ'\,\sigma == \Phi_{02} + \kappa\,\overline{\nu} - \overline{\lambda}\,\rho - \mu\,\sigma - \tau^2 + ð\,\tau,$$
$$ð\,\lambda == -\Phi_{21} - \Psi_3 + \mu\,\pi - \overline{\mu}\,\pi + \nu\,\rho - \nu\,\overline{\rho} + ð'\,\mu,$$
$$ð\,\rho == -\Phi_{01} - \Psi_1 + \kappa\,\mu - \kappa\,\overline{\mu} + \rho\,\tau - \overline{\rho}\,\tau + ð'\,\sigma\}$$

GHP Bianchi identities:

In[381]:= **GHPBianchi**

Out[381]=
$$\{\text{þ}\,\Lambda == \frac{1}{2}(2\,\Psi_3\,\kappa + \Psi_0\,\lambda - \Phi_{00}\,\overline{\mu} - 2\,\Psi_1\,\pi + 2\,\Phi_{11}\,\rho - 3\,\Psi_2\,\rho + \Phi_{02}\,\overline{\sigma} - 2\,\Phi_{10}\,\tau - 2\,\Phi_{01}\,\overline{\tau} + \text{þ}\,\Psi_2 - \text{þ}'\,\Phi_{00} + \eth'\,\Phi_{01} - \eth'\,\Psi_1),\ \text{þ}\,\Phi_{01} == -2\,\Phi_{11}\,\kappa - 3\,\Psi_2\,\kappa - \Phi_{02}\,\overline{\kappa} + \Psi_0\,\pi + \Phi_{00}\,\overline{\pi} + 4\,\Psi_1\,\rho + 2\,\Phi_{01}\,\overline{\rho} + 2\,\Phi_{10}\,\sigma - \text{þ}\,\Psi_1 + \eth\,\Phi_{00} + \eth'\,\Psi_0,$$
$$\text{þ}\,\Phi_{02} == -2\,\Phi_{12}\,\kappa - \Phi_{00}\,\overline{\lambda} + \Psi_0\,\mu + 2\,\Phi_{01}\,\overline{\pi} + \Phi_{02}\,\overline{\rho} + 2\,\Phi_{11}\,\sigma - 3\,\Psi_2\,\sigma + 4\,\Psi_1\,\tau + \text{þ}'\,\Psi_0 + \eth\,\Phi_{01} - \eth\,\Psi_1,$$
$$\text{þ}\,\Phi_{11} == \frac{1}{2}(-2\,(\Phi_{21} + 3\,\Psi_3)\,\kappa - 2\,\Phi_{12}\,\overline{\kappa} - 3\,\Psi_0\,\lambda + \Phi_{00}\,(-2\,\mu + \overline{\mu}) + 6\,\Psi_1\,\pi - 2\,\Phi_{11}\,\rho + 9\,\Psi_2\,\rho + 4\,\Phi_{11}\,\overline{\rho} + 2\,\Phi_{20}\,\sigma - \Phi_{02}\,\overline{\sigma} + 2\,\Phi_{10}\,(\overline{\pi} + \tau) + 2\,\Phi_{01}\,(\pi + \overline{\tau}) - 3\,(\text{þ}\,\Psi_2) + \text{þ}'\,\Phi_{00} + 2\,(\eth\,\Phi_{10}) - \eth'\,\Phi_{01} + 3\,(\eth'\,\Psi_1)),$$
$$\text{þ}\,\Phi_{12} == -\Phi_{22}\,\kappa - \Phi_{10}\,\overline{\lambda} - \Phi_{01}\,\mu + 2\,\Psi_1\,\mu - \Psi_0\,\nu + \Phi_{02}\,\pi + 2\,\Phi_{11}\,\overline{\pi} + \Phi_{12}\,\overline{\rho} + \Phi_{21}\,\sigma - 2\,\Psi_3\,\sigma + 3\,\Psi_2\,\tau + \text{þ}'\,\Psi_1 - \eth\,\Lambda + \eth\,\Phi_{11} - \eth\,\Psi_2,$$
$$\text{þ}\,\Phi_{21} == -3\,\Psi_4\,\kappa - \Phi_{22}\,\overline{\kappa} + 2\,\Phi_{01}\,\lambda - 6\,\Psi_1\,\lambda - 2\,\Phi_{10}\,\mu + 2\,\Phi_{10}\,\overline{\mu} - 2\,\Phi_{00}\,\nu + 2\,\Phi_{11}\,\pi + 9\,\Psi_2\,\pi + \Phi_{20}\,\overline{\pi} - 2\,\Phi_{21}\,\rho + 6\,\Psi_3\,\rho + 2\,\Phi_{21}\,\overline{\rho} - 2\,\Phi_{12}\,\overline{\sigma} + 2\,\Phi_{20}\,\tau + 4\,\Phi_{11}\,\overline{\tau} - 3\,(\text{þ}\,\Psi_3) + 2\,(\text{þ}'\,\Phi_{10}) + \eth\,\Phi_{20} - 2\,(\eth'\,\Phi_{11}) + 3\,(\eth'\,\Psi_2),$$
$$\text{þ}\,\Phi_{22} == 2\,\Phi_{02}\,\lambda - \Phi_{20}\,\overline{\lambda} - 2\,\Phi_{11}\,\mu + 9\,\Psi_2\,\mu + 4\,\Phi_{11}\,\overline{\mu} - 2\,(\Phi_{01} + 3\,\Psi_1)\,\nu - 2\,\Phi_{10}\,\overline{\nu} - 2\,\Phi_{22}\,\rho + \Phi_{22}\,\overline{\rho} - 3\,\Psi_4\,\sigma + 6\,\Psi_3\,\tau + 2\,\Phi_{21}\,(\overline{\pi} + \tau) + 2\,\Phi_{12}\,(\pi + \overline{\tau}) + 2\,(\text{þ}'\,\Phi_{11}) + 3\,(\text{þ}'\,\Psi_2) + \eth\,\Phi_{21} - 3\,(\eth\,\Psi_3) - 2\,(\eth'\,\Phi_{12}),\ \text{þ}\,\Psi_4 == 2\,\Phi_{11}\,\lambda - 3\,\Psi_2\,\lambda + \Phi_{20}\,\overline{\mu} - 2\,\Phi_{10}\,\nu + 4\,\Psi_3\,\pi + \Psi_4\,\rho - \Phi_{22}\,\overline{\sigma} + 2\,\Phi_{21}\,\overline{\tau} + \text{þ}'\,\Phi_{20} - \eth'\,\Phi_{21} + \eth'\,\Psi_3,$$
$$\text{þ}'\,\Lambda == -\Phi_{02}\,\lambda - 3\,\Psi_2\,\mu - 2\,\Phi_{11}\,\overline{\mu} + (\Phi_{01} + 2\,\Psi_1)\,\nu + \Phi_{10}\,\overline{\nu} + \Phi_{22}\,\rho + \Psi_4\,\sigma - (\Phi_{21} + 2\,\Psi_3)\,\tau - \Phi_{12}\,\overline{\tau} - \text{þ}'\,\Phi_{11} - \text{þ}'\,\Psi_2 + \eth\,\Psi_3 + \eth'\,\Phi_{12},$$
$$\text{þ}'\,\Phi_{01} == -2\,\Psi_1\,\mu - 2\,\Phi_{01}\,\overline{\mu} + \Psi_0\,\nu + \Phi_{00}\,\overline{\nu} + 2\,\Phi_{12}\,\rho + 2\,\Psi_3\,\sigma - 2\,\Phi_{11}\,\tau - 3\,\Psi_2\,\tau - \Phi_{02}\,\overline{\tau} - \text{þ}'\,\Psi_1 - 2\,(\eth\,\Lambda) + \eth\,\Psi_2 + \eth'\,\Phi_{02},\ \text{þ}'\,\Phi_{21} == -2\,\Phi_{12}\,\lambda - 4\,\Psi_3\,\mu - 2\,\Phi_{21}\,\overline{\mu} + 2\,\Phi_{11}\,\nu + 3\,\Psi_2\,\nu + \Phi_{20}\,\overline{\nu} - \Psi_4\,\tau - \Phi_{22}\,\overline{\tau} - \text{þ}'\,\Psi_3 + \eth\,\Psi_4 + \eth'\,\Phi_{22}\}$$

Within the GHP formalism any expression must be weighted. The command GHPWeightOf will compute the weight of any expression.

In[382]:= **? GHPWeightOf**

```
GHPWeightOf[expr] returns the GHP boost and spin weight of any expression.
  If the expression is not weighted, then an error message is returned
```

For example: take one of the GHP Bianchi identities:

In[383]:= **GHPBianchi[[2]]**

Out[383]=
$$\text{þ}\,\Phi_{01} == -2\,\Phi_{11}\,\kappa - 3\,\Psi_2\,\kappa - \Phi_{02}\,\overline{\kappa} + \Psi_0\,\pi + \Phi_{00}\,\overline{\pi} + 4\,\Psi_1\,\rho + 2\,\Phi_{01}\,\overline{\rho} + 2\,\Phi_{10}\,\sigma - \text{þ}\,\Psi_1 + \eth\,\Phi_{00} + \eth'\,\Psi_0$$

We compute the GHP weight of the right hand side of this relation.

In[384]:= **GHPWeightOf@%[[2]]**

Out[384]= {3, 1}

Finally we explain how to compute the action of the GHP commutators on any quantity. For example suppose that we wish to compute the commutation of þ and ð' on the scalar Ψ_4 . First of all, we look up in the list of GHP commutators the commutation relation we wish to use. In our case this is

In[385]:=
```
GHPCommutators[[4]]
```

Out[385]=
$$\text{þ}\,\text{ð}'\,\eta^{\{w,\,z\}} - \text{ð}'\,\text{þ}\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\left(w\,(\Psi_{10} + \kappa\,\lambda - \pi\,\rho) - z\left(\overline{\Psi_1} - \overline{\kappa}\,\overline{\mu} + \overline{\pi}\,\overline{\sigma}\right)\right) + \pi\,\left(\text{þ}\,\eta^{\{w,\,z\}}\right) - \overline{\kappa}\,\left(\text{þ}'\,\eta^{\{w,\,z\}}\right) + \overline{\sigma}\,\left(\text{ð}\,\eta^{\{w,\,z\}}\right) + \rho\,\left(\text{ð}'\,\eta^{\{w,\,z\}}\right)$$

Next we replace $\eta^{\{w,\,z\}}$ by Ψ_4 and the weights {w, z} by their values. This is accomplished as follows

In[386]:=
```
% /. {η@LI@{w, z} → PsiCDe4Dyad[],
   w → First@GHPWeightOf@PsiCDe4Dyad[], z → Last@GHPWeightOf@PsiCDe4Dyad[]}
```

Out[386]=
$$\text{þ}\,\text{ð}'\,\Psi_4 - \text{ð}'\,\text{þ}\,\Psi_4 == -4\,\Psi_4\,(\Psi_{10} + \kappa\,\lambda - \pi\,\rho) + \pi\,(\text{þ}\,\Psi_4) - \overline{\kappa}\,(\text{þ}'\,\Psi_4) + \overline{\sigma}\,(\text{ð}\,\Psi_4) + \rho\,(\text{ð}'\,\Psi_4)$$