Adjoint Representation in Graded Geometry

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Motivation

Lie groups

- Groups with a smooth structure, group operations smooth.
- Very well-understood smooth manifolds.
- Abstract nonsense: group objects in $\mathbf{Man}^{\infty}$.
- Essential for understanding symmetries in geometry and physics.

Lie group representations

- Realization of groups as actual transformations.
- Homomorphisms $\rho : G \to \mathrm{GL}(V)$.

Example

The **adjoint representation** is the canonical representation of G on its Lie algebra $\mathfrak{g}$. It (locally) encodes the non-commutativity, crucial for Lie theory itself.

Main goal: We want Ad in $\mathbb{Z}$-graded geometry.

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Linear algebra

Definition

- A **graded vector space** is a sequence $V = (V_k)_{k \in \mathbb{Z}}$ of vector spaces. We write $v \in V$ and $|v| = k$, if $v \in V_k$ for some $k \in \mathbb{Z}$.
- A **graded linear map** $A : V \to W$ of **degree** $|A|$ is a sequence $A = (A_k)_{k \in \mathbb{Z}}$, where $A_k : V_k \to W_{k+|A|}$
- We say that V is finite-dimensional, if $\sum_{k \in \mathbb{Z}} V_k < \infty$.

- **gVect** - the category of *real finite-dimensional* graded vector spaces and *degree zero* graded linear maps.
- $\underline{\mathrm{Lin}}(V, W) \in$ **gVect** - *all* graded linear maps from V to W.
- We write $\mathfrak{gl}(V) := \underline{\mathrm{Lin}}(V, V)$.

Observation

$\mathfrak{gl}(V)$ together with the graded commutator

$$[A, B] := AB - (-1)^{|A||B|} BA$$

forms a graded Lie algebra (of degree 0).

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1. M is a smooth manifold (underlying manifold, body of $\mathcal{M}$)
2. $\mathcal{C}^{\infty}_{\mathcal{M}}$ assigns to each $U \in \mathbf{Op}(M)$ a graded commutative associative algebra $\mathcal{C}^{\infty}_{\mathcal{M}}(U)$ of **functions on $\mathcal{M}$ over U**.
3. $\mathcal{C}^{\infty}_{\mathcal{M}}$ has to form a sheaf - now not important.
4. Locally there is something happening - now not important.

Definition

There is a notion of a **graded smooth map** $\varphi : \mathcal{M} \to \mathcal{N}$.

1. They can be associatively composed, there is the identity $\mathbb{1}_{\mathcal{M}}$;
2. There is an underlying smooth map $\underline{\varphi} : M \to N$.
3. Graded manifolds form a category $\mathbf{gMan}^{\infty}$.
4. There is a *body functor* $\mathfrak{B} : \mathbf{gMan}^{\infty} \to \mathbf{Man}^{\infty}$.

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Diamond functor

- $\mathcal{M}$ has a **graded dimension** $\mathrm{gdim}(\mathcal{M}) = (n_k)_{k \in \mathbb{Z}}$, where n_k is a **number of coordinates of degree** k.
- $V \in \mathbf{gVect}$ has a graded dimension $\mathrm{gdim}(V) = (\dim(V_k))_{k \in \mathbb{Z}}$.
- For any sequence $(a_k)_{k \in \mathbb{Z}}$ write $\neg(a_k)_{k \in \mathbb{Z}} := (a_{-k})_{k \in \mathbb{Z}}$.

Proposition

1. *For any $V \in \mathbf{gVect}$, there is $V_\diamond \in \mathbf{gMan}^\infty$, such that*
$$\mathrm{gdim}(V_\diamond) = \neg\,\mathrm{gdim}(V).$$
2. *Underlying manifold is V_0 with the usual smooth structure.*
3. *To any $A : V \to W$ of degree 0, there is $A_\diamond : V_\diamond \to W_\diamond$.*
4. *We obtain a functor $\diamond : \mathbf{gVect} \to \mathbf{gMan}^\infty$.*

- To any basis $(t_\lambda)_{\lambda=1}^n$ of V there are coordinates $(\mathbb{z}^\lambda)_{\lambda=1}^n$ on $V_\diamond$. One has $|\mathbb{z}^\lambda| = -|t_\lambda|$. This explains the "flip".

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- To any basis $(t_\lambda)_{\lambda=1}^n$ of V there are coordinates $(\mathbb{z}^\lambda)_{\lambda=1}^n$ on $V_\diamond$. One has $|\mathbb{z}^\lambda| = -|t_\lambda|$. This explains the "flip".

Diamond functor

- $\mathcal{M}$ has a **graded dimension** $\text{gdim}(\mathcal{M}) = (n_k)_{k\in\mathbb{Z}}$, where n_k is a **number of coordinates of degree** k.
- $V \in \textbf{gVect}$ has a graded dimension $\text{gdim}(V) = (\dim(V_k))_{k\in\mathbb{Z}}$.
- For any sequence $(a_k)_{k\in\mathbb{Z}}$ write $\neg(a_k)_{k\in\mathbb{Z}} := (a_{-k})_{k\in\mathbb{Z}}$.

Proposition

1. *For any $V \in \textbf{gVect}$, there is $V_\diamond \in \textbf{gMan}^\infty$, such that*
$$\text{gdim}(V_\diamond) = \neg\,\text{gdim}(V).$$
2. *Underlying manifold is V_0 with the usual smooth structure.*
3. *To any $A : V \to W$ of degree 0, there is $A_\diamond : V_\diamond \to W_\diamond$.*
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Graded Lie groups

Observation

gMan$^\infty$ has products $\mathcal{M} \times \mathcal{N}$ and a terminal object $\{*\}$.

Definition

A **graded Lie group** is a group object $(\mathcal{G}, \mu, \iota, e)$ in **gMan**$^\infty$, that is $\mathcal{G} \in$ **gMan**$^\infty$ and graded smooth maps

1. $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ (the multiplication)
2. $\iota : \mathcal{G} \to \mathcal{G}$ (the inverse)
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Operations satisfy group axioms - formulated as commutative diagrams.

Proposition

To any graded Lie group $\mathcal{G}$, there is an **associated graded Lie algebra** $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$*, where $\mathfrak{g} \in$* **gVect** *is $T_e\mathcal{G}$.*

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Actions of graded Lie groups

Definition

An **action of a graded Lie group $\mathcal{G}$ on a graded manifold $\mathcal{M}$** is a graded smooth map $\theta : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$.

It is assumed compatible with (μ, ι, e) in usual way, written as commutative diagrams.

Example

- For any $V \in$ **gVect**, there is **general linear group** $\mathrm{GL}(V)$. It is an open submanifold of $\mathfrak{gl}(V)_\diamond$. Its Lie algebra is $\mathfrak{gl}(V)$.
- To any basis $(t_\lambda)_{\lambda=1}^n$ of V, there are "standard basis coordinates" $(\mathbb{y}^\lambda{}_\kappa)$ on $\mathrm{GL}(V)$ with $|\mathbb{y}^\lambda{}_\kappa| = |t_\lambda| - |t_\kappa|$.
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A **representation of a graded Lie group $\mathcal{G}$ on $V \in$ gVect** is a homomorphism $\rho : \mathcal{G} \to \mathrm{GL}(V)$.

Homomorphism is a graded smooth map compatible with group operations in a usual way, using commutative diagrams.

Observation

- For any representation ρ, the map $\theta := \theta_V \circ (\rho \times \mathbb{1}_{V_\diamond})$ defines a left action of $\mathcal{G}$ on $V_\diamond$. Graded analogue of $\theta(g, v) = \rho(g)v$
- This action is "linear".
- Conversely, every linear action $\theta : \mathcal{G} \times V_\diamond \to V_\diamond$ defines a unique representation.

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To any representation $\rho : \mathcal{G} \to \mathrm{GL}(V)$, there is an induced homomorphism of Lie algebras $\rho' : \mathfrak{g} \to \mathfrak{gl}(V)$, called the **derived representation**.

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Induced actions

- To any $\theta : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$, there is a **tangent action** $\theta_T : \mathcal{G} \times T\mathcal{M} \to T\mathcal{M}$ given by

$$\theta_T = T\theta \times (0_{\mathcal{G}} \times \mathbb{1}_{T\mathcal{M}}),$$

  where $0_{\mathcal{G}} : \mathcal{G} \to T\mathcal{G}$ is the zero section.

- If $(\mathcal{S}, j)$ is an submanifold and $\theta : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ factorizes as

$$\begin{tikzcd} \mathcal{G} \times \mathcal{S} \arrow[r, dashed, "\theta_{\mathcal{S}}"] \arrow[d, "\mathbb{1}_{\mathcal{G}} \times j"] & \mathcal{S} \arrow[d, "j"] \\ \mathcal{G} \times \mathcal{M} \arrow[r, "\theta"] & \mathcal{M} \end{tikzcd},$$

  then $\theta_{\mathcal{S}}$ is an action of $\mathcal{G}$ on $\mathcal{S}$.

- Let $\theta : \mathcal{M} \times \mathcal{G} \to \mathcal{M}$ be a *right* action of $\mathcal{G}$ on $\mathcal{M}$. Then there is an associated left action θ' given by

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Example

Graded Lie group multiplication $\mu : \mathcal{G}\times\mathcal{G} \to \mathcal{G}$ defines a **left (or right) translation** θ_L (or θ_R). *Left* actions θ_L and θ'_R commute, their composite is the **conjugation**

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Adjoint representation

Proposition

For any action θ, suppose that the orbit map $\theta^{(m)} : \mathcal{G} \to \mathcal{M}$ is constant. Then there exists a unique linear action θ^m_T on the fiber manifold $(T_m\mathcal{M})_\diamond$ making the embedding $(T_m\mathcal{M})_\diamond \hookrightarrow T\mathcal{M}$ equivariant.

- The unit e is just a unit e of $G := \underline{\mathcal{G}}$.
- Lie algebra $\mathfrak{g}$ is just the tangent space $T_e\mathcal{G}$.
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The **adjoint action** Ad of $\mathcal{G}$ on $\mathfrak{g}_\diamond$ is defined as

$$\mathrm{Ad} := (\theta_I)^e_T : \mathcal{G} \times \mathfrak{g}_\diamond \to \mathfrak{g}_\diamond.$$

There is the corresponding **adjoint representation** $\rho_{\mathrm{Ad}} : \mathcal{G} \to \mathrm{GL}(\mathfrak{g})$.

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Reality checks

- If $\mathcal{G}$ is Abelian, then $\mathsf{Ad} = p_2$ is just a projection and ρ_{Ad} factorizes through the unit of $\mathrm{GL}(\mathfrak{g})$.
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$$\varphi_L : \mathcal{G} \times \mathfrak{g}_\diamond \to T\mathcal{G}.$$

Similarly, one can define φ_R. Then one can write Ad simply as

$$\mathsf{Ad} = p_2 \circ \varphi_R^{-1} \circ \varphi_L.$$

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Then this map is $\mathcal{G}$-equivariant with respect to the actions:

1. $\theta \otimes \mathrm{Ad}$ on $\mathcal{M} \times \mathfrak{g}_{\diamond}$;
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Translate: "fundamental vector fields are Ad-equivariant".

- There exists an **exponential map** $\exp : \mathfrak{g}_{\diamond} \to \mathcal{G}$. Then there is the following commutative diagram:

$$\begin{tikzcd} \mathcal{G} \times \mathfrak{g}_{\diamond} \arrow[r, "\mathbb{1}_{\mathcal{G}} \times \exp"] \arrow[d, "\mathrm{Ad}"] & \mathcal{G} \times \mathcal{G} \arrow[d, "\theta_I"] \\ \mathfrak{g}_{\diamond} \arrow[r, "\exp"] & \mathcal{G} \end{tikzcd}$$

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