# Noncommutative vector field calculi

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Based on [arXiv:2607.07243] with **Rita Fioresi** and **Emanuele Latini**

# Differential geometry dictionary

| | **Classical** differential geometry | **Quantum** differential geometry |
|---|---|---|
| observables | smooth manifold $M$ | associative unital algebra $A$ |
| differential 1-forms | cotangent bundle $\Gamma(T^*M)$ | FODC $(\Omega^1, \mathrm{d})$ |
| symmetry | Lie group action $M \times G \to M$ | Hopf algebra coaction $\Delta_A \colon A \to A \otimes H$ |
| vector fields | tangent bundle $\Gamma(TM)$ | ??? |
| Lie derivative | $\mathscr{L} \colon \Gamma(TM) \to \mathrm{Der}(\mathscr{C}^\infty(M))$ | ??? |
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Fix a field $\Bbbk$ and an (associative unital) algebra $A$.

**Definition**

We call $(\Omega^1, \mathrm{d})$ a **first order differential calculus** (FODC) on $A$, if

i.) $\Omega^1$ is an $A$-bimodule.

ii.) $\mathrm{d}\colon A \to \Omega^1$ is $\Bbbk$-linear s.t. for all $a, b \in A$ the **Leibniz rule** holds:
$$\mathrm{d}(ab) = (\mathrm{d}a)b + a\mathrm{d}b.$$

iii.) $\Omega^1 = A\mathrm{d}A := \left\{ \sum_i a^i \mathrm{d}b^i \mid a^i, b^i \in A \text{ finite sum} \right\}$.

**Theorem (Woronowicz '89)**

*For every associative unital algebra $A$ there exists a FODC $(\Omega^1_u, \mathrm{d}_u)$ on $A$, where*

$$\Omega^1_u := \ker m_A = \left\{ \sum_i a^i \otimes b^i \;\middle|\; \sum_i a^i b^i = 0 \right\} \subseteq A \otimes A$$

*and $\mathrm{d}_u(a) = 1 \otimes a - a \otimes 1$ for all $a \in A$.*

*Moreover, every FODC $(\Omega^1, \mathrm{d})$ on $A$ is a quotient of $(\Omega^1_u, \mathrm{d}_u)$.*

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# Vector fields in NCG

$A$ associative unital algebra. Then $\mathrm{End}_{\Bbbk}(A)$ is an $A$-bimodule via

$$(a \cdot \phi \cdot b)(c) = a\phi(bc)$$

for all $\phi \in \mathrm{End}_{\Bbbk}(A)$ and $a, b, c \in A$.

**Definition (Borowiec '96)**

A **first order vector field calculus** (FOVC) on $A$ is a tuple $(\mathfrak{X}, \mathscr{L})$, where

i.) $\mathfrak{X}$ is an $A$-bimodule.

ii.) $\mathscr{L} \colon \mathfrak{X} \to \mathrm{End}_{\Bbbk}(A)$ is a left $A$-linear map satisfying for all $X \in \mathfrak{X}$ and $a, b \in A$ the Leibniz rule

$$\mathscr{L}_X(ab) = \mathscr{L}_X(a)b + \mathscr{L}_{X \cdot a}(b)$$

iii.) $\mathscr{L} \colon \mathfrak{X} \hookrightarrow \mathrm{End}_{\Bbbk}(A)$ is injective.

A morphism of FOVCi is an $A$-bilinear map $\phi \colon \mathfrak{X} \to \mathfrak{X}'$ such that

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### Example (Classical case/sanity check)

For a **commutative** algebra $A$, let

$$\mathfrak{X} := \mathrm{Der}(A) = \{X \in \mathrm{End}_{\Bbbk}(A) \mid X(ab) = X(a)b + aX(b) \text{ for all } a, b \in A\}$$

with $A$-bimodule structure $\qquad (a \cdot X)(b) := aX(b) =: (X \cdot a)(b)$

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For $A = \mathscr{C}^\infty(M)$ this gives $\mathfrak{X} \cong \Gamma^\infty(TM)$.

### Theorem (Fioresi-Latini-TW '26)

*For any associative unital algebra $A$ there* ***exists*** *a FOVC $(\mathfrak{X}_u, \mathscr{L}^u)$ defined by $\mathfrak{X}_u := \{X \in \mathrm{End}_{\Bbbk}(A) \mid X(1) = 0\}$ with $A$-actions*

$$(a \cdot X)(b) := aX(b) \qquad \text{and} \qquad (X \cdot a)(b) := X(ab) - X(a)b$$

*for all $X \in \mathfrak{X}_u$, $a, b \in A$ and $\mathscr{L}^u \colon \mathfrak{X}_u \hookrightarrow \mathrm{End}_{\Bbbk}(A)$ the canonical inclusion.*

*Moreover, $(\mathfrak{X}_u, \mathscr{L}^u)$ is* ***universal*** *in the sense that for any FOVC $(\mathfrak{X}, \mathscr{L})$ on $A$ there is a morphism of FOVCi $(\mathfrak{X}, \mathscr{L}) \to (\mathfrak{X}_u, \mathscr{L}^u)$.*

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# Directed graphs on finite sets

**Example**

$X$ finite set and $A := \mathbb{C}[X] = \mathrm{span}_{\mathbb{C}}\{\delta_x \colon X \to \mathbb{C} \mid \delta_x(y) := \delta_{x,y}\}$.

Consider a directed graph (no loops, no multiple edges) and define $\mathfrak{X} := \mathrm{span}_{\mathbb{C}}\{\chi_{x\to y} \mid x \to y \text{ edge}\}$

with $A$-bimodule structure $\quad f \cdot \chi_{x\to y} \cdot g := f(x)g(y)\chi_{x\to y}$

and $\quad \mathscr{L}_{\chi_{x\to y}}(f) := (f(x) - f(y))\delta_x, \quad f, g \in A.$

Then $(\mathfrak{X}, \mathscr{L})$ is a FOVC on $A$.

**Proposition (Fioresi-Latini-TW '26)**

*Let $X$ be a finite set and $A := \mathbb{C}[X]$. Then,*

$$\left\{\begin{matrix}\text{FOVCi}\\ \text{on } A\end{matrix}\right\} \xleftrightarrow{1:1} \left\{\begin{matrix}\text{directed graphs on } X\\ \text{as above}\end{matrix}\right\}$$

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# Duality

**Theorem (Fioresi-Latini-TW '26)**

*There is an adjunction of categories*

$$\begin{tikzcd} \underline{\mathrm{FODC}}_A \arrow[rr, bend left, "(\cdot)^*"] \arrow[rr, phantom, "\perp"] & & (\underline{\mathrm{FOVC}}_A)^{\mathrm{op}} \arrow[ll, bend left, "{}^*(\cdot)"] \end{tikzcd}$$

*Explicitly,*

i.) *Given a FODC* $(\Omega^1, \mathrm{d})$ *on* $A \Rightarrow$ *FOVC* $(\mathfrak{X}, \mathscr{L})$ *on* $A$ *via*

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For fgp $A$-modules the adjunction becomes an **equivalence**!

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# Covariant FOVCi

$H$ Hopf algebra

$A$ right $H$-comodule algebra, i.e., coaction $\Delta_A\colon A \to A \otimes H$ is algebra morphism

**Definition**

A FOVC $(\mathfrak{X}, \mathscr{L})$ on $A$ is **right $H$-covariant** if $\mathfrak{X}$ is right $H$-covariant $A$-bimodule, i.e.

$$\Delta_{\mathfrak{X}}(a \cdot X \cdot b) = \Delta_A(a)\Delta_{\mathfrak{X}}(X)\Delta_A(b)$$

and the following diagram commutes, where $\widetilde{\mathscr{L}}(X \otimes a) := \mathscr{L}_X(a)$.

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Morphisms of right $H$-covariant FOVCi have to be right $H$-colinear, in addition.

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$G$ finite group $\rightsquigarrow$ $H := \mathbb{C}[G]$ is Hopf algebra with

$$\Delta(\delta_g) = \sum\nolimits_{h \in G} \delta_h \otimes \delta_{h^{-1}g} \quad , \quad \varepsilon(\delta_g) = \delta_{e,g} \quad , \quad S(\delta_g) = \delta_{g^{-1}}.$$

Directed graph on $G$ (no loops, no multiple edges)
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### Definition

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# Quantum principal bundles and vertical vector fields

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*Let $A$ be a QPB and $\mathfrak{g}$ a quantum tangent space on $H \Rightarrow (\mathfrak{X}_A, \mathscr{L}^A)$ is a FOVC on $A$.*

*If $\mathfrak{g}$ is a bicovariant quantum tangent space on $H$*
*$\Rightarrow (\mathfrak{X}_A, \mathscr{L}^A)$ is a right $H$-covariant FOVC on $A$.*

# Quantum principal bundles and vertical vector fields

A **quantum principal bundle** (QPB) is a right $H$-comodule algebra $A$ such that

$$B := A^{\mathrm{co}H} := \{a \in A \mid \Delta_A(a) = a \otimes 1\} \subseteq A$$

is **Hopf–Galois**, i.e., such that

$$A \otimes_B A \to A \otimes H, \qquad a \otimes_B a' \mapsto a\Delta_A(a')$$

is a bijection$\rightsquigarrow$ **"translation map"** $\tau \colon H \to A \otimes_B A$, $\tau(h) =: h^{\langle 1\rangle} \otimes_B h^{\langle 2\rangle}$.

For generic $\mathfrak{g} \subseteq (H^\circ)^+$ define

$$\mathfrak{X}_A := A \otimes \mathfrak{g} \qquad a \cdot (a' \otimes \chi_i) \cdot a'' := aa' a''_0 f^j_i(a''_1) \otimes \chi_j$$
$$\mathscr{L}^A \colon \mathfrak{X}_A \to \mathrm{End}_{\Bbbk}(A) \qquad \mathscr{L}^A_\chi(a) := a_0 \chi(a_1)$$

**Proposition (Fioresi-Latini-TW '26)**

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# Base vector fields and the Atiyah sequence

For a right $H$-comodule algebra $A$ with FOVC $(\mathfrak{X}, \mathscr{L})$ on $A$ define $B := A^{\mathrm{co}H}$. We call

$$\mathfrak{X}_B := \{X \in \mathfrak{X} \mid \mathscr{L}_X(B) \subseteq B\} \Big/ \{X \in \mathfrak{X} \mid \mathscr{L}_X(B) = 0\}$$

the **base vector fields**.

**Lemma**

*$\mathfrak{X}_B$ is a FOVC on $B$ with Lie derivative $\mathscr{L}^B \colon \mathfrak{X}_B \to \mathrm{End}_{\Bbbk}(B)$, $\mathscr{L}^B_{[X]}(b) := \mathscr{L}_X(b)$.*

**Definition**

Given a QPB $B := A^{\mathrm{co}H} \subseteq A$ and a bicovariant quantum tangent space $\mathfrak{g}$ on $H$ we call a right $H$-covariant FOVC $(\mathfrak{X}, \mathscr{L})$ on $A$ a **principal FOVC** if the **Atiyah sequence**

$$0 \to A \otimes \mathfrak{g} \xrightarrow{\phi} \mathfrak{X} \xrightarrow{\psi} \mathfrak{X}_B A \to 0$$
$$X \mapsto [X_0 \cdot (X_1)^{\langle 1 \rangle}](X_1)^{\langle 2 \rangle}$$

is well-defined and exact.

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## Definition

Given a QPB $B := A^{\mathrm{co}H} \subseteq A$ and a bicovariant quantum tangent space $\mathfrak{g}$ on $H$ we call a right $H$-covariant FOVC $(\mathfrak{X}, \mathscr{L})$ on $A$ a **principal FOVC** if the **Atiyah sequence**

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is well-defined and exact.

# An example: 3D FOVC on $\mathrm{SL}_q(2)$

$q \in \mathbb{C}$ not zero or a root of unity. $A := \mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$ is generated by $\alpha, \beta, \gamma, \delta$ modulo the Manin relations

$$\beta\alpha = q\alpha\beta, \quad \gamma\alpha = q\alpha\gamma, \quad \delta\beta = q\beta\delta, \quad \delta\gamma = q\gamma\delta, \quad \delta\alpha - \alpha\delta = (q - q^{-1})\beta\gamma, \quad \gamma\beta = \beta\gamma$$

and the quantum determinant relation $\alpha\delta - q^{-1}\beta\gamma = 1$.

Right $H = \mathcal{O}(U(1))$-comodule algebra via $\Delta_A \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \otimes \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$.

Let $\mathfrak{X} := \mathrm{span}_A\{\chi_0, \chi_\pm\}$ with

$$(a \otimes \chi_0) \cdot f = q^{-2|f|} af \otimes \chi_0, \qquad (a \otimes \chi_\pm) \cdot f = q^{-|f|} af \otimes \chi_\pm,$$

for all $a \in A$ and $f \in \{\alpha, \beta, \gamma, \delta\}$, where $|\alpha| = |\gamma| = 1$ and $|\beta| = |\delta| = -1$. Moreover, the Lie derivative is defined by

$$\mathscr{L}_{\chi_0}(\alpha) = q^{-2}\alpha, \qquad \mathscr{L}_{\chi_0}(\beta) = -\beta,$$
$$\mathscr{L}_{\chi_0}(\gamma) = q^{-2}\gamma, \qquad \mathscr{L}_{\chi_0}(\delta) = -\delta,$$

$$\mathscr{L}_{\chi_+}(\alpha) = q^2\beta, \quad \mathscr{L}_{\chi_+}(\beta) = 0, \qquad \mathscr{L}_{\chi_-}(\alpha) = 0, \quad \mathscr{L}_{\chi_-}(\beta) = q^{-1}\alpha,$$
$$\mathscr{L}_{\chi_+}(\gamma) = q^2\delta, \quad \mathscr{L}_{\chi_+}(\delta) = 0, \qquad \mathscr{L}_{\chi_-}(\gamma) = 0, \quad \mathscr{L}_{\chi_-}(\delta) = q^{-1}\gamma.$$

On $H = \mathcal{O}(U(1))$ we choose the bicovariant quantum tangent space $\mathfrak{g} = \mathrm{span}_{\mathbb{C}}\{\vartheta\}$ with corresponding bicovariant FOVC $(\mathfrak{X}_H, \mathscr{L}^H)$ determined by

$$(t^n \otimes \vartheta) \cdot t = q^{-2} t^{n+1} \otimes \vartheta, \qquad \mathscr{L}^H_\vartheta \begin{pmatrix} t \\ t^{-1} \end{pmatrix} = \begin{pmatrix} q^{-2} t \\ -t^{-1} \end{pmatrix}.$$

$B = A^{\mathrm{co}H} = \mathbb{C}_q[\mathbb{S}^2]$ Podleś sphere

$\rightsquigarrow$ base vector fields $\mathfrak{X}_B = \mathrm{span}_B\{[\alpha^2 \chi_+], [\beta^2 \chi_-]\}$.

vertical vector fields $\mathfrak{X}_{\mathrm{ver}} := A \otimes \mathfrak{g}$ with $\mathscr{L}^{\mathrm{ver}}_{a \otimes \vartheta}(c) = a c_0 \vartheta(c_1)$ satisfies

$$\mathscr{L}^{\mathrm{ver}}_{1 \otimes \theta} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha\theta(t) & \beta\theta(t^{-1}) \\ \gamma\theta(t) & \delta\theta(t^{-1}) \end{pmatrix} = \begin{pmatrix} q^{-2}\alpha & -\beta \\ q^{-2}\gamma & -\delta \end{pmatrix} = \mathscr{L}_{\chi_0} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

i.e., $\mathfrak{X}_{\mathrm{ver}} \hookrightarrow \mathfrak{X}$. Moreover, the Atiyah sequence is exact

$$0 \to A \otimes \mathfrak{g}_H \xrightarrow{\phi} \mathfrak{X}_A \xrightarrow{\psi} \mathfrak{X}_B A \to 0$$

and thus we have a **principal** FOVC on $A$.

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Thank you for your attention!