

Intro to the large charge expansion

Susanne Reffert
University of Bern

based on arXiv:1505.01537, 1610.04495, 1707.00710, 1809.06371, 1902.09542, 1905.00026,
1909.08642, 1909.02571, 2008.03308, 2010.07942, 2102.12488, 2110.07617, 2110.07616,
2203.12624, 2211.15318, 2311.14793, 2403.18898, 2501.10505, 2510.26876, 2604.15422

L. Alvarez-Gaume (SCGP), D. Banerjee (Kolkata), S. Beane (Washington),
Sh. Chandrasekharan (Duke), N. Dondi (ICTP), S. Hellerman (IPMU),
Kalogerakis (Bern), Adèle Le Borgne, R. Moser (Bern), Daniil Krichevskiy
(Stavanger), O. Loukas, D. Orlando (INFN Torino), V. Pellizzani (Bern),
F. Sannino (Odense/Napoli), T. Schmidt (Bern), Ian Swanson, M. Watanabe (U.Tokyo)



Introduction

Strongly coupled physics is notoriously difficult to access, especially analytically.

We do not have small parameters in which to do a perturbative expansion. Our most basic notions of field theory are of a perturbative nature.

Make use of symmetries, look at special limits/subsectors where things simplify/look like standard perturbation theory.

Examples:

- large- N limit, 't Hooft limit
- ϵ expansion
- supersymmetric sectors
- large spin
- integrability
-

Introduction

Study theories with a **global symmetry** group.

Hilbert space of the theory can be decomposed into sectors of fixed charge Q .

Study subsectors with **large charge Q** .

Best case scenario: Large charge Q becomes **controlling parameter in a perturbative expansion!**

Effective theory at large Q :

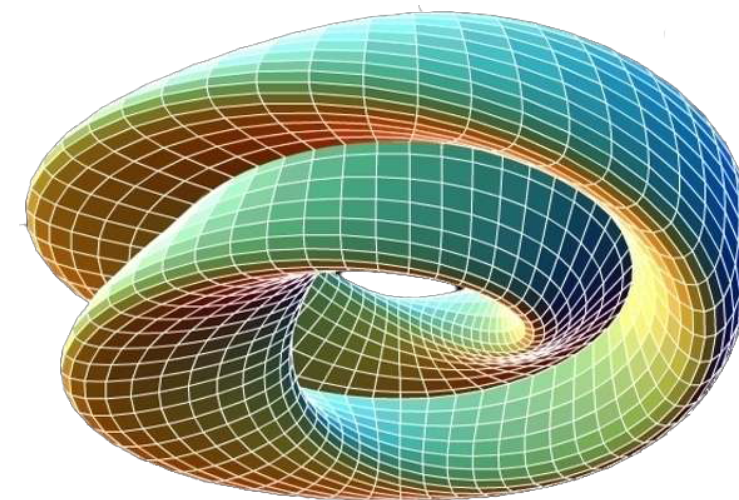
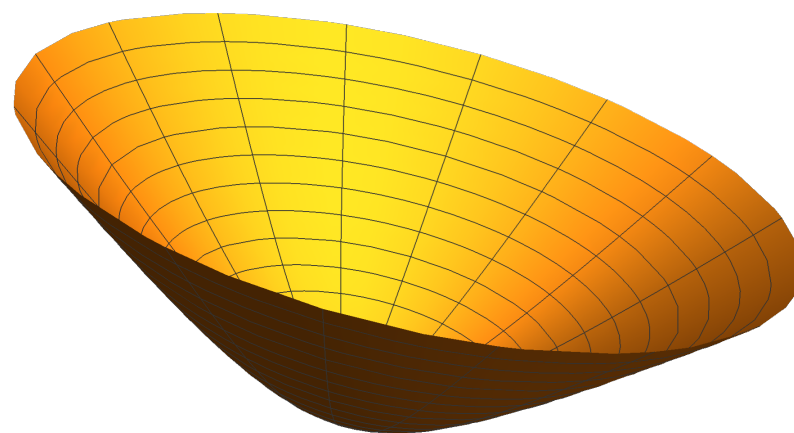
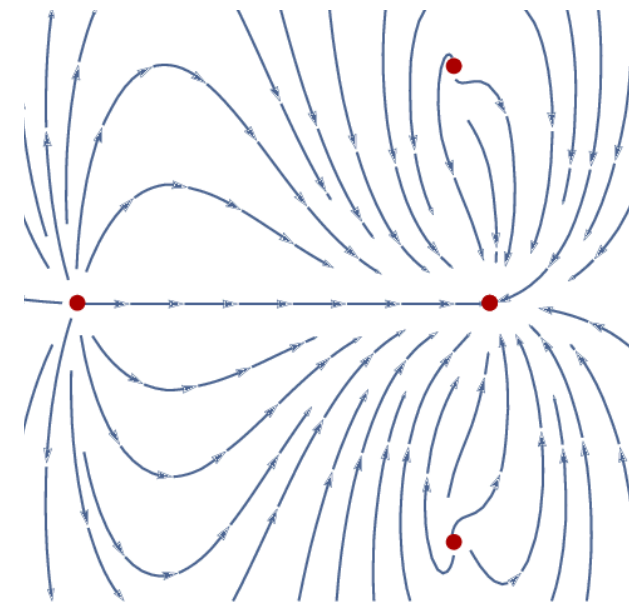
vacuum + Goldstone + $1/Q$ -suppressed corrections

Working at large charge Q **always** leads to simplifications. For hard problems, large charge may however not be enough (combine with other limits, etc.)

Introduction

Conformal field theories (CFTs) play an important role in theoretical physics:

- fixed points in RG flows
- critical phenomena
- quantum gravity (via AdS/CFT)
- string theory (VWS theory)



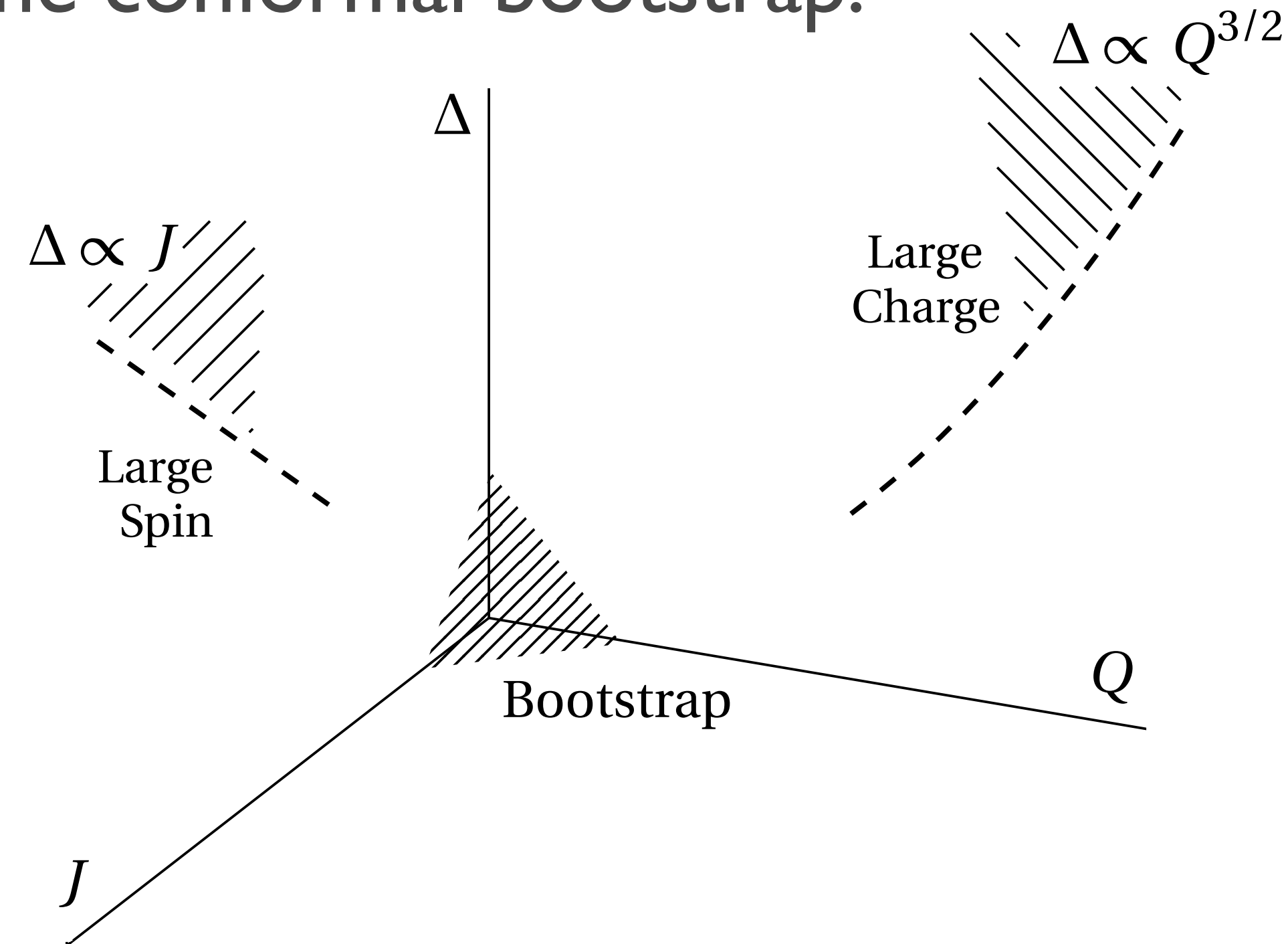
But: CFTs **do not have any intrinsic scales**, most have by naturalness couplings of $O(1)$.

Possibilities: analytic (2d), conformal bootstrap ($d > 2$), lattice calculations, non-perturbative methods...

Prime candidate for the large-charge approach.

Introduction

The large charge expansion is **complementary** to other CFT approaches like large spin and the conformal bootstrap:

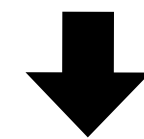


Both bootstrap and large spin are based on crossing symmetry

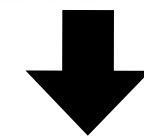
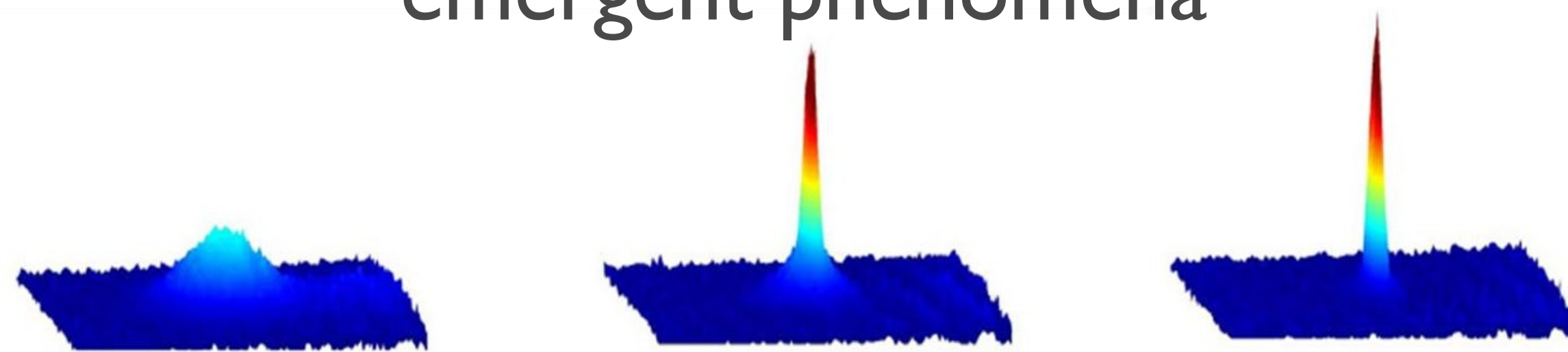
$$\sum_k \begin{array}{c} \mathcal{O}_1 \quad \mathcal{O}_4 \\ \diagdown \quad \diagup \\ \mathcal{O}_k \\ \diagup \quad \diagdown \\ \mathcal{O}_2 \quad \mathcal{O}_3 \end{array} = \sum_k \begin{array}{c} \mathcal{O}_1 \quad \mathcal{O}_4 \\ \diagdown \quad \diagup \\ \mathcal{O}_k \\ \diagup \quad \diagdown \\ \mathcal{O}_2 \quad \mathcal{O}_3 \end{array}$$

Introduction

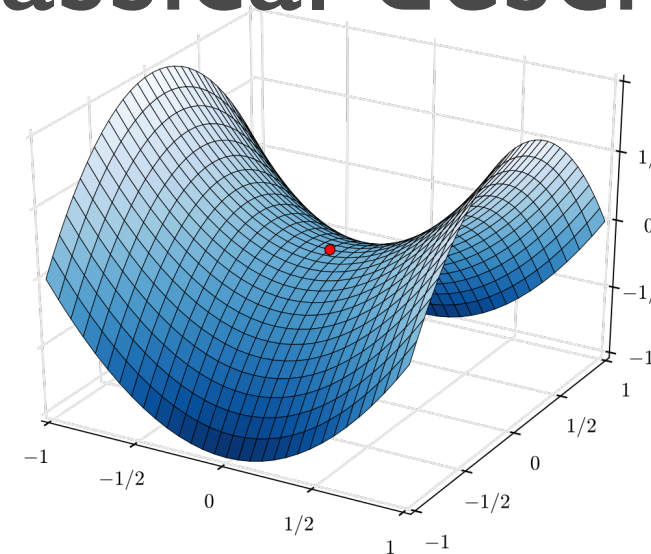
Consider systems with large quantum number
many degrees of freedom



emergent phenomena



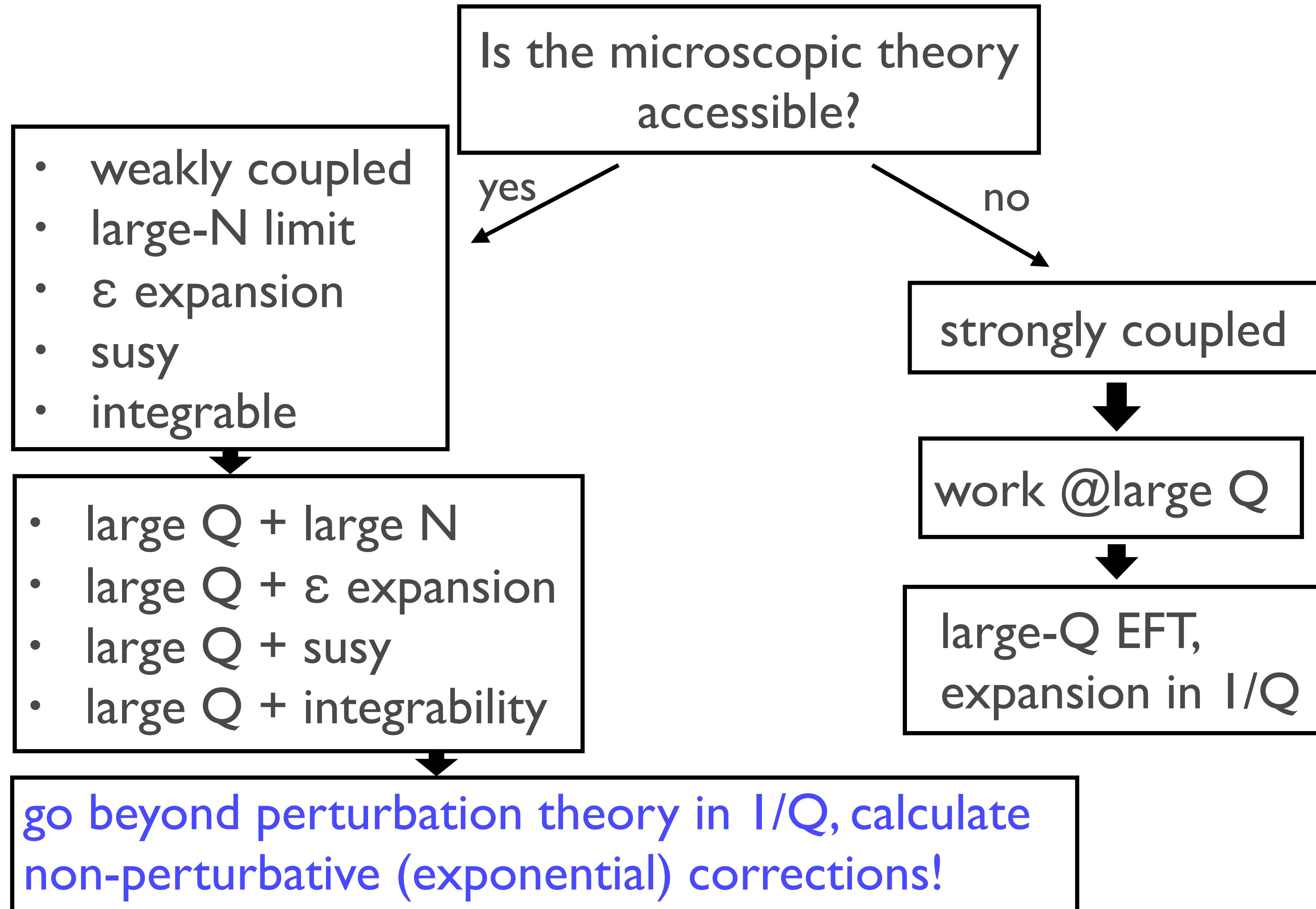
semiclassical description



e.g. superfluid

works especially well for strongly coupled systems!

Introduction



Introduction

There seem to be 2 main categories of behavior for systems at large quantum number:

Superfluid

isolated vacuum

- Wilson-Fisher CFT
- NRCFT (unitary Fermi gas)
- $N=2$ SCFT in 3d
- asymptotically safe model in 4d
- NJL model

EFT of the moduli space

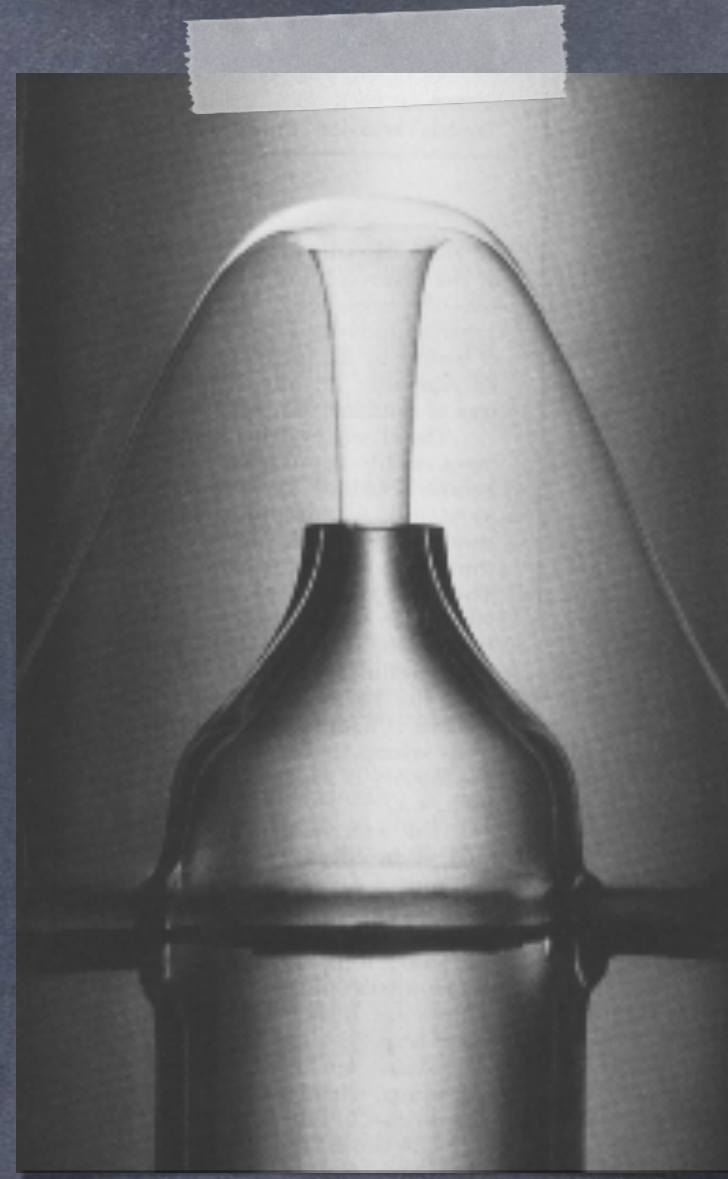
moduli space of vacua

- free boson
- $N=2$ theories in 4d

Introduction

To which models can we **apply** the large Q expansion?

- $O(N)$ vector model in 3D
- NJL in 3D
- non-relativistic CFTs
- integrable models
- SCFTs
- Abelian Higgs model
- ...



The $O(2)$ model

The O(2) model

Simplest example: O(2) model in (2+1) dimensions

$$\mathcal{L}_{UV} = \partial_\mu \phi^* \partial^\mu \phi - g^2 (\phi^* \phi)^2$$

Flows to **Wilson-Fisher fixed point in IR.**

Assume that also the IR DOF are encoded by **cplx scalar**

$$\varphi_{IR} = a e^{i\chi} \quad \text{Global U(1) symmetry: } \chi \rightarrow \chi + \text{const.}$$

Look at scales: put system in box (2-sphere) of scale R

Second scale given by U(1) charge Q:

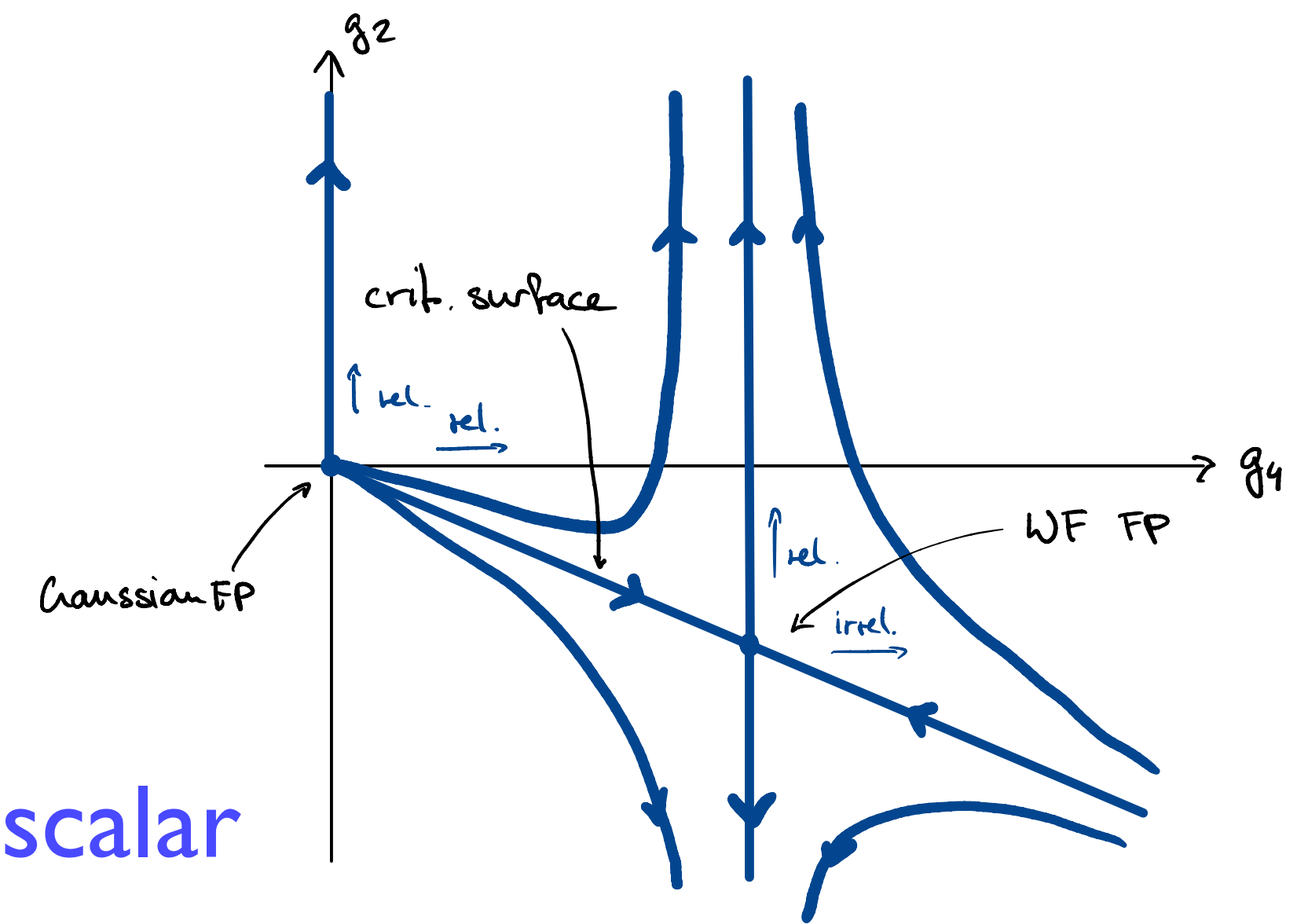
$$\rho^{1/2} \sim Q^{1/2}/R$$

Study the CFT at the fixed point in a sector with

$$\frac{1}{R} \ll \Lambda \ll \frac{Q^{1/2}}{R} \ll g^2$$

UV scale

cut-off of effective theory



The $O(2)$ model

Fixing the charge breaks symmetries:

$$SO(3, 2) \times O(2) \rightarrow SO(3) \times D \times O(2) \rightsquigarrow SO(3) \times D'$$

$D' = D - \mu O(2)$ 

Broken $U(1)$ - **superfluid!**

Dynamics is described by a single Goldstone field χ :

$$\mathcal{L}_{LO} = k_{3/2} (\partial_\mu \chi \partial^\mu \chi)^{3/2}$$

 can get this purely by dimensional analysis

Lowest-energy solution: **homogeneous ground state**

$$\chi = \mu t, \quad \text{non-const. vev} \quad \mu \sim \rho^{1/2}$$

Beyond LO: use **dimensional analysis, parity and scale invariance** to determine (tree-level) operators in effective action (Lorentz scalars of scaling dimension 3, including couplings to geometric invariants)

The O(2) model

Use ρ -scaling to determine which terms are not suppressed:

$$\partial\chi \sim \rho^{1/2}, \quad \partial \dots \partial\chi \sim \rho^{-1/4}$$

Result for NLSM action in D=3:

$$\mathcal{L} = k_{3/2}(\partial_\mu\chi\partial^\mu\chi)^{3/2} + k_{1/2}R(\partial_\mu\chi\partial^\mu\chi)^{1/2} + \mathcal{O}(Q^{-1/2})$$

← LO Lagrangian
← curvature coupling

dimensionless parameters
← suppressed by inverse powers of Q

Energy of classical ground state at fixed charge:

$$E_\Sigma(Q) = \frac{c_{3/2}}{\sqrt{V}} Q^{3/2} + \frac{c_{1/2}}{2} R\sqrt{V} Q^{1/2} + \mathcal{O}(Q^{-1/2})$$

2 dimensionless parameters
cannot be calculated within EFT!

dependence on manifold

The $O(2)$ model

Expand action around GS to second order in fields: $\chi = \mu t + \hat{\chi}$

$$\mathcal{L} = k_{3/2}\mu^3 + k_{1/2}R\mu + (\partial_t\hat{\chi})^2 - \frac{1}{2}(\nabla_{S^2}\hat{\chi})^2 + \dots$$

Compute zeros of inverse propagator for fluctuations and get dispersion relation:

$$\omega_{\vec{p}} = \frac{|\vec{p}|}{\sqrt{2}} \leftarrow \text{dictated by conf. invariance } 1/\sqrt{d}$$

$\Rightarrow \chi$ is indeed a “conformal” Goldstone

Are also the **quantum effects** controlled?

Yes! All effects except Casimir energy of χ are suppressed (negative ρ -scaling).

Effective theory at large Q :

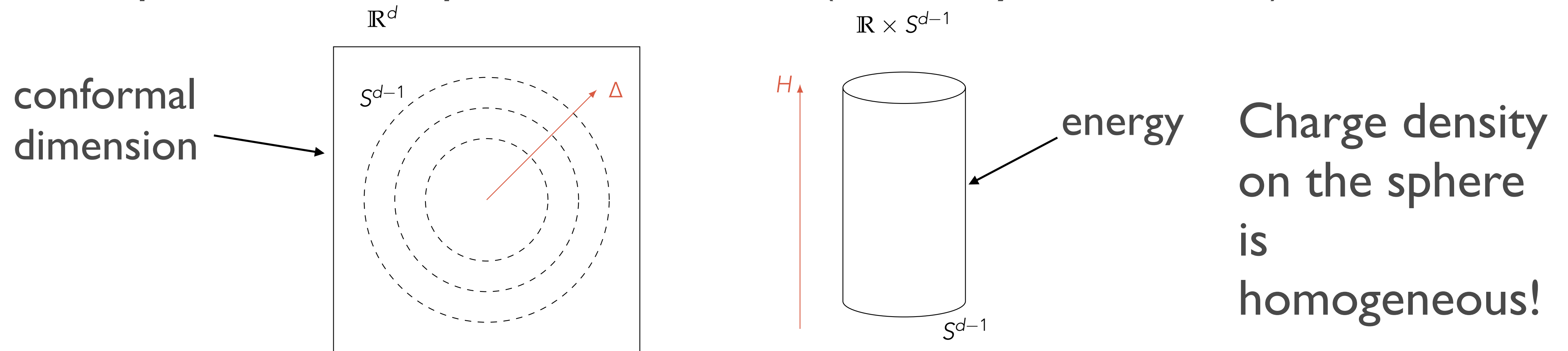
vacuum + Goldstone + $1/Q$ -suppressed corrections

The $O(2)$ model

We're ready to calculate observables:

CFT: **conformal data** (scaling dim. + 3pt coefficients)!

Use state-operator correspondence of CFT (flat vs. cylinder frame):



Scaling dimension of lowest operator of charge Q :

$$D(Q) = R_0(E_0 + E_{Cas}) = c_{3/2} Q^{3/2} + c_{1/2} Q^{1/2} - 0.0937 \cdots + \mathcal{O}(Q^{-1/2})$$

Annotations for the equation:

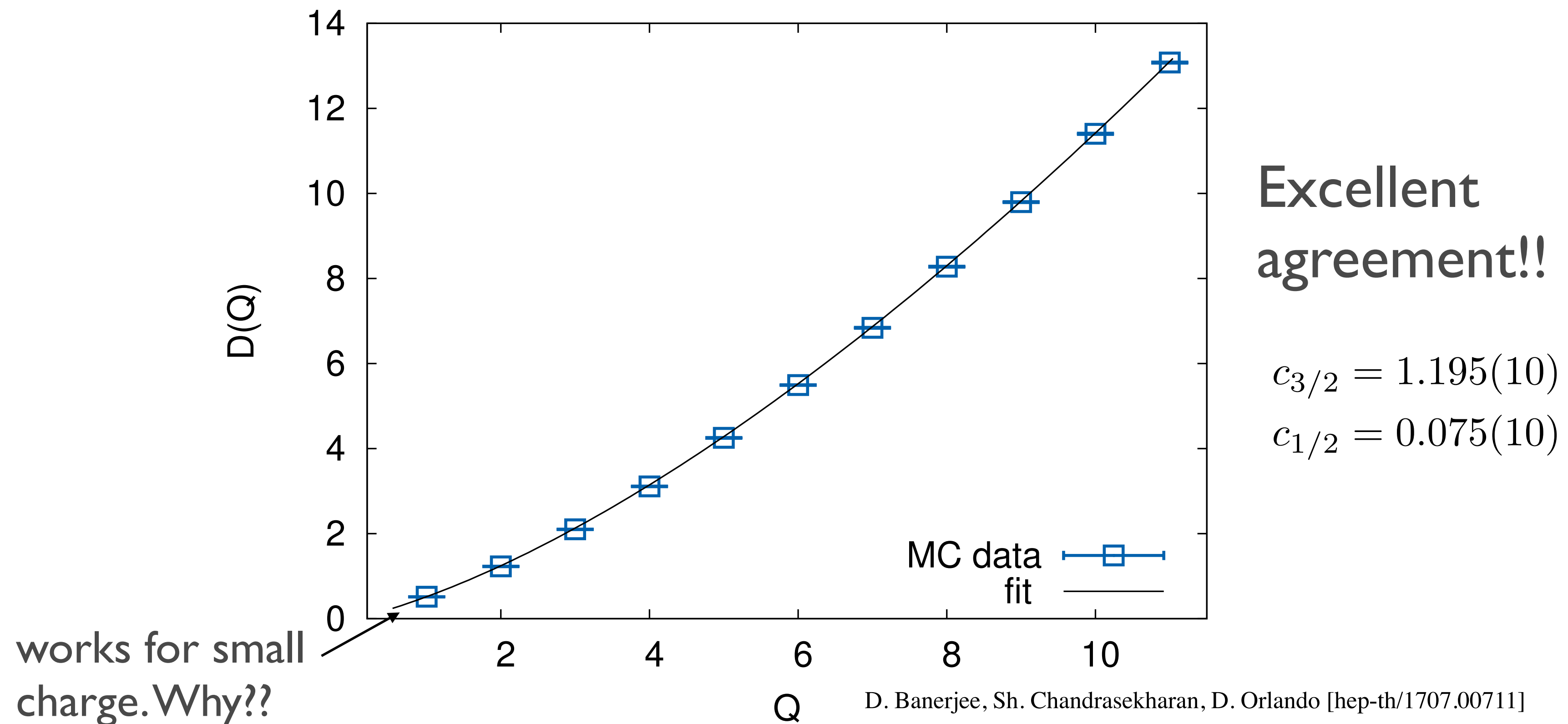
- An arrow points from "energy of class. ground state" to E_0 .
- An arrow points from "quantum correction from Casimir energy of Goldstone" to E_{Cas} .

The $O(2)$ model

Testing our prediction:

$$D(Q) = \frac{c_{3/2}}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi} c_{1/2} Q^{1/2} - 0.094 + \mathcal{O}(Q^{-1/2})$$

Independent calculation on the lattice:



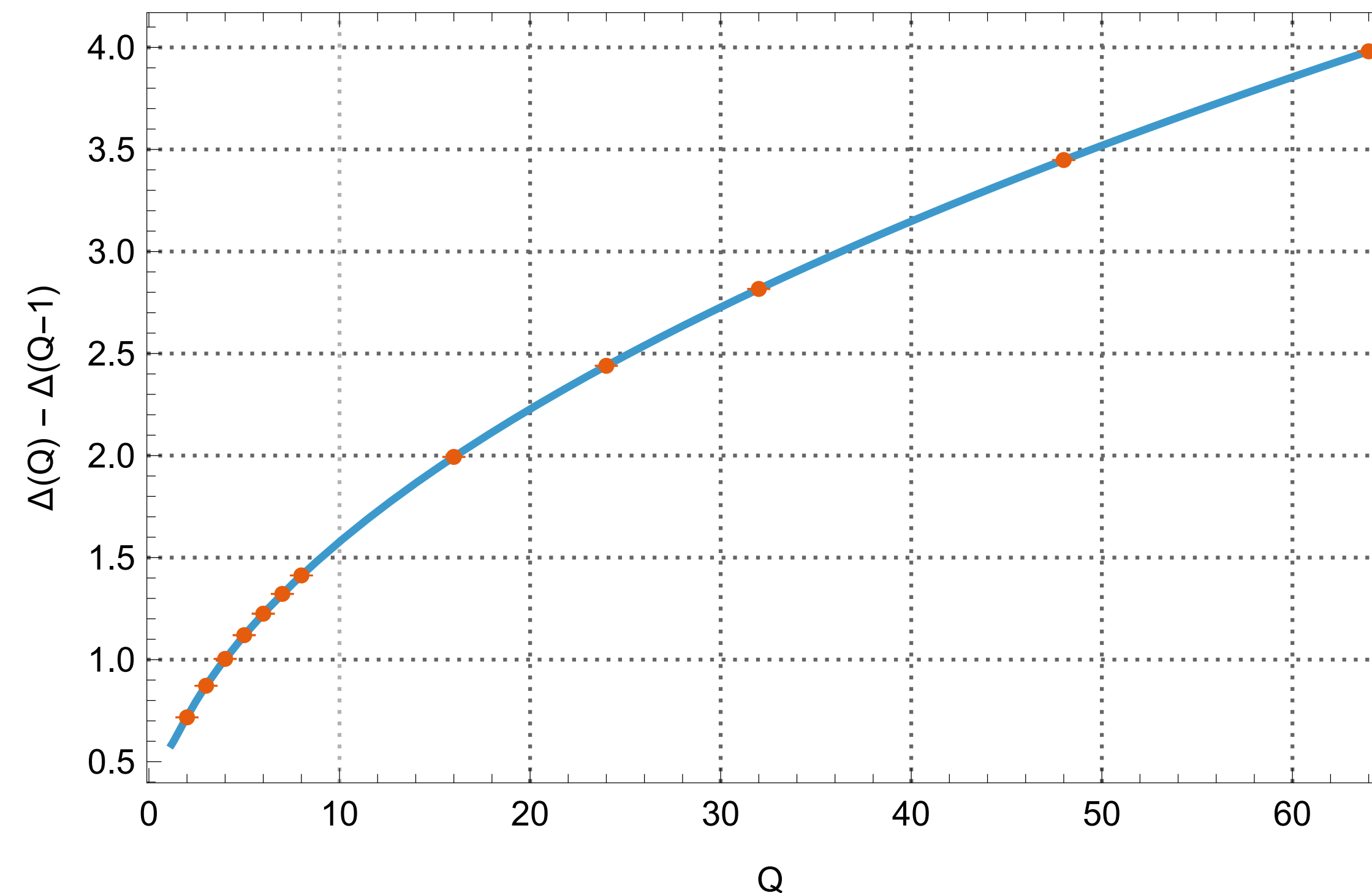
Large-charge expansion works extremely well for $O(2)$.

The O(2) model

Testing our prediction:

$$D(Q) = \frac{c_{3/2}}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi} c_{1/2} Q^{1/2} - 0.094 + \mathcal{O}(Q^{-1/2})$$

Comparing with new, better lattice results:



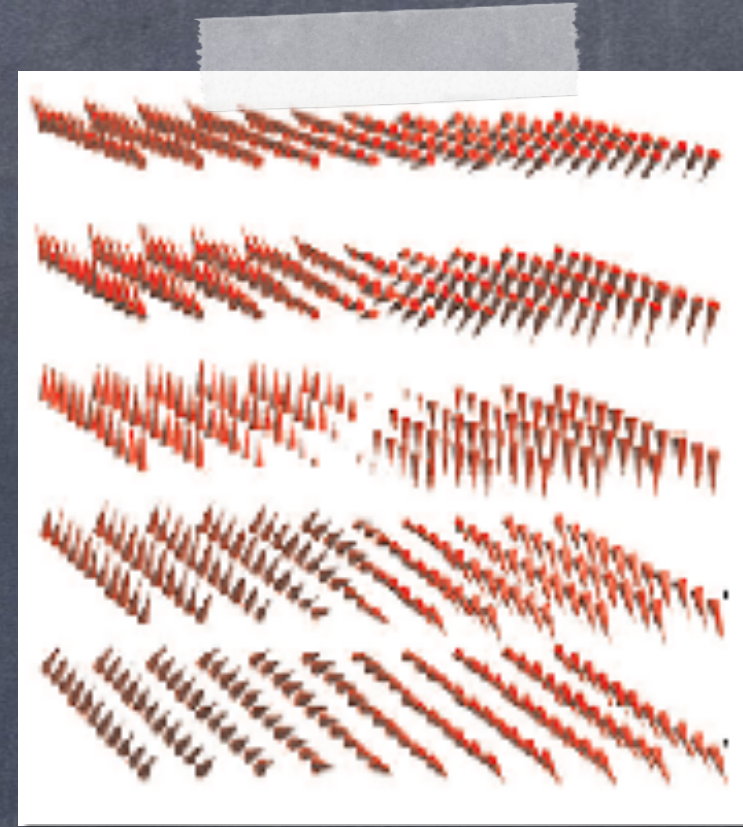
$$c_{3/2} = 0.331623(25)$$

$$c_{1/2} = 0.27670(62)$$

Also confirms the
Casimir energy!

Hasenbusch, [2510.19473](#)

Large-charge expansion works extremely well for O(2).



Beyond $O(2)$:
3d $O(2N)$ vector model

Beyond $O(2)$

Where else can we apply the large-charge expansion?

Obvious generalization in 3d: $O(2N)$ vector model

non-Abelian global symmetry group: new effects

Different symmetry breaking patterns possible, inhomogeneous ground states possible.

Homogeneous case: same form of ground state,

$$SO(3, 2) \times O(2N) \rightarrow SO(3) \times D \times U(N) \rightarrow SO(3) \times D' \times U(N - 1)$$

We expect $\dim[U(N)/U(N-1)] = 2N-1$ Goldstone d.o.f.

On top of the conformal Goldstone of $O(2)$, a new sector with $N-1$ non-relativistic type II Goldstones and $N-1$ massive modes with $m=2\mu$ appears.

The $O(2N)$ vector model

Dispersion relation:

$$\omega = \frac{p^2}{2\mu} + \mathcal{O}\mu^{-3}$$

The non-relativistic Goldstones **count double**.

Nielsen and Chadha; Murayama and Watanabe

Counting type I and type II modes, indeed,

$$1 + 2(N - 1) = 2N - 1 = \dim(U(N)/U(N - 1))$$

Non-relativistic Goldstones contribute to the conformal dimensions only at higher order.

The ground-state energy is again determined by a **single relativistic Goldstone!**

Same formula for scaling dimensions as for $O(2)$:

$$D(Q) = \frac{c_{3/2}}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi} c_{1/2} Q^{1/2} - 0.094 + \mathcal{O}(Q^{-1/2})$$

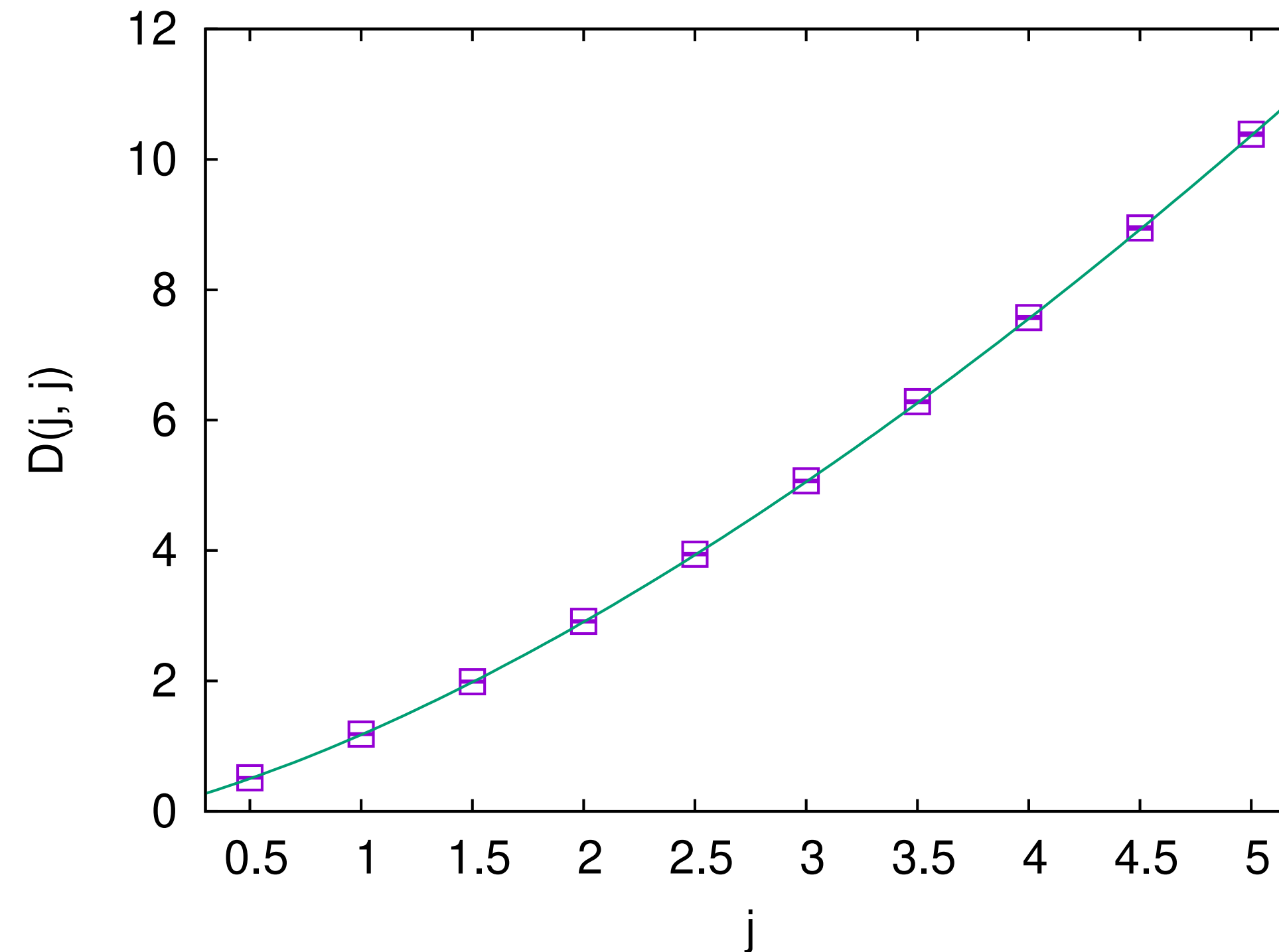
$\swarrow$ N-dependent $\swarrow$ universal for $O(2N)$
 $\nwarrow$ verified at large N for CP(N-1) model de la Fuente

The $O(2N)$ vector model

Testing our prediction:

$$D(Q) = \frac{c_{3/2}}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi} c_{1/2} Q^{1/2} - 0.094 + \mathcal{O}(Q^{-1/2})$$

Lattice data for $O(4)$ model:



$$c_{3/2} = 1.068(4)$$
$$c_{1/2} = 0.083(3)$$

D. Banerjee, Sh. Chandrasekharan, D. Orlando, S.R. 1902.09542

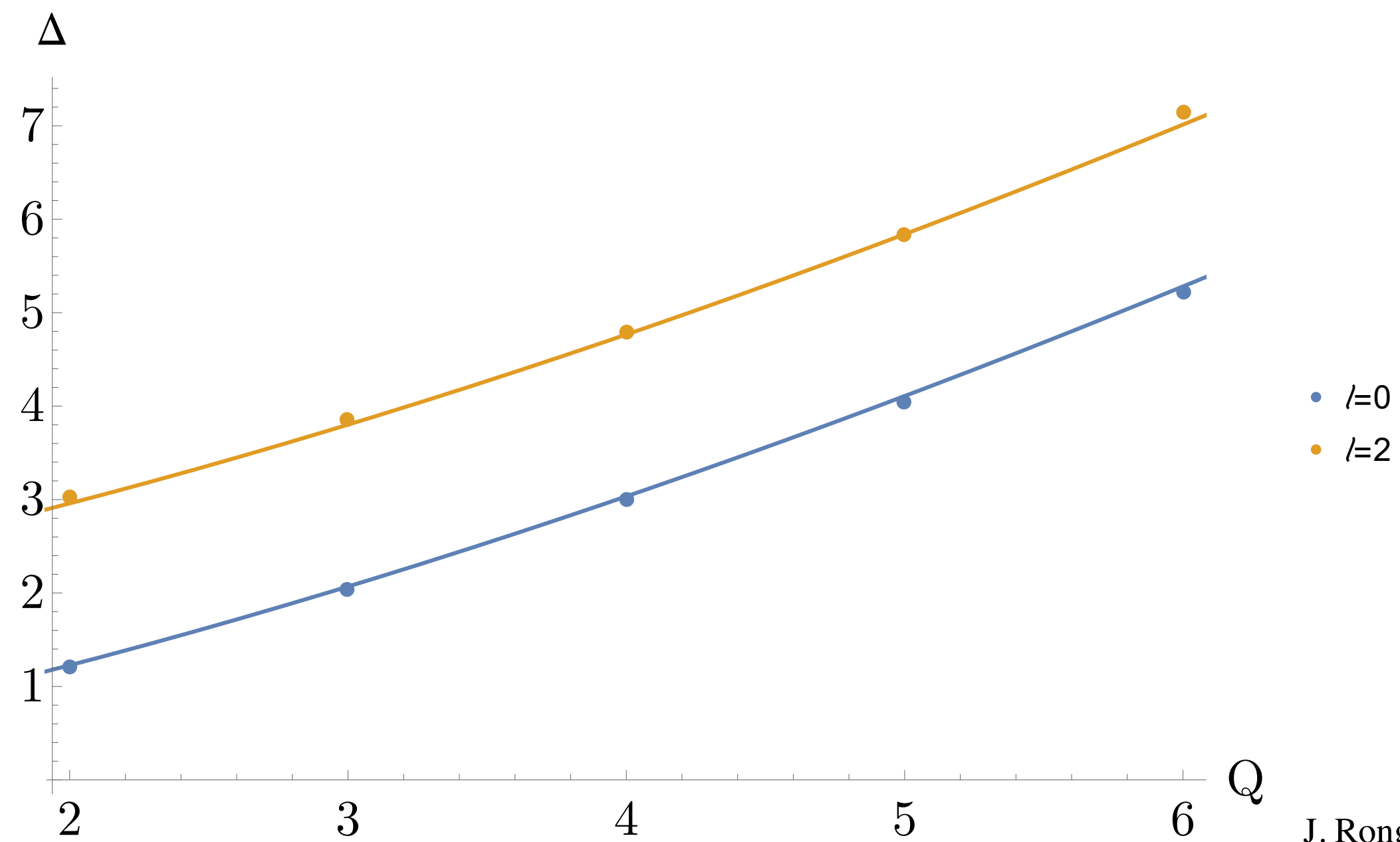
Again excellent agreement with large- Q prediction!

The $O(2N)$ vector model

Testing our prediction:

$$D(Q) = \frac{c_{3/2}}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi} c_{1/2} Q^{1/2} - 0.094 + \mathcal{O}(Q^{-1/2})$$

Numerical bootstrap data for $O(3)$ model:



J. Rong, N. Su, 2311.00933

Again excellent agreement with large- Q prediction!

The large-N limit

Standard large-N methods, expand path integral at fixed charge around saddle point (no EFT!)

Extra control parameter at large N: can go further!

Start in the UV with

$$S[\phi_i] = \sum_{i=1}^N \int dt d\Sigma \left[g^{\mu\nu} (\partial_\mu^i \phi_i)^\dagger (\partial_\nu^i \phi_i) + r(\phi_i^\dagger \phi_i) + \frac{u}{2N} (\phi_i^\dagger \phi_i)^2 \right]$$

For $r=R/8$, this flows to the WF fixed pt in the IR, $u \rightarrow \infty$

Scaling dimension for $Q/N \gg 1$:

$$\frac{\Delta(Q)}{2N} = \frac{2}{3} \left(\frac{Q}{2N} \right)^{3/2} + \frac{1}{6} \left(\frac{Q}{2N} \right)^{1/2} - \frac{7}{720} \left(\frac{Q}{2N} \right)^{-1/2} - \frac{71}{181440} \left(\frac{Q}{2N} \right)^{-3/2} + \dots$$

same Q-scaling as in EFT

L. Alvarez-Gaume, D. Orlando, S.R. 1909.02571

Small charge limit, $Q/N \ll 1$:

$$\frac{\Delta(Q)}{Q} = \frac{1}{2} + \frac{4}{\pi^2} \frac{Q}{N} + \mathcal{O} \left(\frac{Q}{2N} \right)^2$$

engineering dimension of ϕ

In this limit, the operator of charge Q is φ^Q

The large-N limit

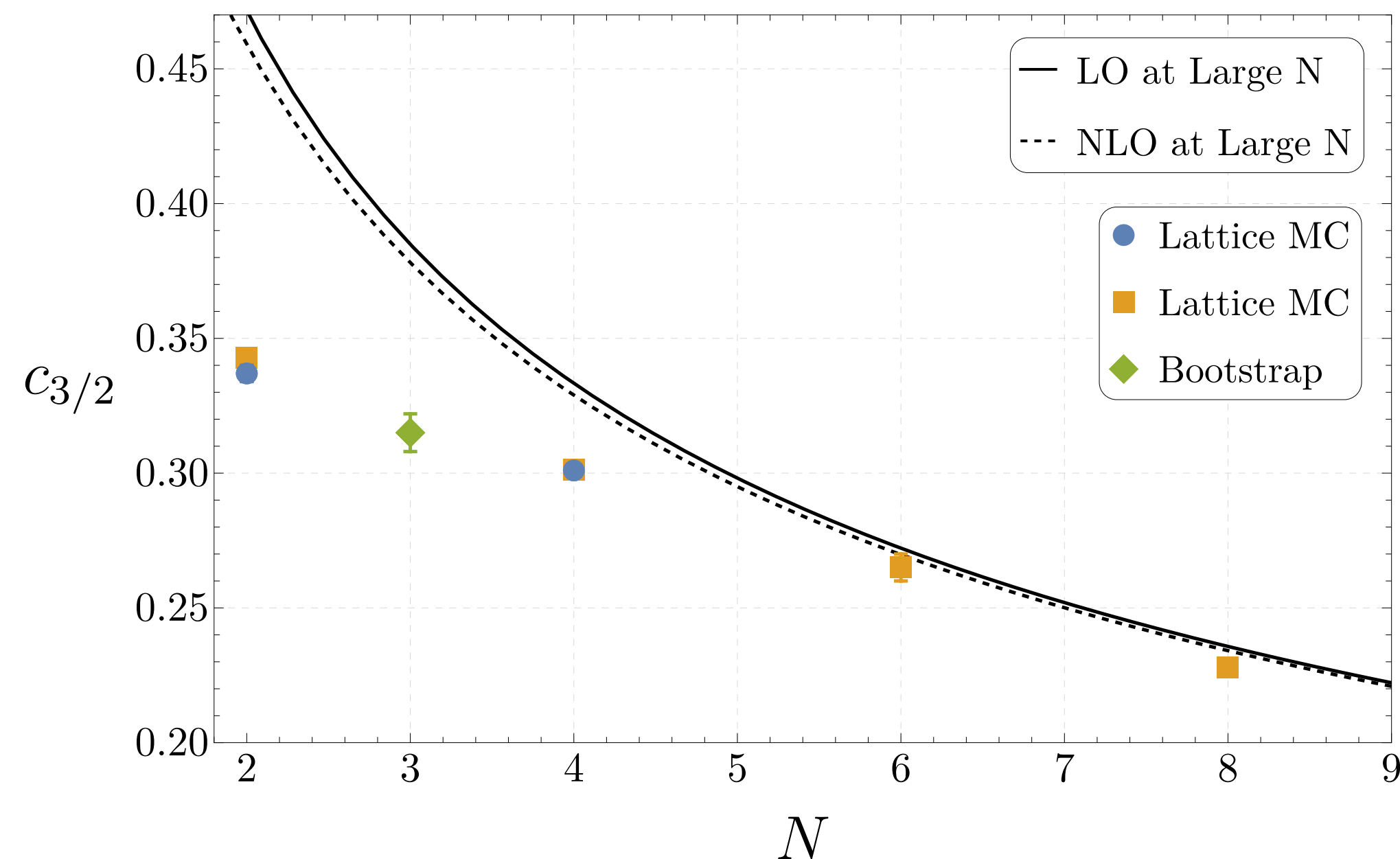
NLO in N: reproduce dispersion relations of Goldstones.

Find **coefficients of the expansion** (leading order in N):

$$c_{3/2} = \frac{1}{3} \sqrt{\frac{2}{N}}$$

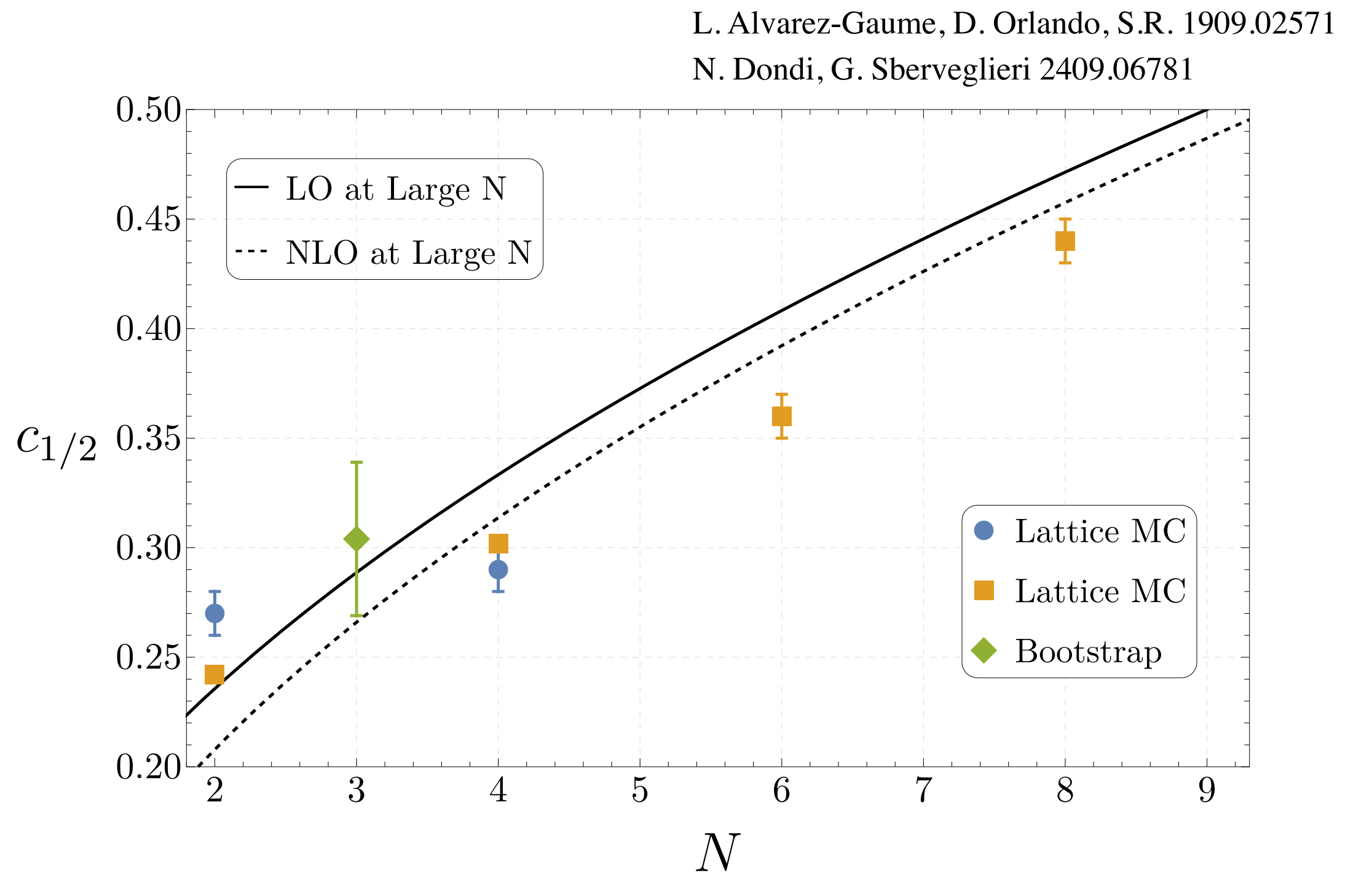
$$c_{1/2} = \frac{1}{3} \sqrt{\frac{N}{2}}$$

Comparison of results:



D. Banerjee, Sh. Chandrasekharan, D. Orlando, S.R. 1902.09542

Singh, arXiv:2203.00059 [hep-lat]



L. Alvarez-Gaume, D. Orlando, S.R. 1909.02571
N. Dondi, G. Sberveglieri 2409.06781

J. Rong, N. Su, 2311.00933

Resurgence analysis

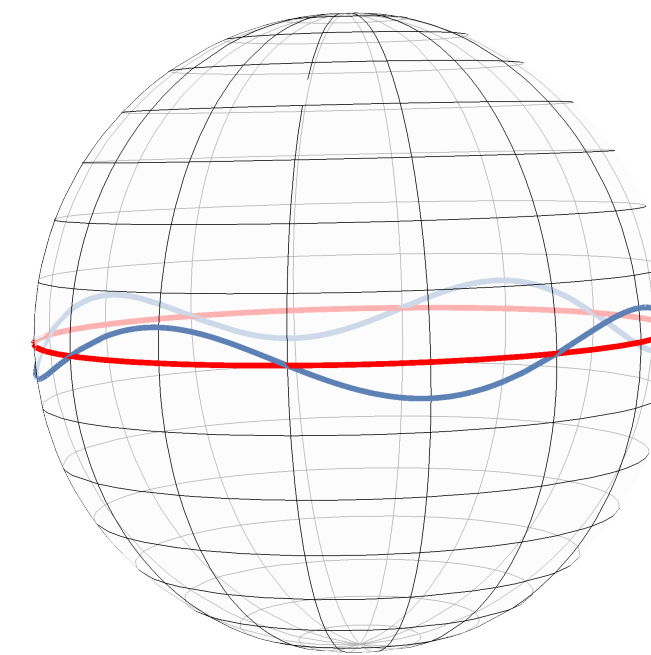
Since we can compute all the coefficients of the large- Q expansion, we can do a resurgence analysis to relate the large and small-charge regimes.

Asymptotic series which diverges as $(2L)!$

We can write the transseries. Find **non-perturbative corrections**:

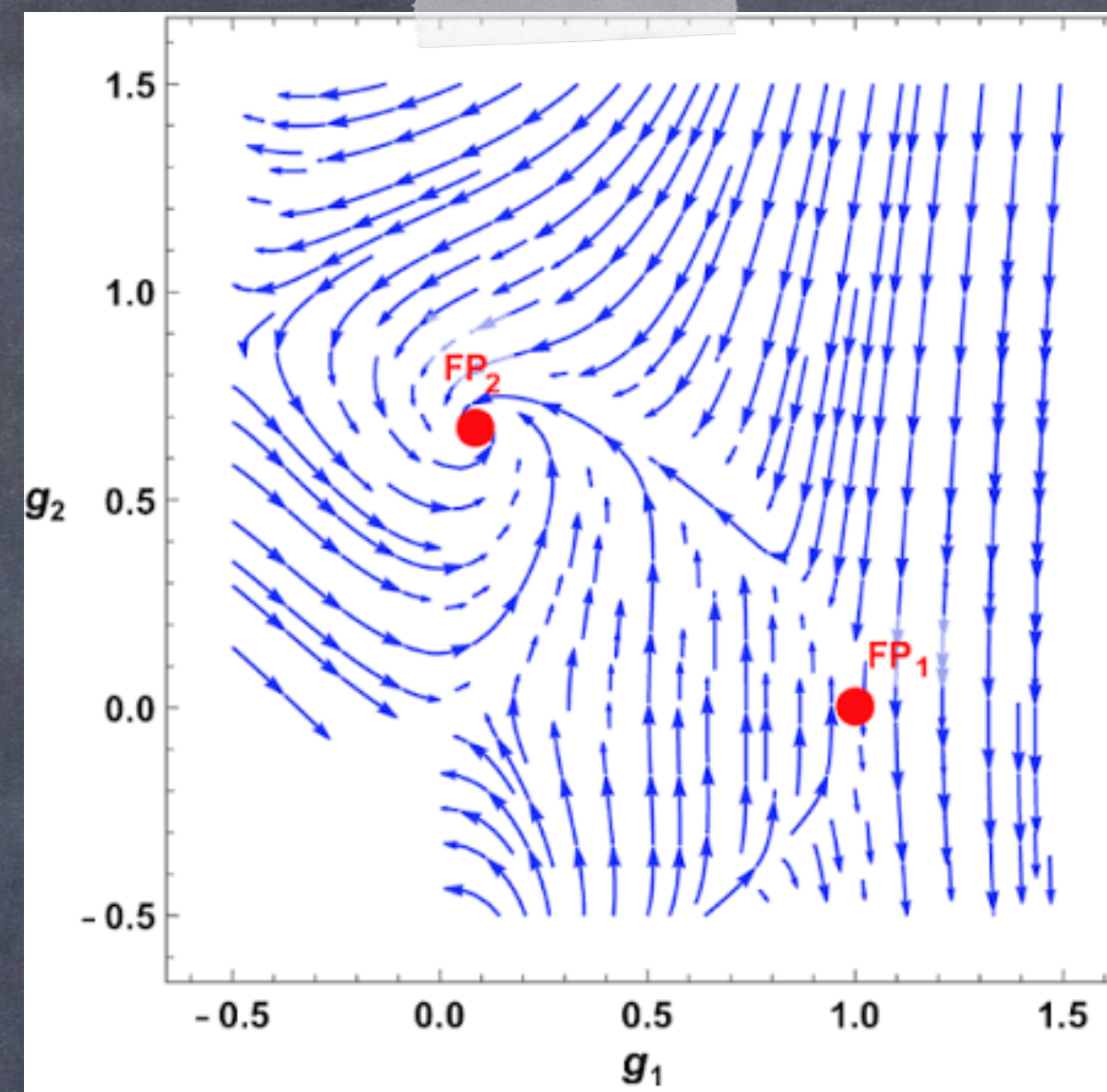
$$e^{-2\pi k \sqrt{Q/(2N)}}$$

Geometric interpretation: particles of mass μ propagating on the equator of the 2-sphere.



CFT + resurgence: This picture must work for any N !

The **optimal truncation** is $\mathcal{O}(\sqrt{Q})$ terms. This explains why the comparison to the lattice calculation works so well.



Extension to fermionic
and gauge theories

Fermions@large Q

Will large Q work for fermionic models?

Antipin, Bersini, Panopoulos;

Let's start with the multicomponent **Nambu-Jona-Lasinio (NJL)** model, also known as the **chiral Gross-Neveu (GN)** model in 3D:

$$S_{\text{cGN}} = - \int d^3x \left[\bar{\psi}_a i \not{\partial} \psi_a + \frac{g}{2N} \left((\bar{\psi}_a \psi_a)^2 + (\bar{\psi}_a i \gamma_5 \psi_a)^2 \right) \right]$$

There are two conserved currents:

$$j^\mu = \bar{\psi} \gamma^\mu \psi, \quad j^{5\mu} = \bar{\psi} \gamma^\mu \gamma^5 \psi$$

We can study this model at large N with standard methods.

We find that **only the axial charge** gives rise to a condensate at criticality.

Scaling dimension:

$$\begin{aligned} \frac{\Delta}{N} &= \overset{\text{large } Q/N}{\frac{\sqrt{2}}{3} \left(\frac{Q}{\kappa N} \right)^{3/2}} + \frac{1}{3\sqrt{2}} \left(\frac{Q}{\kappa N} \right)^{1/2} + \dots \\ &\overset{\text{small } Q/N}{=} \frac{1}{2} \frac{Q}{N} + \frac{1}{\pi^2} \left(\frac{Q}{N} \right)^2 + \dots \end{aligned}$$

Dondi, Hellerman, Kalogerakis, Moser, Orlando, S.R.,
2211.15318

Fermions@large Q

Like for the scalar case, we get a **condensate at fixed charge**, but not WF universality class.

Can go to a different frame using the Pauli-Gürsey transformation:

$$\psi_a \mapsto \frac{1}{2}(1 - \gamma^5)\psi_a + \frac{1}{2}(1 + \gamma^5)C\bar{\psi}_a^T$$
$$S_{\text{BCS}} = - \int d^3x \left[\bar{\psi}_a i \not{\partial} \psi_a + \frac{g}{2N} (\bar{\psi}_a C \bar{\psi}_a^T) (\psi_b^T C \psi_b) \right]$$

This model gives rise to superconductivity from Cooper pair formation!

The condensate consists of **Cooper pairs - superconductor!**

The end result is similar to the scalar case in the sense that we have an EFT in terms of Goldstones fluctuating around a condensate.

Semiclassical description for cusped Wilson lines

Consider two Wilson lines meeting in a cusp in the critical Abelian Higgs model in $d=4-\varepsilon$ dimensions: $\lambda^* \sim \varepsilon$, $e^* \sim \sqrt{\varepsilon}$

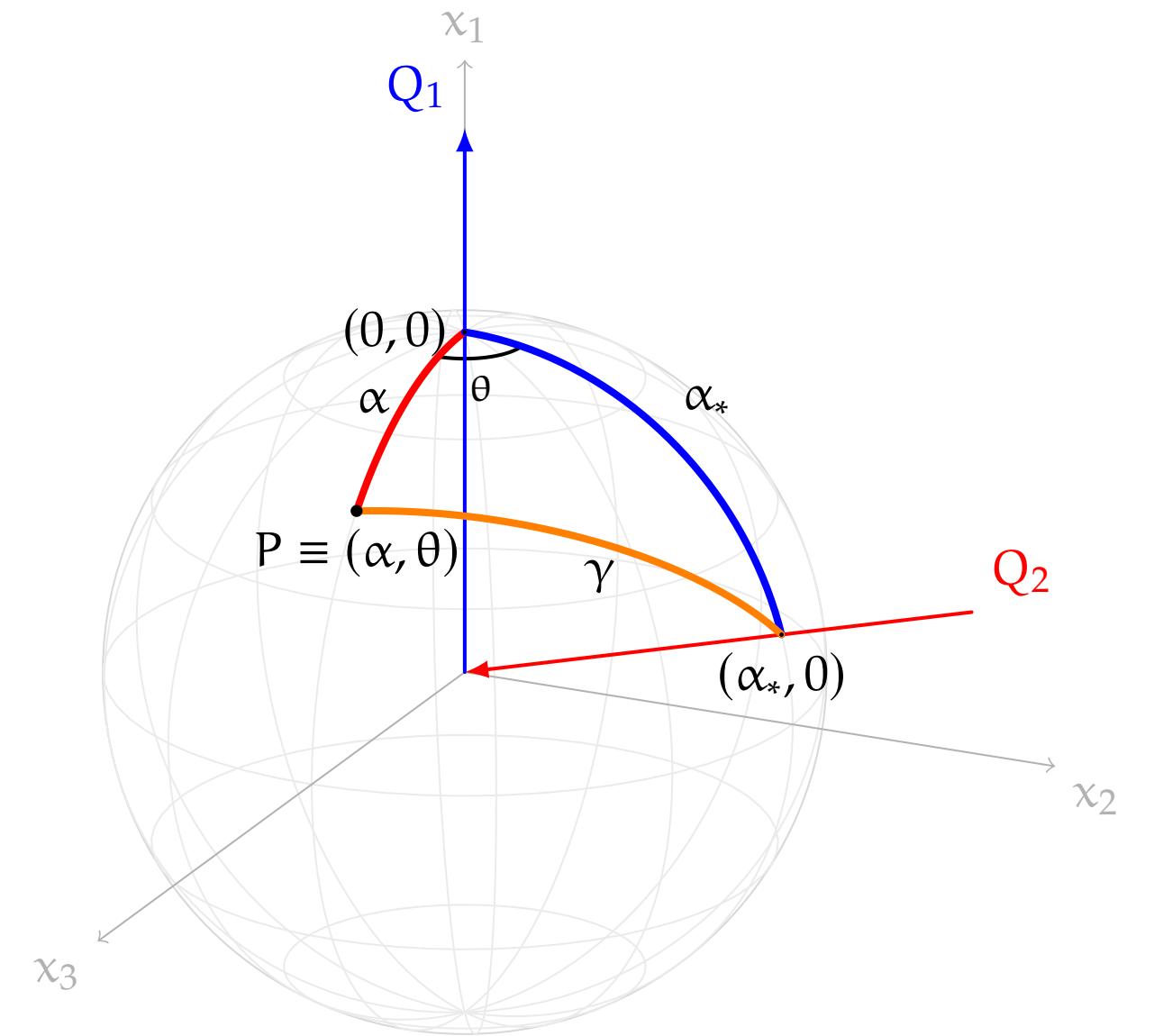
$$S_{\text{AH}}[A, \phi] = \int d^4x \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D^\mu \phi_a)^* D_\mu \phi_a + \frac{(4\pi)^2}{6} \lambda (\phi_a^* \phi_a)^2 \right]$$

These lines can also carry charges.

Wilson line as trajectory of a heavy particle: cusps as sudden change in velocity.

The cusp anomalous dimension measures the strength of the divergence produced by a cusp in a Wilson line:

$$\log \left\langle \begin{array}{c} \text{blue arrow } Q_1 \\ \text{red arrow } Q_2 \end{array} \right\rangle = -\Gamma_{Q_1 Q_2}(\alpha_*) \log \frac{L}{a} + \dots$$



Semiclassical description for cusped Wilson lines

At $\alpha_* = \pi$, the cusp becomes a straight line with a charge-changing insertion, so Γ gives the scaling dimension of the lowest defect-changing operator.

In the double-scaling limit

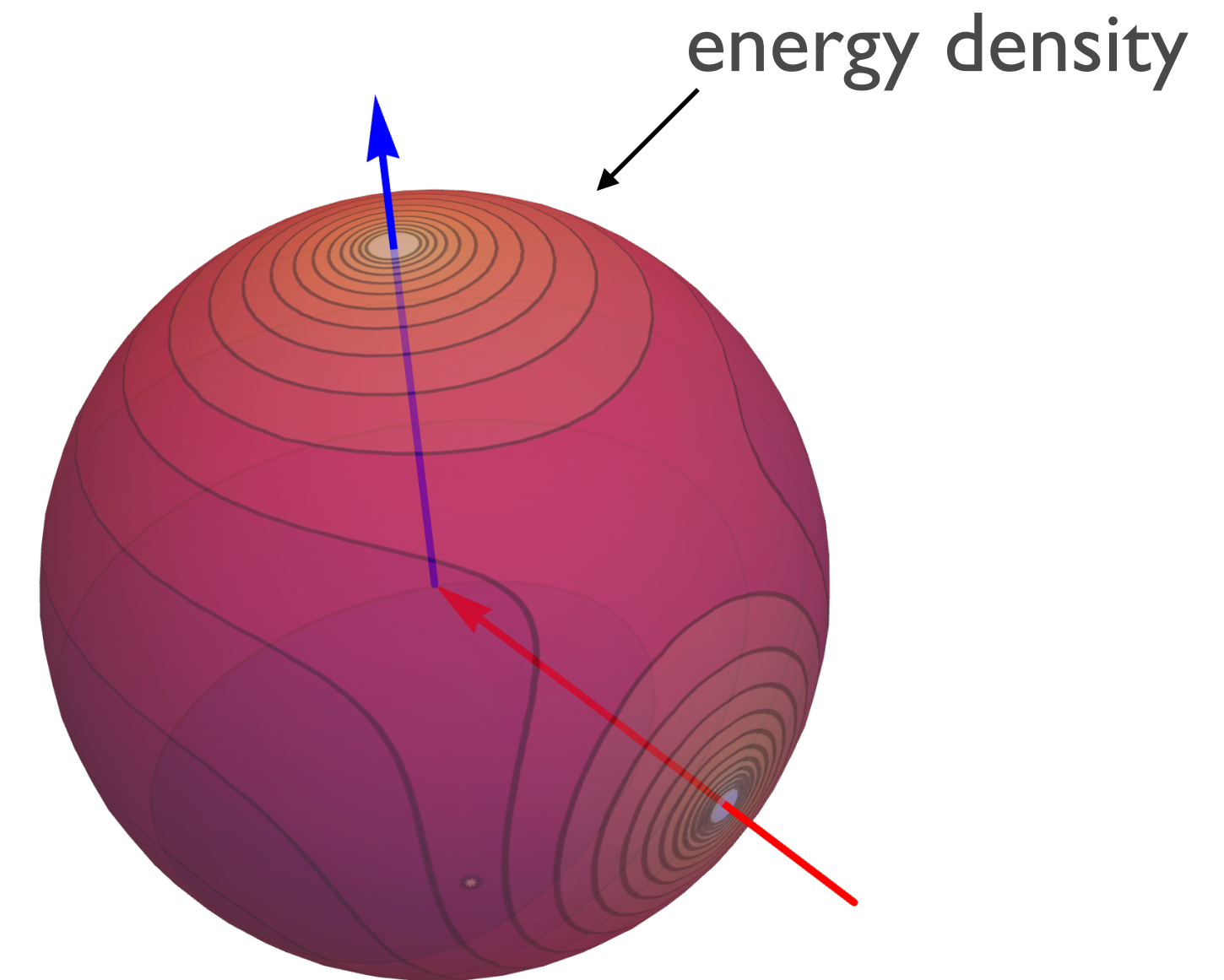
$$Q \rightarrow \infty, \epsilon \rightarrow 0 \quad Q\epsilon \text{ fixed}$$

the problem becomes semi-classical.

Main result (expansion in Q and G):

$$\Gamma_{q_1 q_2}(\alpha_*) = Q \left[\Gamma_{q_1 q_2}^{(-1,0)} - \frac{3(q_1 - q_2)^2 + 4q_1 q_2 (1 + (\pi - \alpha_*) \cot(\alpha_*))}{16\pi^2} G + \Gamma_{q_1 q_2}^{(-1,2)}(\alpha_*) G^2 + \mathcal{O}(G^3) \right] + \Gamma_{q_1 q_2}^{(0,0)} + \mathcal{O}(Q^0 G, 1/Q)$$

$$q_1 = \frac{Q_1}{Q}, \quad q_2 = \frac{Q_2}{Q}, \quad G = Qe^2, \quad \kappa = \frac{(4\pi)^2}{6} Q\lambda$$





Summary

Summary

Concrete examples where a strongly-coupled CFT simplifies significantly at large charge.

$O(2N)$ model in 3d: in the limit of large $U(1)$ charge Q , we computed the conformal dimensions in a controlled perturbative expansion:

- Excellent agreement with lattice results for $O(2)$, $O(4)$
- large Q and large N : path integral at saddle pt., more control than in EFT, can calculate coefficients
- can follow the flow away from conformal point, find the full effective potential

NJL model: similar results, condensate due to Cooper pairs.

GN model at large N : condensate suppressed at large N .

Problem of cusped Wilson line becomes semiclassical in double-scaling limit.

Further directions

- Further study of supersymmetric models at large R-charge (higher-dim. moduli spaces)

Hellerman, Maeda, Orlando, Reffert, Watanabe;
Argyres et al.

- Connection to holography (gravity duals)

Loukas, Orlando, Reffert, Sarkar;
De la Fuente, Zosso;
Giombi, Komatsu, Offertaler;
Perlmutter et al.

- Operators with spin; connection to large-spin results

Cuomo, de la Fuente, Monin, Pirtskhalava, Rattazzi; Cuomo

- Use/check large-charge results in conformal bootstrap

Jafferis and Zhiboedov; Rong and Su

- Further lattice simulations: inhomogeneous sector, general $O(N)$

Chandrasekharan et al.;
Singh

- CFTs in other dimensions (2, 5, 6)

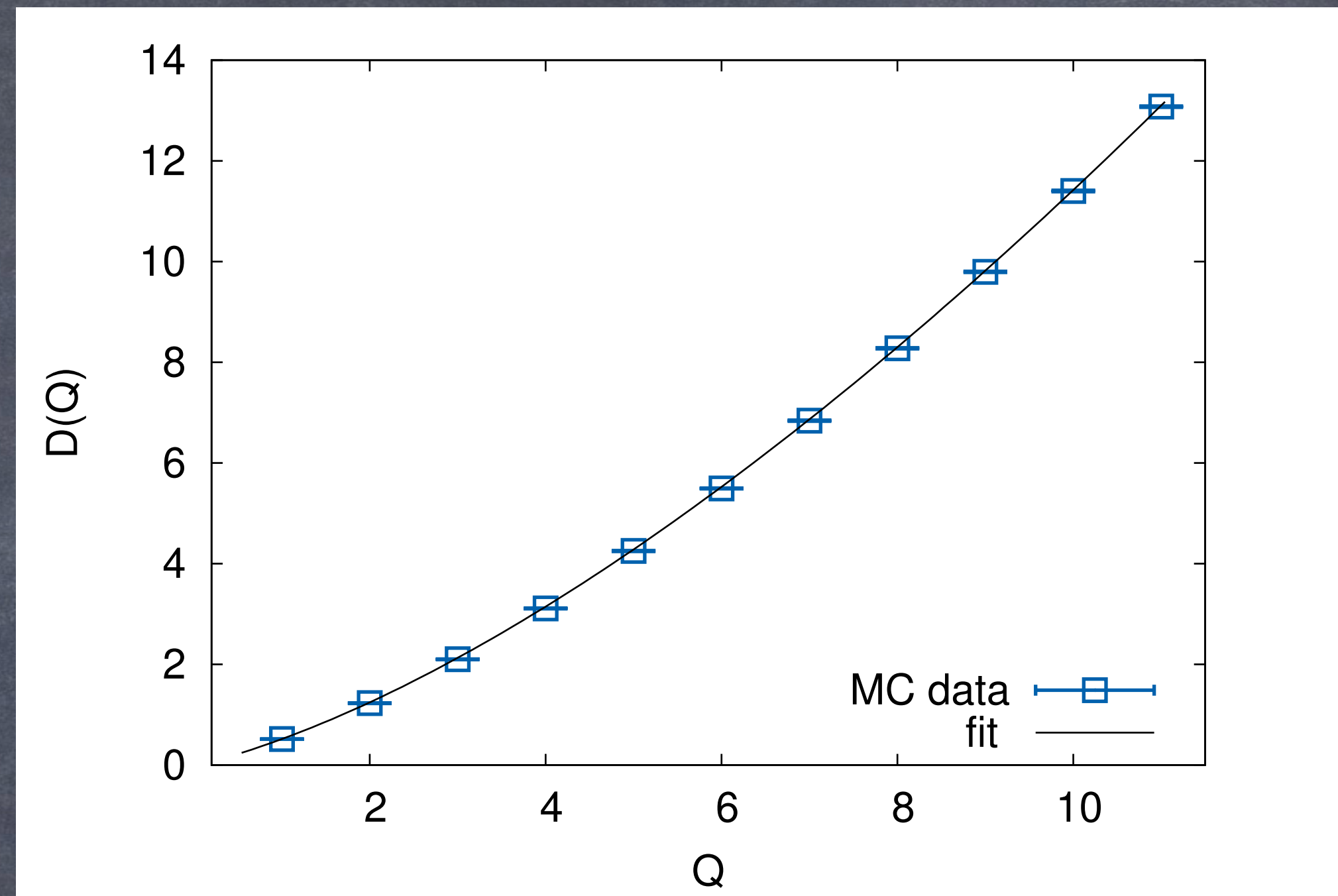
Komargodski, Mezei, Pal, Raviv-Moshe;
Araujo, Celikbas, Reffert, Orlando;
Moser, Orlando, Reffert

- Integrability and large Q

Dodelson, Hellerman, Watanabe, Yamazaki;
Ashwinkumar, Orlando, S.R., Sberveglieri,
2501.18458

Further directions

- Chern-Simons matter theories @large charge
Watanabe
- $4-\varepsilon$ expansion @large charge
Arias-Tamargo, Rodriguez-Gomez, Russo;
Badel, Cuomo, Monin, Rattazzi; Watanabe;
Antipin et al.
- going away from the conformal point
Orlando, Reffert, Sannino;
Orlando, Pellizzani, Reffert
- non-relativistic CFTs
Favrod, Orlando, Reffert; Kravec, Pal;
Orlando, Pellizzani, Reffert;
Hellerman, Swanson; Pellizzani
- Boundary CFTs at large Q
Cuomo, Mezei, Raviv-Moshe
- Swampland, weak gravity conjecture
Aharony, Palti; Antipin et al.
Orlando, Palti
- Study fermionic theories. Can large-charge approach be used for QCD
(e.g. large baryon number)?
Komargodski, Mezei, Pal, Raviv-Moshe;
Antipin, Bersini, Panopoulos;
Dondi, Hellerman, Kalogerakis, Moser, Orlando, Reffert;
- Gauge theories @large charge, cusped Wilson lines, Standard Model
Antipin, Bersini, Sannino et al.
Bersini, Orlando, S.R., Woods



Thank you for your
attention!