Wavefunction of the universe from a Lorentzian path integral in three-dimensional gravity

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Motivation and historical overview

The wavefunction of the universe

- A quantum theory of gravity would treat the universe in its entirety as a quantum system.

  $\Rightarrow$ It must be described by a wavefunction, encoding information about its evolution, as well as its boundary conditions.

- Two ways to determine the wavefunction of the universe (WfU)

$$\left.\begin{array}{l} \text{1. Wheeler–DeWitt equation} \\ \text{2. Gravitational path integral} \end{array}\right\} \text{\& boundary conditions}$$

Motivation and historical overview

Proposals for the boundary conditions

- Most dominant proposals:
  - Hartle-Hawking: no-boundary proposal
  - Vilenkin: quantum tunneling proposal

- Ψ in both proposals can be framed as Euclidean path integrals over compact geometries from "nothing".

- The Euclidean path integral representation of the two proposals can be obtained via analytic continuation from Lorentzian time HH: $t \to -i\tau$ & Linde-V: $t \to +i\tau$
  - Euclidean path integrals are ill-defined
    1. HH: unbounded from below due to kinetic term in the action
    2. LV: unbounded from above due to potential term in the action
  - The contour choices in both proposals are arbitrary.

Motivation and historical overview

Wavefunction as a Lorentzian path integral and Picard-Lefechetz

- Recent work defines the WfU as a Lorentzian path integral [Feldbrugge, Lehners & Turok '17]

$$\Psi[h_1, \phi_1; h_2, \phi_2] = \int_{\substack{(g,\phi)|_{\Sigma_1}=(h_1,\phi_1) \\ (g,\phi)|_{\Sigma_2}=(h_2,\phi_2)}} \mathcal{D}g_{\mu\nu}\mathcal{D}\phi \, e^{iS[g_{\mu\nu},\phi]/\hbar}$$

where the path integral is taken over $g_{\mu\nu}$ and ϕ on $\mathcal{M}$ satisfying the BCs on $\partial\mathcal{M} = \Sigma_1 \cup \Sigma_2$.

  - Utilize Picard-Lefschetz theory to choose contour of integration for absolute convergence.

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The wavefunction of the universe as a 1D integral

Classical minisuperspace model in 3D gravity

- We are interested in a homogeneous and isotropic 3D spacetime

$$\mathrm{d}s^2 = -N^2(x^0)\,(\mathrm{d}x^0)^2 + a^2(x^0)\,\mathrm{d}\Omega_2^2$$

where

  - $N(x^0)$ is the lapse function
  - $a(x^0)$ is the scale factor
  - $\mathrm{d}\Omega_2^2$ is the metric on the unit 2-sphere

- The Einstein-Hilbert action reduces to

$$S[N,a] = 4\pi \int_{x_i^0}^{x_f^0} \mathrm{d}x^0 \Big(-\frac{\dot{a}^2}{N} + N - \Lambda N a^2 \Big)$$

where $a = a(x^0)$, $N = N(x^0)$ and Λ is the cosmological constant.

The wavefunction of the universe as a 1D integral

Defining the wavefunction

- Define the WfU as

$$\Psi(a_0) = \int \frac{\mathcal{D}N}{\mathrm{Vol}(\mathrm{Diff}[-N^2])} \int_{\substack{a(x_i^0)=0 \\ a(x_f^0)=a_0}} \mathcal{D}a \; e^{iS[N,a]/\hbar}$$

with $a(x_i^0) = 0$ and $a(x_f^0) = a_0$.

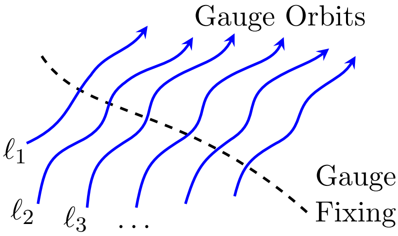
  - ➢ Divide $\mathcal{D}N$ by the Volume of the time-reparametrization group to avoid overcounting physically equivalent configurations.
  - ➢ Interpretation as the probability amplitude for creating a spherical two-dimensional universe of scale factor a_0 from a configuration in which the universe is a vanishing 2-sphere, i.e. "nothing".

The wavefunction of the universe as a 1D integral

Gauge-fixing & the path integral over a

- $S[N, a]$ is invariant under $x^0 \to \xi^0(x^0) = x^0_\xi$
- Given $g_{00}(x^0) \equiv -N^2(x^0)$ on $[x^0_i, x^0_f]$, a diffeomorphism generates a new metric $g_{00}(\xi^0)$, with the proper-time interval preserved

$$\ell = \int_{x^0_i}^{x^0_f} \mathrm{d}x^0 \sqrt{-g^x_{00}(x^0)} = \int_{\xi^0_i=\xi^0(x^0_i)}^{\xi^0_f=\xi^0(x^0_f)} \mathrm{d}\xi^0 \sqrt{-g^\xi_{00}(\xi^0)} \quad \in \mathbb{R}_+$$



The wavefunction of the universe as a 1D integral

Gauge-fixing & the path integral over a

- Choose time coordinate t and lapse function $N(t) = \ell$ on $t \in [0,1]$ [Partouche, Toumbas & de Vaulchier '21]

$$\int \mathcal{D}N \to \int_0^\infty \mathrm{d}\ell \, \Delta_{\mathrm{FP}}(\ell) \int \mathcal{D}(\text{orbit of equivalence class } \ell)$$

where $\Delta_{\mathrm{FP}}(\ell) = 1$ (does not depend on ℓ).

- The WfU reduces to a 1D integral and a path integral over a

$$\Psi(a_0) = \int_0^\infty \mathrm{d}\ell \int_{\substack{a(0)=0 \\ a(1)=a_0}} \mathcal{D}a \, e^{iS[\ell,a]/\hbar}$$

where the path integral of the orbit cancels with the volume.
Next step: to compute the path integral over a.

The wavefunction of the universe as a 1D integral

Gauge-fixing & the path integral over a

- [Feldbrugge, Lehners & Turok '17] computes the WfU in 4D by reducing $\Psi(a_0)$ to

$$\Psi(a_0) = \int_0^\infty \mathrm{d}\ell \int_{\substack{a(0)=0\\ a(1)=a_0}} \mathcal{D}a\, e^{iS_{4D}[\ell,a]/\hbar}$$

they rescale $\ell \to \ell/a(t)$

$$\ell\,\mathrm{d}t = \frac{\ell}{a(t)}\,\mathrm{d}u$$

resulting in their Lagrangian being quadratic in $q(t) \equiv a^2(t)$

$$S_{4D}[q,\ell] = \int_0^{u_1} \mathrm{d}u\,\ell\left(\frac{3\dot{q}^2}{4\ell^2} + 3q - \Lambda q^2\right), \quad u_1 = \int_0^1 \mathrm{d}t\,a(t)$$

Approximate path integral over q as Gaussian, neglecting contributions from $\sqrt{q(t)}$ present in $u_1 \Rightarrow$ 3D-exact results

The wavefunction of the universe as a 1D integral

Gauge-fixing & the path integral over a

- Solve classical EoM for $a(t)$ with $a(0) = 0$ and $a(1) = a_0$

$$a_{\rm cl}(t) = a_0 \, \frac{\sinh\left(\sqrt{\Lambda}\ell t\right)}{\sinh\left(\sqrt{\Lambda}\ell\right)}$$

- Consider fluctuations around classical solution: $a(t) = a_{\rm cl}(t) + \delta a(t)$

$$\int_{\substack{\delta a(0)=0\\ \delta a(1)=0}} \mathcal{D}(\delta a)\, e^{iS[\ell,\delta a]/\hbar} = \left(\frac{-4\pi\sqrt{\Lambda}}{\pi i\hbar \sinh\left(\sqrt{\Lambda}\ell\right)}\right)^{1/2} \equiv f(\ell) \quad \text{Gaussian}$$

- Write the WfU as a 1D, oscillatory integral over ℓ

$$\Psi(a_0) = \int_0^\infty \mathrm{d}\ell \; f(\ell)\; e^{iS_{\rm cl}(\ell)/\hbar}$$

where

$$S_{\rm cl}(\ell) \equiv S[\ell, a_{\rm cl}] = 4\pi\left[\ell - a_0^2\sqrt{\Lambda}\coth\left(\sqrt{\Lambda}\ell\right)\right]$$

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The Picard-Lefschetz method for oscillating integrals

- Consider an one-dimensional, oscillatory integral of the form

$$I = \int_D \mathrm{d}\ell \; f(\ell) \; e^{\mathcal{I}(\ell)}$$

  where

  - D is a contour of integration in the complex plane.
  - $f(\ell)$ and $\mathcal{I}(\ell)$ are complex functions.

  PL theory can be applied when $f(\ell)$ and $\mathcal{I}(\ell)$ are holomorphic functions of ℓ.

The Picard-Lefschetz method for oscillating integrals

- Main Idea: deform D to the sum of steepest-descent contours, each emanating from a critical point of $\mathcal{I}(\ell)$

$$D \overset{\text{Cauchy}}{\sim} \mathcal{C} = \sum_\sigma n_\sigma \mathcal{J}_\sigma \quad \longrightarrow \quad I = \sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \mathrm{d}\ell \; f(\ell)\, e^{\mathcal{I}(\ell)}$$

where

- D is the original contour of integration in the complex plane.
- $\mathcal{K}_\sigma$ and $\mathcal{J}_\sigma$ are dual and Lefschetz thimbles, satisfying $\frac{\mathrm{d}\ell}{\mathrm{d}\lambda} = \pm \frac{\partial \mathcal{I}}{\partial \bar{\ell}}$
- $n_\sigma = \mathrm{Int}(D, \mathcal{K}_\sigma) \in \{0, \pm 1\}$ is the intersection number between D and $\mathcal{K}_\sigma$.

For $D \sim \mathcal{C}$, no singularities must be crossed and the endpoints must be kept fixed.

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Applying Picard-Lefschetz theory to 3D gravity

Plan for computing the 1D integral

- Recall that we are faced with the integral

$$\Psi(a_0) = \int_D \mathrm{d}\ell \, f(\ell) \, e^{iS_{\mathrm{cl}}(\ell)/\hbar}$$

with $D = (0, \infty)$, and

$$f(\ell) = \left(\frac{-4\pi\sqrt{\Lambda}}{\pi i \hbar \sinh(\sqrt{\Lambda}\ell)} \right)^{1/2} \qquad S_{\mathrm{cl}}(\ell) = \left[\ell - \sqrt{\Lambda} a_0^2 \coth\left(\sqrt{\Lambda}\ell\right) \right]$$

Proceed with the classically forbidden region ($\sqrt{\Lambda} a_0 < 1$)

- Determine saddle points of $iS_{\mathrm{cl}}(\ell)/\hbar$.
- Plot $\mathcal{J}_\sigma$ and $\mathcal{K}_\sigma$ for each σ.
- Apply $n_\sigma = \mathrm{Int}(D, \mathcal{K}_\sigma)$ to determine the contributions to $\Psi(a_0)$

Applying Picard-Lefschetz theory to 3D gravity

The saddle points of $S_{\rm cl}(\ell)$

- The saddle points satisfy

$$\frac{\partial S_{\rm cl}}{\partial \ell} = 0 \;\Leftrightarrow\; \ell_\sigma = \frac{i}{\sqrt{\Lambda}}\left\{(-1)^\sigma \arcsin\left(\sqrt{\Lambda}a_0\right) + \left\lfloor \frac{\sigma+1}{2} \right\rfloor \pi\right\}$$

where $\sigma \in \mathbb{Z}$ (infinitely many saddle points).

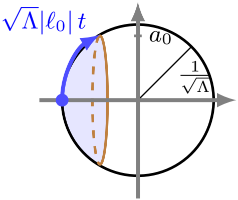
- Euclidean metrics for 3D space

$$\mathrm{d}s_\sigma^2 = \frac{1}{\Lambda}\left[\mathrm{d}\left(\sqrt{\Lambda}|\ell_\sigma|\, t\right)^2 + \sin^2\left(\sqrt{\Lambda}|\ell_\sigma|\, t\right)\mathrm{d}\Omega_2^2\right]$$

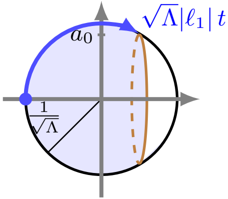
depicted as a necklace composed of beads with radius $1/\sqrt{\Lambda}$, terminating with a cap.

Applying Picard-Lefschetz theory to 3D gravity

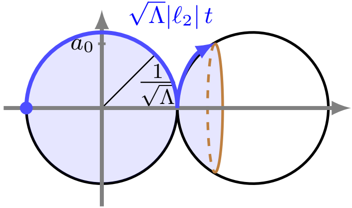
The saddle points of $S_{\rm cl}(\ell)$



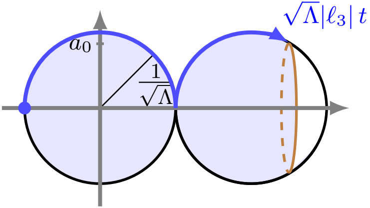
$\sigma = 0$



$\sigma = 1$



$\sigma = 2$



$\sigma = 3$

Applying Picard-Lefschetz theory to 3D gravity

The steepest-descent and -ascent contours

- For the Lefschetz and dual thimbles $\mathcal{J}_\sigma$ and $\mathcal{K}_\sigma$, solve

$$\frac{\mathrm{d}\ell}{\mathrm{d}\lambda} = \mp\frac{4i\pi}{\hbar}\left[1 + \frac{a_0^2\Lambda}{\sinh^2(\sqrt{\Lambda}\bar{\ell})}\right] \quad \text{around } \ell_\sigma \; \forall\,\sigma \in \mathbb{Z}$$

with $-$ sign for $\mathcal{J}_\sigma$ and $+$ sign for $\mathcal{K}_\sigma$.

  - Along $\mathcal{J}_\sigma$ ($\mathcal{K}_\sigma$) the real part of the exponent $iS_{\mathrm{cl}}(\ell)/\hbar$ decreases (increases) rapidly, while the imaginary part remains constant.
  - Use $\mathcal{J}_\sigma$ and $\mathcal{K}_\sigma$ to write $D \sim \mathcal{C} = \sum_\sigma n_\sigma \mathcal{J}_\sigma$, and

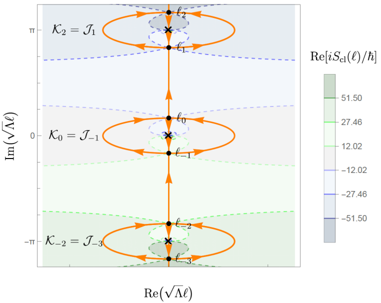
$$\Psi(a_0) = \sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \mathrm{d}\ell\, f(\ell)\, e^{iS_{\mathrm{cl}}(\ell)/\hbar}$$

where $n_\sigma = \mathrm{Int}(D, \mathcal{K}_\sigma)$.

Applying Picard-Lefschetz theory to 3D gravity

The steepest-descent and -ascent contours

- Plots of $\mathcal{J}_\sigma$ and $\mathcal{K}_\sigma$ for $(\Lambda, a_0) = (1, 1/2)$, obtained numerically



- Arrows indicate decreasing $\mathrm{Re}[iS_{\mathrm{cl}}(\ell)/\hbar]$
- Dotted curves: constant $\mathrm{Re}[iS_{\mathrm{cl}}(\ell)/\hbar]$
- As $\ell \to i\infty$, $\mathrm{Re}[iS_{\mathrm{cl}}(\ell)/\hbar] \to -\infty$
- Branches of $\mathcal{J}_\sigma$ coincide with branches of $\mathcal{K}_{\kappa\neq\sigma}$

  $\Rightarrow$ Stokes phenomenon

Applying Picard-Lefschetz theory to 3D gravity

Stokes phenomenon

- Steepest-ascent contours stop at other saddle points instead of reaching infinity or the singular points of $S_{\text{cl}}(\ell)$ (usual domain)

  $\Rightarrow n_\sigma$ not well-defined.

- Introduce small parameter η by letting $\hbar = |\hbar| e^{i\eta}$
  - Of course, this situation is unphysical $\Rightarrow$ eventually take $\eta \to 0$ limit
  - Flow equations change into

$$\frac{\mathrm{d}\ell}{\mathrm{d}\lambda} = \mp \frac{i\omega_2}{|\hbar|} e^{i\eta} \left[1 + \frac{\Lambda a_0^2}{\sinh^2(\sqrt{\Lambda}\bar{\ell})} \right]$$

  As a result, steepest-ascent and -descent contours from different saddle points disentangle.

Applying Picard-Lefschetz theory to 3D gravity

Results

- After $\eta \to 0$, the semiclassical factor
  - in the classically forbidden region ($0 < \sqrt{\Lambda}a_0 < 1$) is

$$\Psi(a_0) \approx \frac{i}{(1-\Lambda a_0^2)^{1/4}} \exp\left[-\frac{4\pi}{\hbar\sqrt{\Lambda}}\left\{\arcsin\left(\sqrt{\Lambda}a_0\right)+\sqrt{\Lambda}a_0\sqrt{1-\Lambda a_0^2}\right\}\right]$$

  - in the classically allowed region ($\sqrt{\Lambda}a_0 > 1$) is

$$\Psi(a_0) = \frac{e^{-\frac{2\pi^2}{\hbar\sqrt{\Lambda}}}\, e^{i\frac{\pi}{4}}}{(\Lambda a_0^2-1)^{1/4}} \exp\left[\frac{4\pi i}{\hbar\sqrt{\Lambda}}\left\{\operatorname{arccosh}\left(\sqrt{\Lambda}a_0\right)-\sqrt{\Lambda}a_0\sqrt{\Lambda a_0^2-1}\right\}\right]$$

  up to corrections $\mathcal{O}(\hbar)$.

- The contour at the semiclassical level looks like that of Vilenkin.

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Summary & future directions

- Summary of results:

  1. Picard-Lefschetz theory provides a way to determine the WfU (starting from a Lorentzian path integral).
  2. In 3D gravity, the semiclassical factor favors Vilenkin's proposal

     $\Rightarrow$ No rescaling of time-interval required.

     $\Rightarrow$ Stokes phenomenon occurs, resolved by introducing η.

Summary & future directions

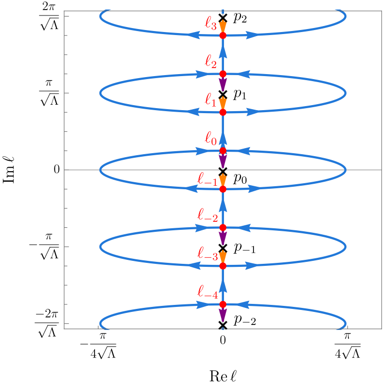
- Summary of results:
  1. Picard-Lefschetz theory provides a way to determine the WfU (starting from a Lorentzian path integral).
  2. In 3D gravity, the semiclassical factor favors Vilenkin's proposal
     - $\Rightarrow$ No rescaling of time-interval required.
     - $\Rightarrow$ Stokes phenomenon occurs, resolved by introducing η.
- Future directions:
  - ➢ Compute WfU in 2D Jackiw–Teitelboim gravity.
  - ➢ Compute WfU in 4D expanding universes.

Thank you for your attention

Applying PL theory to 3D gravity

The steepest-descent and -ascent contours

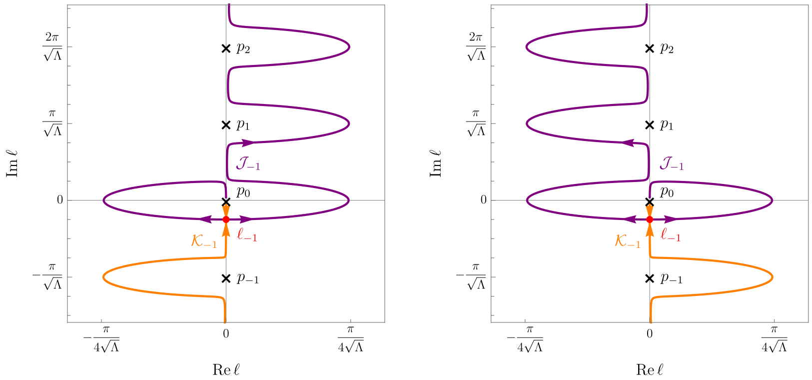
- Plots of $\mathcal{J}_\sigma$ and $\mathcal{K}_\sigma$ for $(\Lambda, a_0) = (1, 1/\sqrt{2})$, obtained numerically



Applying PL theory to 3D gravity

The steepest-descent and -ascent contours

- Plots of $\mathcal{J}_\sigma$ (purple) and $\mathcal{K}_\sigma$ (orange) for $\sigma = -1$

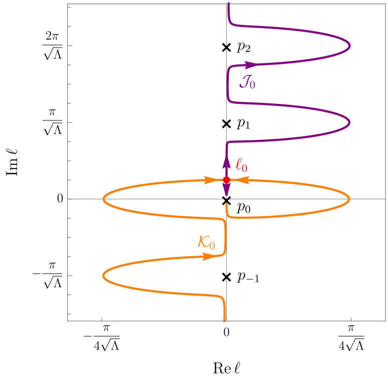


$\eta < 0$ $\qquad$ $\eta > 0$

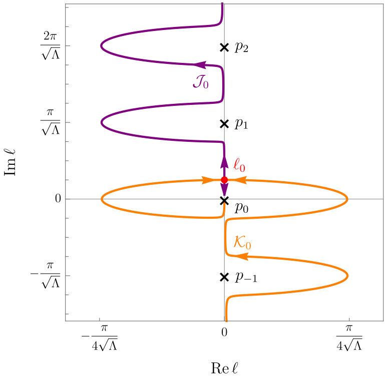
Applying PL theory to 3D gravity

The steepest-descent and -ascent contours

- Plots of $\mathcal{J}_\sigma$ (purple) and $\mathcal{K}_\sigma$ (orange) for $\sigma = 0$



$\eta < 0$

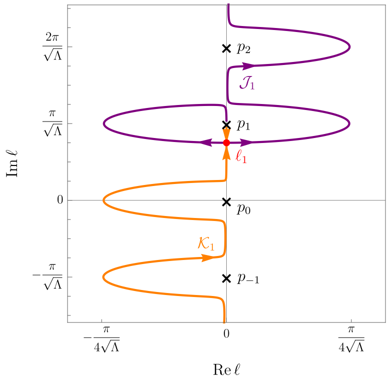


$\eta > 0$

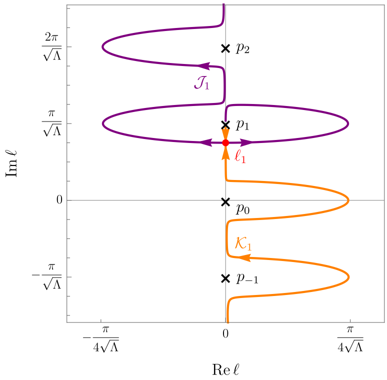
Applying PL theory to 3D gravity

The steepest-descent and -ascent contours

- Plots of $\mathcal{J}_\sigma$ (purple) and $\mathcal{K}_\sigma$ (orange) for $\sigma = +1$



$\eta < 0$

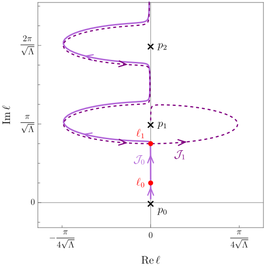


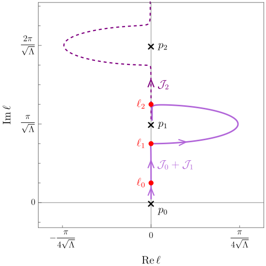
$\eta > 0$

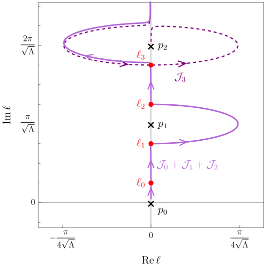
Applying PL theory to 3D gravity

The steepest-descent and -ascent contours

- Apparent ambiguity: $n_\sigma(\eta < 0) \neq n_\sigma(\eta > 0)$ $\forall\ \sigma = 2\kappa,\ \kappa \in \mathbb{Z}$
- Overlapping curves cancel due to opposite orientations ($\eta > 0$)



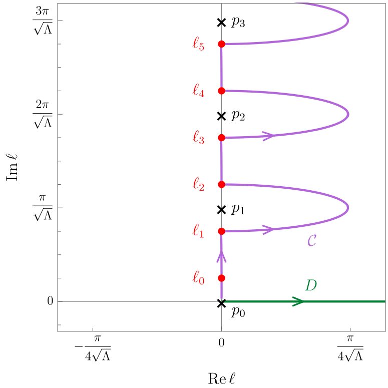




Applying PL theory to 3D gravity

The steepest-descent and -ascent contours

- The resulting contour for $\eta \to 0^+$ is the same as $\mathcal{J}_0$ for $\eta \to 0^-$



- We can define

$$C \equiv \lim_{\eta \to 0^-} n_\sigma(\eta)\mathcal{J}_\sigma(\eta)$$

or

$$C \equiv \lim_{\eta \to 0^+} n_\sigma(\eta)\mathcal{J}_\sigma(\eta)$$

since the last two limits are equal.

$\Rightarrow$ Ψ is also continuous.