Accretion, mergers, and metastability of fuzzy spheres in a three-matrix model

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The goal

Take a matrix model with many (fuzzy-sphere) vacua and figure out whether the lowest energy one can actually be reached.

The question in one slide

We picked a three-matrix model that contains a multitude of fuzzy sphere states, with the largest one being the only stable one.

Is that state reachable in a finite amount of *time*?

It depends — on the dispersion of the initial configuration.

- **cold** start $\Rightarrow$ stuck in an *onion* state,
- **hot** start $\Rightarrow$ the global minimum.

Crucial mechanism: accretion, driven by fluctuations of the sphere centres and instabilities.

Why who choice this model?

It is simple, numerically accessible, and well-researched.

The real target is IKKT:

- many saddle points, quantum spaces of various dimensions,
- candidates for near-realistic cosmological backgrounds.

Are such backgrounds long-lived enough to support physics?

Here: the same question, in a model simple enough to simulate.

The model

$$S[A] = N\ \mathrm{tr}\left(-\frac{1}{4}[A_i, A_j]^2 + \frac{2}{3}\, i\alpha\, \epsilon_{ijk}\, A_i A_j A_k\right)$$

$$Z = \int dA\, e^{-S}$$

Three traceless Hermitian $N \times N$ matrices, $i = 1, 2, 3$. Yang–Mills plus a Myers term, invariant under $A_i \to A_i + a_i \mathbf{1}$.

Two phases, with $\tilde{\alpha} = \alpha\sqrt{N}$ and $\tilde{\alpha}_{crit} \approx 2.1$:

- $\tilde{\alpha} < \tilde{\alpha}_{crit}$: diagonal matrices, $S = 0$,
- $\tilde{\alpha} > \tilde{\alpha}_{crit}$: the fuzzy sphere $A_i = \alpha L_i$.

We keep $\alpha = 1$, deep in the fuzzy-sphere phase.

Alekseev, Recknagel, Schomerus; Azuma, Bal, Nagao, Nishimura 2004.

A plethora of classical solutions

$$[X_i, X_j] = \lambda i \varepsilon_{ijk} X_k, \quad X_i X_i = \frac{\lambda^2 \left(N^2 - 1\right)}{4} \mathbf{1}$$

so any block-diagonal stack of fuzzy spheres solves the model:

$$A_i = \begin{pmatrix} X_i^{(n_1)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & X_i^{(n_K)} \end{pmatrix} \equiv \mathbf{X}(n_1, \ldots, n_K), \quad N = \sum_{k=1}^{K} n_k$$

$$S_0(n_1, ..., n_K) = -\frac{N\alpha^4}{24} \sum_k n_k (n_k^2 - 1)$$

Minimised by $\mathbf{X}(N)$, a single largest sphere. The competitors are all partitions of N.

Simulation setup

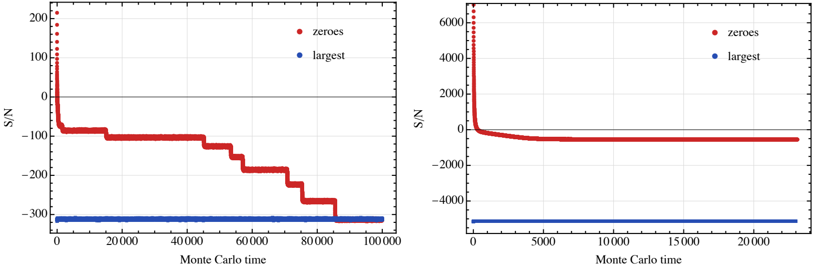
Hamiltonian Monte Carlo, deliberately without enhancements such as cluster flipping — those algorithms exist to hop between vacua, and the hopping is exactly what we want to watch.

Two observables suffice:

- S/N,
- the eigenvalues of the radial operator $X^2 = X_i X_i$.

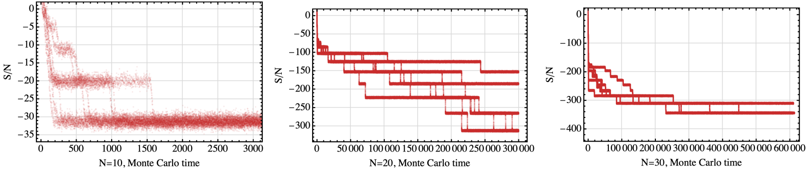
Metastability: the system stays near a given point for a large, but not infinite, number of HMC steps.

Reaching the minimum



Two runs each, one from the zero configuration and one from $\mathbf{X}(N)$, with $\alpha = 1$. Left $N = 20$, right $N = 50$.

Not an accident



Ensembles of 8 runs, $N = 10, 20, 30$. Only $N = 10$ reaches the lowest energy state; long plateaus are common everywhere.

What is the system doing on those plateaus?

A tempting explanation: entropy

There is one maximal sphere, but many ways to split N:

$$P(N) \sim \frac{1}{4N\sqrt{3}} \exp\left(\pi\sqrt{\frac{2N}{3}}\right)$$

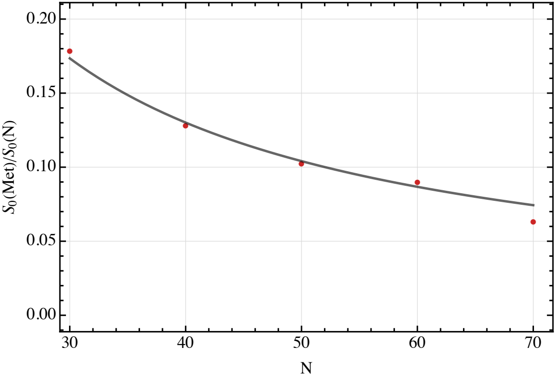
If partitions were sampled uniformly, the typical shape would be the Vershik curve, with $x = n_k/\sqrt{N}$ and $y = k/\sqrt{N}$:

$$e^{-\pi x/\sqrt{6}} + e^{-\pi y/\sqrt{6}} = 1 \quad \Rightarrow \quad S_0(Ver) \approx -\frac{N^3\alpha^4}{10}$$

This gives a prediction: $S_0(Ver)/S_0(\mathbf{X}(N)) = 2.4/N$,

with both the size and the number of spheres scaling as $\sqrt{N}$.

But the plateaus are not Vershik states

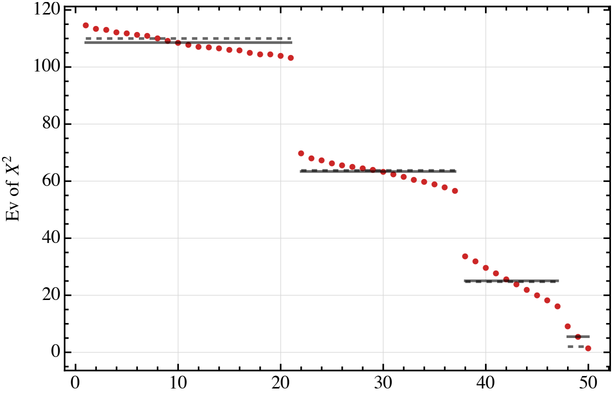


Ratio $S_0(Met)/S_0(\mathbf{X}(N))$ against N, averaged over the last 1000 steps after the plateau is reached.

The N^{-1} trend is confirmed, but the coefficient is 5.20, not 2.4.

Not typical partitions — far fewer, far larger spheres.

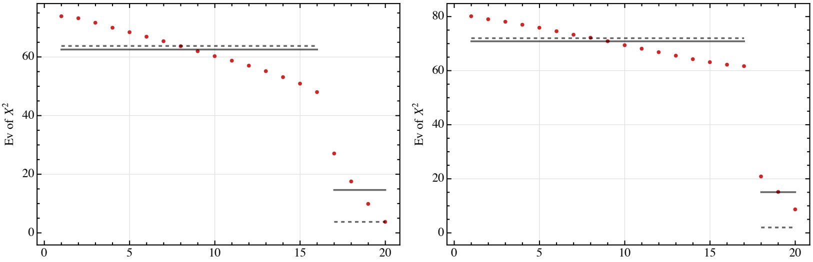
Disecting the states, eigenvalues of $\mathbf{X}^2$



Average $\mathbf{X}$ over 10 steps so the vacuum shows through, then diagonalise X^2. Each sphere gives a plateau at $\alpha^2(n_k^2-1)/4$ (dashed); solid lines are cluster means.

For $N=50$ this state is $\mathbf{X}(21,16,10,3)$. Vershik would want $\sim\sqrt{50}$ spheres of size $\sim\sqrt{50}$.

The elementary transition is an accretion



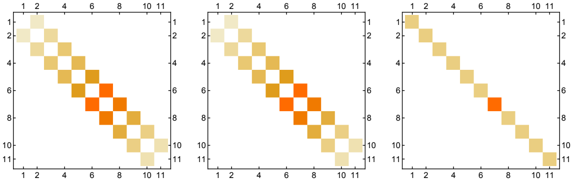
Radial-operator eigenvalues at steps 56 000 and 57 000, $N = 20$.

$$\mathbf{X}(\ldots, n_r, \ldots, n_s, \ldots) \to \mathbf{X}(\ldots, n_r - 1, \ldots, n_s + 1, \ldots)$$

The larger sphere absorbs a single unit from the smaller one. An **accretion**, not a merger.

The obstacle, in the matrix elements

$$\mathbf{dX} = \mathbf{X}(n, n+1) - \mathbf{X}(n-1, n+2)$$

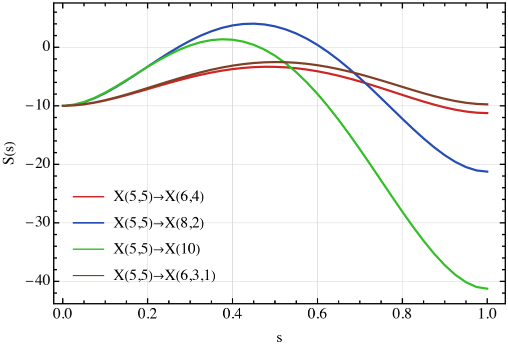


Amplitudes of the difference matrices for $n = 5$.

dX_3 is diagonal with all entries $-1/2$ except one, which equals n. Growing a block requires a single new element of size $\sim n/2$. With Gaussian HMC momenta this is exponentially suppressed.

The same statement as an energy barrier

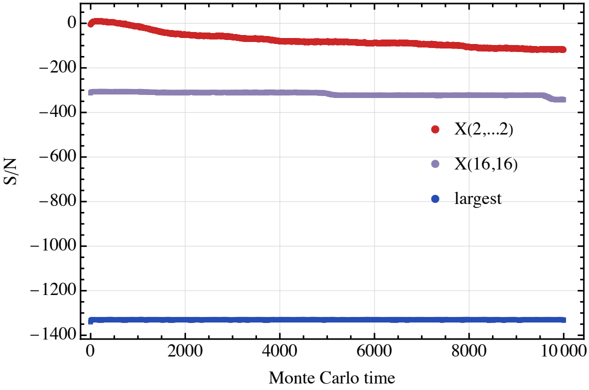
$$D(s) = S\left((1-s)\,\mathbf{X}_1 + s\,\mathbf{X}_2\right), \quad 0 \leq s \leq 1$$



Here $\mathbf{X}_1 = \mathbf{X}(5,5)$ and $\mathbf{X}_2 = \mathbf{X}(6,4), \mathbf{X}(8,2), \mathbf{X}(10), \mathbf{X}(6,3,1)$.

The cheapest route is the smallest accretion, and it is still expensive. For $\mathbf{X}(50,50) \rightarrow \mathbf{X}(49,51)$ the available fluctuation is $|\delta S/S| \approx 0.0001$, while $|\delta S/S| \approx 0.76$ is needed.

And mergers do not happen



A merger would be $\mathbf{X}(\ldots, r, s, \ldots) \to \mathbf{X}(\ldots, r+s, \ldots)$.

Tested from $\mathbf{X}(2,\ldots,2)$ and from $\mathbf{X}(16,16)$: none observed, only gradual steps.

D-branes, and how to look at them

Diagonal elements give the positions of $D0$-branes, fuzzy spheres are $D2$-branes, and the cascade is a sequence of brane accretions and mergers.

Close $D0$-branes form $D2$-branes, their centres tend to align into concentric shells, and overlapping $D2$-branes are unstable.

To visualise a configuration, define the displacement Hamiltonian

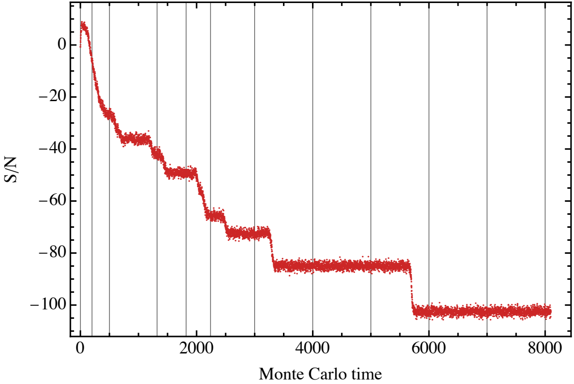
$$H(x) = \frac{1}{2}\sum_{a=1}^{D}(Y_a - x_a I_N)^2,$$

scan through $\mathbb{R}^D$, keep the lowest eigenvalues and plot

$$Y(x) = \left(\langle x|Y_1|x\rangle, \ldots, \langle x|Y_D|x\rangle\right).$$

Schneiderbauer, Steinacker 2016.

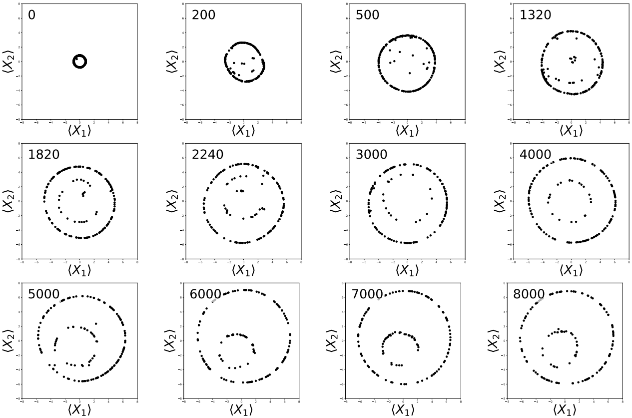
Cascading and plateauing: numerical point of view



$N = 20$, initiated in $\mathbf{X}(2, \ldots, 2)$: the smallest possible concentric spheres. Grid lines mark the states visualised next.

Accretion drives the cascade, fluctuations make the gap smaller.

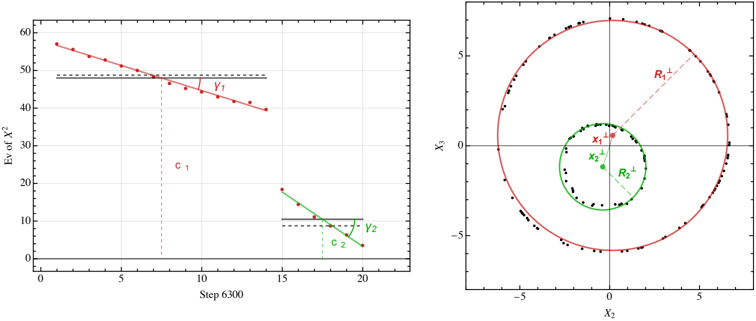
Cascading and plateauing: membrane point of view



Two concentric fuzzy spheres form and grow; the position of the smaller one fluctuates considerably.

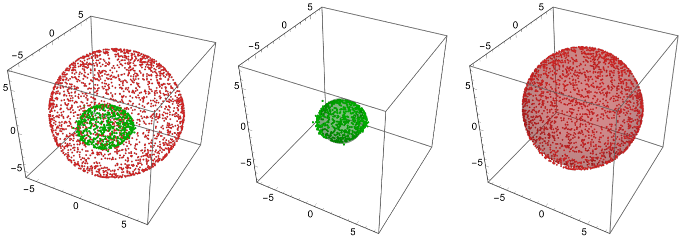
Metastability is stiffness, not entropy.

Two readings of the same configuration



Step 6300 of the previous run. Left: two distinct levels in the radial operator, fitted for the offset and the radius. Right: the visualised points, projected on the $X_1 = 0$ plane and fitted.

Radial levels against quantum geometry



	$\lvert x_1 \rvert$	R_1	$\lvert x_2 \rvert$	R_2
radial levels	0.66(1)	6.90(2)	1.45(7)	2.9(4)
quantum geometry	0.673(7)	6.405(3)	1.379(9)	2.479(6)

Few-percent agreement between two entirely different extractions.

One-loop effective action for two spheres

Background $\mathbf{X} = \alpha\, \mathrm{diag}\left(L^{(n_1)} + x^{(1)}, L^{(n_2)} + x^{(2)}\right)$ with $n_1 \gg n_2$:

$$W = \sum_{j_{min}}^{j_{max}} \log w_j, \quad w_j = \frac{h(j,j+1)h(j,-j-1)}{h(j,j)h(j,-j)} \prod_{m=-j}^{j} h(j,m)$$

with $h(j,m) = \xi^2 + 2\xi m + j(j+1)$ and $\xi_i = x_i^{(1)} - x_i^{(2)}$.

For a sphere inside a sphere the minimum is at $\xi_i = 0$ – they are concentric – and for $W = W_0 + \frac{1}{2} M_\xi^2 \xi_i^2 + \ldots$ we have the mass as

$$M_\xi^2 \;=\; \sum_{j=j_{min}}^{j_{max}} (2j+1) \left(\frac{2}{3\,j(j+1)} - \frac{8}{j^2(j+1)^2} \right).$$

For well-separated spheres there is a logarithmic attraction: long-range, and with no barrier.

Azuma, Nagao, Nishimura 2007.

Measuring the centre fluctuations

For a state $\alpha(L_i + x_i\mathbf{1})$ the radial operator is

$$X_i X_i = \alpha^2 \left(L^2 + 2L_i x_i + x^2\right),$$

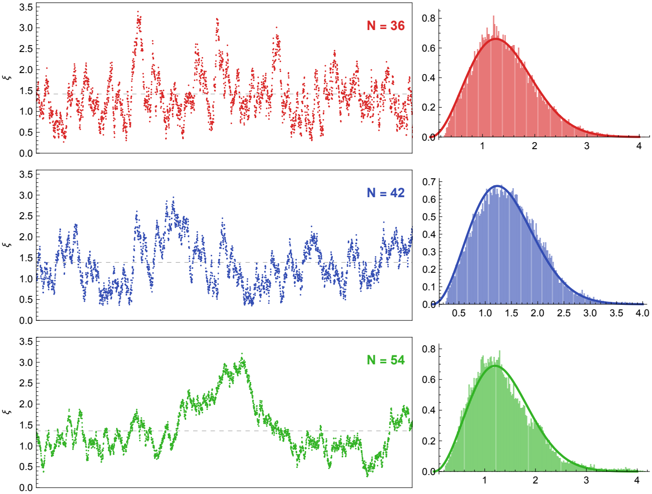
so each level is lifted by x^2 and, choosing $\vec{x} = (0,0,x)$, tilted by $2\alpha x m$. The tilt measures $|x|$ accurately.

The shifts of a pair are opposite, so $|\xi| = |x_1| + |x_2|$ is Maxwell-distributed with mean $\sqrt{8/(\pi M_\xi^2)}$ — a prediction with no free parameters.

	$\langle\xi\rangle_{mes}$	$\langle\xi\rangle_{pred}$	τ_{aut}
$N = 36$	1.44(3)	1.42	~ 3600
$N = 42$	1.44(4)	1.39	~ 6900
$N = 54$	1.36(6)	1.36	$\sim 19\ 400$

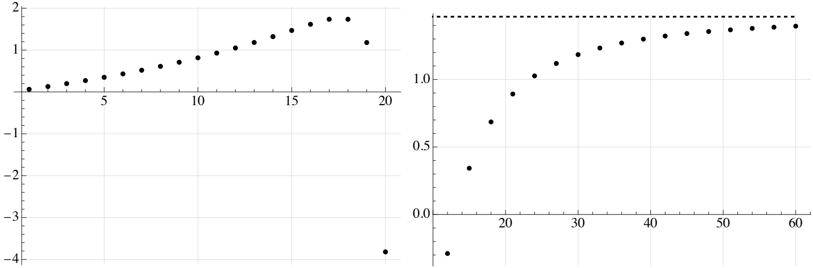
Splits $n_1 = 2N/3$, jackknife errors. Also $\langle n_1|x_1|/n_2|x_2|\rangle = 1.0(1)$, as $\mathrm{Tr}X = 0$ requires.

Trajectories and histograms, nothing fitted



$|x_i|$ measured every 100 steps during runs of 3×10^6 steps. The amplitudes barely change with N, while the autocorrelation time grows rapidly. Right: measured histograms against the one-loop prediction.

Why the accretions stop



Left: the effective mass for $N = 42$ and various splits $\mathbf{X}(42 - n_2, n_2)$. Right: the $n_1 = 2n_2$ splitting and its large-N limit.

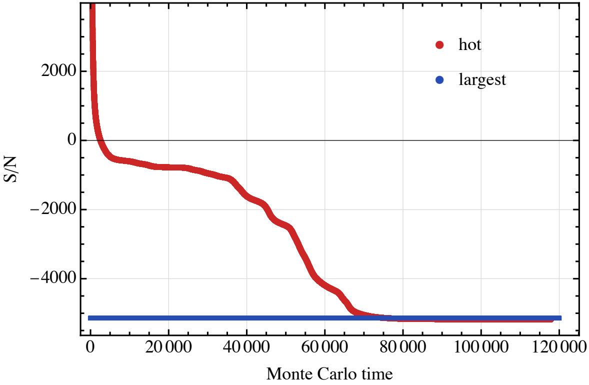
An accretion needs $|\xi| \sim R_1 - R_2$. The amplitude saturates while the gap grows linearly. For $n_1 = bN$ with $b > 1/2$,

$$\lim_{N\to\infty} M_\xi^2(b) = -\frac{4}{3}\log\left(2b-1\right),$$

negative as $b \to 1/2$: the known instability of overlapping spheres.

We understand the plateaus now. Can we see a path to the minimum state $\mathbf{X}(N)$?

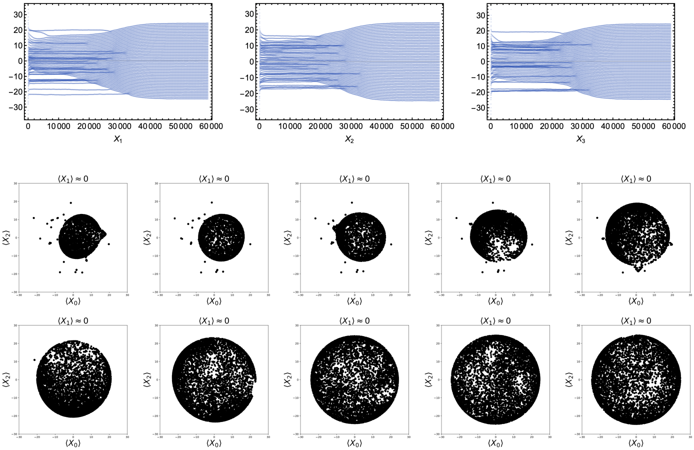
Hot initial conditions



Hermitian matrices with normally distributed entries, dispersion 3 instead of $1/\sqrt{2}$, giving well-separated $D0$-branes.

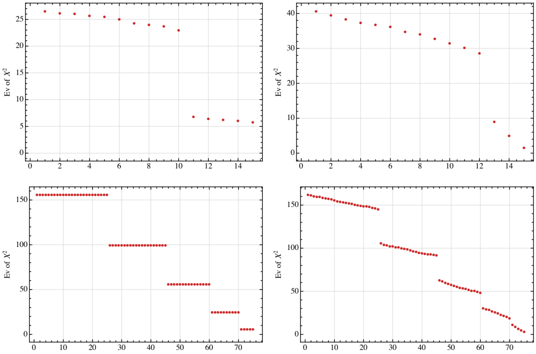
A few of them seed a sphere and the rest fall in from outside. Slow, long-range, but with no barriers to cross.

Infall, seen directly



$N = 50$ from a hot state: HMC trajectories of the three matrices, and the geometry visualised every 1000th step. The lowest energy state is achieved.

The fuzzy onion



Top: $\mathbf{X}(10, 5)$ at steps 100 and 1000. Bottom: $\mathbf{X}(25, 20, 15, 10, 5)$ at steps 100 and 200 000.

Concentric spheres $n_k = \Delta k$ describe a rotationally invariant 3D quantum space. The two-sphere system decays after $\sim 10^3$ steps; the five-sphere one survives $\sim 232\,000$.

Outer layers stabilise the inner ones.

Stiffening with layers

Generalising to K blocks, the interaction is a sum of pairwise terms:

$$W = W_0 + \frac{1}{2} \sum_{I<J} M_{IJ}^2 \left| \vec{\xi}^{(IJ)} \right|^2 + \dots$$

For the innermost sphere, with the others near the origin, $\mathbf{x}^{(2)} = \mathbf{x}^{(3)} = \dots = 0$

$$W_1 = \frac{2}{3} \left(\log \frac{K(K+1)}{2} - \frac{q}{\Delta^2} \right) \left(\mathbf{x}^{(1)} \right)^2, \quad q = \frac{355}{12} \approx 30.$$

The mass diverges as $K \to \infty$: the onion becomes rigid. For finite K the system is unstable below

$$K_C \approx e^{15/\Delta^2}.$$

Conclusions

- The model is surprisingly rich.
- Both cold initial configurations and multiple copies of the same fuzzy sphere lead to a slowing cascade of metastable states.
- Onion state seems stable in the infinite size limit.
- Hot initial states lead to the largest fuzzy sphere state.

Conclusions

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Thank you for your attention!