# Multi-Matrix invariants, degeneracy graphs and determinants.

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Corfu workshop on "Noncommutative and Generalised Geometries and their Physical Applications" 09/2026

Based on :
Garreth Kemp and Sanjaye Ramgoolam, “Gauge-string duality, monomial bases and graph determinants ” **arXiv:2603.05259[hep-th]**
Garreth Kemp and Sanjaye Ramgoolam, , “BPS states, conserved charges and centres of symmetric group algebras,” **arXiv:1911.11649[hep-th]**, JHEP 01 (2020)

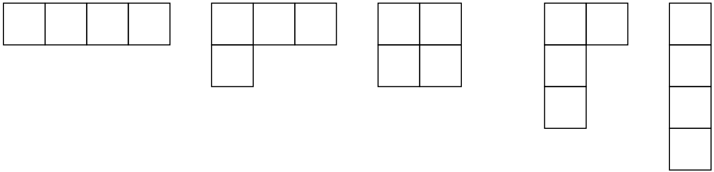
## Half BPS states in $\mathcal{N}=4$ SYM : Young diagram basis.

$\mathcal{N}=4$ SYM with $U(N)$ gauge group, dual by AdS/CFT, to string theory on $AdS_5 \times S^5$.

Half-BPS states. Polynomial gauge invariants in one complex matrix field $Z$, adjoint of $U(N)$.

Single and Multi-traces, e.g. degree 4 :
$\mathrm{tr}Z^4, \mathrm{tr}Z^3\mathrm{tr}Z, (\mathrm{tr}Z^2)^2, \mathrm{tr}Z^2(\mathrm{tr}Z)^2, (\mathrm{tr}Z)^4$

Good basis for degree $n$, finite $N$: Schur Polynomials $\chi^R(Z)$.
Young diagrams with $n$ boxes, $R : l(R) \leq N$.



## Young diagram basis: Orthogonality, Giant gravitons.

In the free field inner product, computed using

$$\langle Z^i_j (Z^\dagger)^k_l \rangle = \delta^i_l \delta^k_j$$

The Schur polynomials are orthogonal. For degree $n$,

$$\langle \chi^R(Z) \chi^S(Z^\dagger) \rangle = \delta^{RS} \, \frac{n! \mathrm{Dim}_N R}{d_R}$$

Non-renormalization theorems.

Natural candidates for duals of giant gravitons in the $AdS_5 \times S^5$

Corley, Jevicki,Ramgoolam, 2001.

# Schur Operators: Permutations and characters

$$\mathcal{O}_\sigma(Z) = \sum_{i_1,\cdots,i_n} Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_n}_{i_{\sigma(n)}}$$

$$(\mathrm{tr}Z)^2 = Z^{i_1}_{i_1} Z^{i_2}_{i_2}$$
$$(\mathrm{tr}Z^2) = Z^{i_1}_{i_2} Z^{i_2}_{i_1}$$

$$\chi^R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^R(\sigma) \mathcal{O}_\sigma(Z)$$

## Schur Operators: Group algebra and projectors.

The group algebra $\mathbb{C}(S_n)$ :

$$\sum_{\sigma \in S_n} c_\sigma \ \sigma$$

The centre of the group algebra has a basis of projectors

$$P_R = \frac{d_R}{n!} \sum_\sigma \chi^R(\sigma) \ \sigma$$

Tensor product space: $V_N^{\otimes n}$

$$e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_n}$$

Then

$$\chi^R(Z) = \frac{1}{d_R} \mathrm{tr}_{V_N^{\otimes n}} \left( Z^{\otimes n} P_R \right)$$

## Centre of group algebra: Conjugacy class basis

Conjugacy classes in symmetric group $S_n$: Cycle structures of permutations, parametrised by partitions $p \vdash n$.

Example: $S_4$

$$
\begin{aligned}
p &= [4]: & (1,2,3,4) \\
p &= [3,1]: & (1,2,3)(4) \\
p &= [2,2]: & (1,2)(3,4) \\
p &= [2,1,1]: & (1,2)(3)(4) \\
p &= [1,1,1,1]: & (1)(2)(3)(4)
\end{aligned}
$$

## Centre of group algebra: Conjugacy class and cycle operators.

Linear basis for centre of group algebra of $S_n$: sums over classes $\mathcal{C}_p$

$$\mathcal{T}_p = \sum_{\sigma \in \mathcal{C}_p} \sigma$$

Two bases: $\{T_p\}$ and $\{P_R\}$

$$\mathcal{T}_p P_R = \frac{\chi^R(T_p)}{d_R} P_R$$

**Cycle operators :**

$$T_k \equiv \mathcal{T}_{[k,1^{n-k}]}$$

## LLM geometries : Young diagrams and supergravity solutions with AdS asymptotics.

"Bubbling AdS space and 1/2 BPS geometries," Lin, Lunin and Maldacena 2004.

- Schur polynomial states correspond to regular half-BPS super-gravity geometries.
- Determined by function taking values zero and one on a plane.
- Can be interpreted as phase space occupation numbers of $N$-fermions in a harmonic oscillator.
- Related to the $Z$ sector matrix quantum mechanics.

## LLM geometries : Asymptotics and Casimirs of $U(N)$.

Integrability versus information loss: A Simple example,

Balasubramanian, Czech, Larjo, Simon, 2006.

- The Young diagrams $R$ have $U(N)$ Casimir eigenvalues $C_i(R)$.
- These can be read off from the asymptotic multipole moments of the SUGRA fields of the LLM solution.
- $N$ Casimirs determine that $R$. But restricting to moments constrained by Planck scale, only partial information about the $R$ - hence about the microstate of the SUGRA solution.

## Fixed degree and symmetric group formulation

BPS states, conserved charges and centres of symmetric group algebras,

G. Kemp and S. Ramgoolam, 2020.

- Consider the operators of fixed degree $n$. Let us assume $n$ is less that $N$.
- Take advantage of Schur-Weyl duality, which implies that the $U(N)$ Casimirs are related by a triangular system to the $T_k$ and their products.
- For fixed $n$, how many of $\{T_2, T_3, \ldots T_{k_*(n)}\}$ do we need to distinguish the Young diagram projectors $P_R$.
- We found that $k_*(n)$ grows slowly with $n$, e.g. up $n = 79$, $k_*(n) \leq 6$.

## Distinct eigenvalues : the powers generate the algebra

In cases where $T_2$ suffices to generate the centre, there is a very nice concrete formula, which allows us to prove that the projectors can be written in terms of monomials

$$P_R = \prod_{S \neq R} \left( \frac{T_2 - a_S(T_2)}{a_R(T_2) - a_S(T_2)} \right)$$

Projectors form a basis

## Algebraic meaning of the minimal detecting sets

- Detecting sets form non-linearly generating sets

$$P_R = \sum_{\vec{m}} C_{m_2, m_3, \ldots} T_2^{m_2} T_3^{m_3} \cdots T_{k_*(n)}^{m_{k_*(n)}}$$

- This raises the question: How explicitly do we write the projectors in terms of monomials in the $T$'s, i.e. a monomial basis for $\mathcal{Z}(\mathbb{C}(S_n))$ in terms of the cycle operators.
- The question can be phrased in the context of general commutative semisimple algebras.
- It has a nice connection to a class of degeneracy graphs, which turn out to be crucial to solving the question.
- We solved the question and gave a basis in the recent paper. I will give an outline explanation of the solution.

## OUTLINE OF TALK

- The case of one generator.
- Graph : the two-layer case and the $L$-layer case.
- The general $L$-layer graphs - and the monomial basis.
- Applications: symmetric groups (centres and JM)
- Applications: multi-matrix gauge invariant operators.

## One generator case : low order symmetric group examples

For the cases, $n = 2, 3, 4, 5, 7$, the lists of normalised characters of $T_2$ alone suffice to distinguish the irreps.
$n = 5$:

| $R$ | $[5]$ | $[4,1]$ | $[3,2]$ | $[3,1,1]$ | $[2,2,1]$ | $[2,1,1,1]$ | $[1^5]$ |
|---|---|---|---|---|---|---|---|
| $\frac{\chi^R(T_2)}{d_R}$ | 10 | 5 | 2 | 0 | $-2$ | $-5$ | $-10$ |

## One generator case : Vandermonde and inversion

The monomial basis in this case is:

$$\{1, T_2, T_2^2, \cdots , T_2^6\}$$

$$T_2^A = \sum_R \left( \frac{\chi^R(T_2)}{d_R} \right)^A P_R = \sum_R \mathcal{M}_{R,A} P_R$$

The matrix relating the monomial basis to the $P_R$ is a Van-der-Monde matrix of the form $[x_R^A]$ and has a determinant

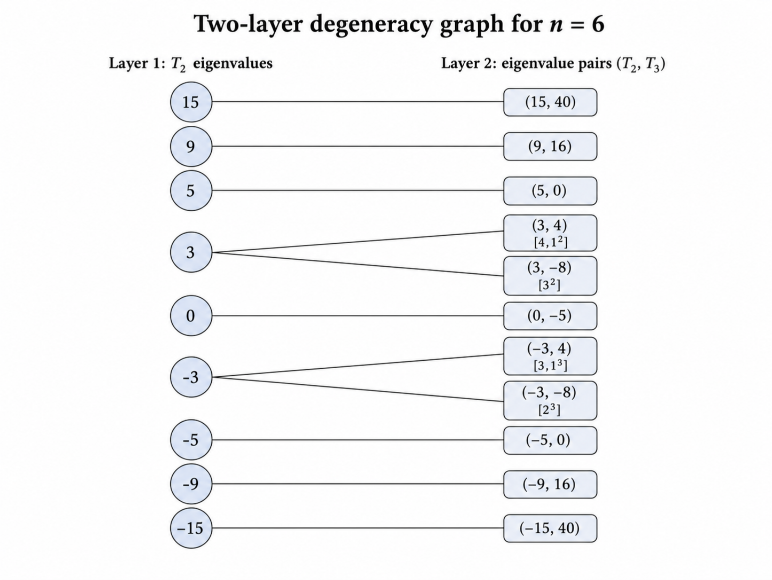
$$\prod_{R<S} (x_R - x_S)$$

Hence invert and get $P_R$ in terms of monomials ( product formula from before).

| $R \vdash 6$ | $[6]$ | $[5,1]$ | $[4,2]$ | $[4,1^2]$ | $[3^2]$ | $[3,2,1]$ | $[3,1^3]$ | $[2^3]$ | $[2^2,1^2]$ | $[2,1^4]$ | $[1^6]$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{\chi^R(T_2)}{d_R}$ | 15 | 9 | 5 | 3 | 3 | 0 | $-3$ | $-3$ | $-5$ | $-9$ | $-15$ |

| $R$ | $[4,1^2]$ | $[3^2]$ | $[3,1^3]$ | $[2^3]$ |
|---|---|---|---|---|
| $\frac{\chi^R(T_2)}{d_R}$ | 3 | 3 | $-3$ | $-3$ |
| $\frac{\chi^R(T_3)}{d_R}$ | 4 | $-8$ | 4 | $-8$ |

# Two generator case : Degeneracy graph



## $L$-generator case : $L$-layer graph

For the case, where we have generators $\mathcal{C}_1, \mathcal{C}_2, \cdots \mathcal{C}_L$ forming a minimal generating set for an algebra, we will have a graph

- $L$ layers, with numbers of nodes $\{D_1, D_2, \cdots, D_L\}$.
- $D_1 < D_2 < \cdots < D_L$.
- A partition of $D_2$ into $D_1$ parts ; A composition of $D_3$ into $D_2$ parts etc.
- This data $\{p_1, c_2, c_3, \cdots, c_{L-1}\}$ determines a commutative associative semi-simple algebra.

$D_L$ is the dimension of the algebra. For $\mathcal{Z}(\mathbb{C}(S_n))$, it is $p(n)$ if $L = k_*(n) - 1$.

## $L$-generator case : A condition defined by descendant structure

We define $\mathrm{Monom}(d_2, d_3, \cdots, d_L)$ as a set of monomials

$$\mathbf{Monom}(d_2, d_3, \cdots, d_L) = \{1, \mathcal{C}_1, \mathcal{C}_1^2, \cdots, \mathcal{C}_1^{|\mathcal{S}^{(1)}_{[\vec{d}]}|-1}\} \times \mathcal{C}_2^{d_2-1}\mathcal{C}_3^{d_3-1}\cdots\mathcal{C}_L^{d_L-1}$$

$\mathcal{S}^{(1)}_{[\vec{d}]} \equiv \mathcal{S}^{(1)}_{[d_2,d_3,\cdots,d_L]}$ - selected by a condition on the nodes of layer 1, defined in terms of descendant numbers constrained by $\vec{d}$.

In words .... **or more** ...

## $L$-generator case : Monomial basis

Our main conjecture is the following.
**Monomial Basis Conjecture:** A basis of $\mathcal{A}_L$ is the disjoint union

$$\mathcal{S}(\mathcal{A}_1 \to \cdots \to \mathcal{A}_L) \equiv \bigsqcup_{d_2, d_3, \cdots, d_L} \mathbf{Monom}(d_2, d_3, \cdots, d_L)$$

where $d_i \in \{1, \cdots, d_i^{\max}\}$.
The dimension of $\mathcal{A}_L$ is equal to the number of distinct eigenvalue lists of length $L$. For the algebras $\langle T_2, T_3, \cdots, T_{L+1} \rangle$, the algebras are semi-simple and have a basis of projectors. Each projector is specified by a node in the final layer, which in turn specifies a unique path to the first layer.

In the 2026 paper, we show that the monomial basis specified has the **correct dimension, equal to the number of projectors**, i.e. equal to the number of nodes in the final layer.

$p(n)$ in terms of degeneracy graph.

## Matrix of monomial expansion coefficients in projectors

Conjecture specifies the exponents $\{m_k\}$.

$$\prod_{k=1}^{L} \mathcal{C}_k^{m_k} = \sum_{a_L \in [D_L]} \left( \prod_{k=1}^{L} (x_{a_k}^{(k)})^{m_k} \right) \mathcal{P}_{a_L}^{(L)}$$

$$\mathcal{M}_{a_L,\mathbf{m}} = \left( \prod_{k=1}^{L} (x_{a_k}^{(k)})^{m_k} \right)$$

Projectors can be expressed in terms of monomials, if the matrix is invertible.

## Computational evidence : Factored determinant

**Determinant Conjecture 1:** The matrix $\mathcal{M}$ above has a determinant

$$\det(\mathcal{M}) = \pm 1 \prod_{i=1}^{L} \prod_{a=1}^{D_{i-1}} \prod_{p<q \in B_a^{(i)}} (x_p^{(i)} - x_q^{(i)})^{\mathrm{Exponent}(i,p,q)}$$

with positive integer exponents, $\mathrm{Exponent}(i, p, q) > 0$ for all $i \in \{1, 2, \cdots, L\}$, for all $a \in [D_{i-1}]$ and every pair $p, q \in B_a^{(i)}$ (same parent).

A generalisation of the Vandermonde determinant.

The arxiv upload has the relevant sage files.

# Factored determinant : Example 1

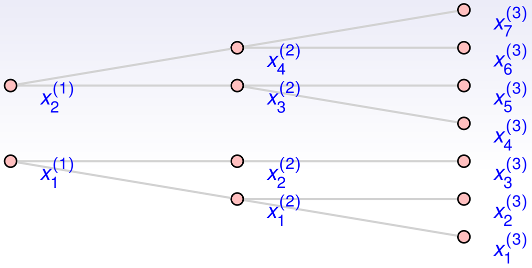

List of monomials

$$\{1,\ \mathcal{C}_1,\ \mathcal{C}_2,\ \mathcal{C}_1\mathcal{C}_2\}$$

$$M = \begin{pmatrix} 1 & x_1^{(1)} & x_1^{(2)} & x_1^{(1)}x_1^{(2)} \\ 1 & x_1^{(1)} & x_2^{(2)} & x_1^{(1)}x_2^{(2)} \\ 1 & x_2^{(1)} & x_3^{(2)} & x_2^{(1)}x_3^{(2)} \\ 1 & x_2^{(1)} & x_4^{(2)} & x_2^{(1)}x_4^{(2)} \end{pmatrix}.$$

$$\det M \;=\; -\left(x_1^{(1)} - x_2^{(1)}\right)^2 \left(x_1^{(2)} - x_2^{(2)}\right)\left(x_3^{(2)} - x_4^{(2)}\right)$$

# Factored determinant : Example 2




$$1,\ \mathcal{C}_1,\ \mathcal{C}_3,\ \mathcal{C}_1\mathcal{C}_3,\ \mathcal{C}_2,\ \mathcal{C}_1\mathcal{C}_2,\ \mathcal{C}_2\mathcal{C}_3.$$

## Factored determinant : Example 2

The matrix is

$$M = \begin{pmatrix} 1 & x_1^{(1)} & x_1^{(3)} & x_1^{(1)}x_1^{(3)} & x_1^{(2)} & x_1^{(1)}x_1^{(2)} & x_1^{(2)}x_1^{(3)} \\ 1 & x_1^{(1)} & x_2^{(3)} & x_1^{(1)}x_2^{(3)} & x_1^{(2)} & x_1^{(1)}x_1^{(2)} & x_1^{(2)}x_2^{(3)} \\ 1 & x_1^{(1)} & x_3^{(3)} & x_1^{(1)}x_3^{(3)} & x_2^{(2)} & x_1^{(1)}x_2^{(2)} & x_2^{(2)}x_3^{(3)} \\ 1 & x_2^{(1)} & x_4^{(3)} & x_2^{(1)}x_4^{(3)} & x_3^{(2)} & x_2^{(1)}x_3^{(2)} & x_3^{(2)}x_4^{(3)} \\ 1 & x_2^{(1)} & x_5^{(3)} & x_2^{(1)}x_5^{(3)} & x_3^{(2)} & x_2^{(1)}x_3^{(2)} & x_3^{(2)}x_5^{(3)} \\ 1 & x_2^{(1)} & x_6^{(3)} & x_2^{(1)}x_6^{(3)} & x_4^{(2)} & x_2^{(1)}x_4^{(2)} & x_4^{(2)}x_6^{(3)} \\ 1 & x_2^{(1)} & x_7^{(3)} & x_2^{(1)}x_7^{(3)} & x_4^{(2)} & x_2^{(1)}x_4^{(2)} & x_4^{(2)}x_7^{(3)} \end{pmatrix}.$$

The determinant is

$$\det M = (x_1^{(1)} - x_2^{(1)})^3 (x_1^{(2)} - x_2^{(2)}) (x_3^{(2)} - x_4^{(2)})^2 (x_1^{(3)} - x_2^{(3)}) (x_4^{(3)} - x_5^{(3)}) (x_6^{(3)} - x_7^{(3)})$$

Consistent with Conjecture 2 (truncated graph conjecture) for the determinant.

## Application to non-commutative algebras via maximally commutative sub-algebras

- By inverting the matrix $\mathcal{M}_{a_L,\mathbf{m}}$, we can write the projectors labelled by $R$ in terms of the monomials labelled by $m$.
- The construction works for any commutative associative semi-simple algebra (which always have a basis of projectors).

- Example : The algebra generated by the Jucys-Murphy elements $J_2, J_3, \cdots, J_n \in \mathbb{C}(S_n)$.

$$J_k = (1, k) + (2, k) + \cdots + (k-1, k)$$

- By using the known eigenvalues of $J_k$ in irreps of $S_n$, it is easy to construct the degeneracy graphs (similar to Bratteli diagrams, but (labelled) Young tableaux as loops) and build the monomial basis.

# Diagonal matrix units for $\mathbb{C}(S_n)$

Matrix elements of permutations $\sigma \in S_n$ in irreducible representations $R$ can be used to construct the matrix basis for $\mathbb{C}(S_n)$.

$$Q^R_{ij} = \frac{d_R}{n!} \sum_{\sigma \in S_n} D^R_{ij}(\sigma)\, \sigma$$

These multiply as elementary matrices in blocks labelled by $R$

$$Q^R_{ij} Q^S_{kl} = \delta^{RS} \delta_{jk} Q^R_{il}$$

And give the decomposition of $\mathbb{C}(S_n)$ - Maschke's theorem for finite group algebras ; Wedderburn-Artin theorem for non-commutative associative semi-simple algebras.

The diagonal ones $Q^R_{ii}$ span a maximally commutative algebra which can be constructed by using the JM elements and the degeneracy-graph/monomial-basis inversion.

## Algebras for 2-matrix models

Matrix Units for NCASS-algebras give rise to orthogonal bases of observables in matrix quantum mechanics
For gauge invariant observables of degree $(m, n)$ in two matrices $X, Y$, there are two representations theoretic orthogonal bases:

- $\mathcal{A}(m, n)$: the sub-algebra of $\mathbb{C}(S_{m+n})$ which commutes with $S_m \times S_n$.
- Sub-algebra of $B_N(m, n)$ (walled Brauer algebra) which commutes with $S_m \times S_n$.

These are associative and semi-simple. Have bases of matrix units. Given generators of a maximal commutative sub-algebra, the diagonal matrix units follow from the monomial basis algorithm.
**Question:** natural generators of maximal commutative sub-algebras of $\mathcal{A}(m, n)$ and its walled Brauer analogue?
(in progress with M. Studziński)

## Outlook

1. This line of research was motivated by the sequence $k_*(n)$. Finding the asymptotic growth of this for large $n$ will be useful for many of these directions mentioned. Heuristic estimate $k_*(n) \sim n^{1/4}$ - close to lower bound $\frac{n^{1/4}}{\sqrt{\log n}}$ and seems compatible with data up to $n \sim 80$.
2. Also interesting for AdS/CFT is $k_*(n, N)$.
3. The orthogonal multi-matrix bases related to matrix units of $\mathcal{A}(m, n)$ and Brauer have should wider applicability in multi-matrix quantum mechanics ( beyond the original AdS/CFT motivations) – e.g. BFSS ? IKKT ?