MUR
Funded by the European Union
Research supported by PNRR Young Researchers,
Project number: MSCA2024-0000100

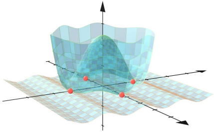
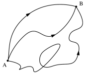
Università di Genova

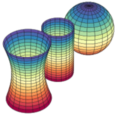
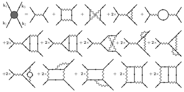
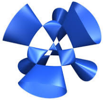
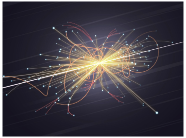
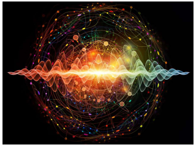
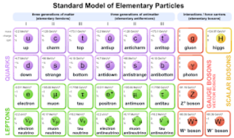
BV theory for Spectral Triples

Roberta A. Iseppi

Workshop on Noncommutative and Generalised Geometry and their Physical Applications

Corfù, 16 September 2026

Spectral Triples and Gauge Theories

Def: A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of:

- an involutive unital algebra $\mathcal{A}$, faithfully represented as operators on a Hilbert space $\mathcal{H}$, $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$;
- a separable Hilbert space $\mathcal{H}$;
- a self-adjoint operator $D : \mathcal{H} \to \mathcal{H}$, with dense domain, such that the resolvent $(i - D)^{-1}$ is a compact operator and the commutators $[D, a]$ are bounded for each $a \in \mathcal{A}$

Spectral Triples and Gauge Theories

Def: A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of:

- an involutive unital algebra $\mathcal{A}$, faithfully represented as operators on a Hilbert space $\mathcal{H}$, $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$;
- a separable Hilbert space $\mathcal{H}$;
- a self-adjoint operator $D : \mathcal{H} \to \mathcal{H}$, with dense domain, such that the resolvent $(i - D)^{-1}$ is a compact operator and the commutators $[D, a]$ are bounded for each $a \in \mathcal{A}$

Additional structure can be added, such as $\mathbb{Z}/2$-grading $\gamma : \mathcal{H} \to \mathcal{H}$ or a real structure, that is, an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$ such that:

$$\underbrace{[a, Jb^* J^{-1}] = 0 \quad \forall a, b \in \mathcal{A}}_{\text{commutation relation}} \qquad \text{and} \qquad \underbrace{[[D, a], Jb^* J^{-1}] = 0, \quad \forall a, b \in \mathcal{A}}_{\text{first-order condition}}$$

Spectral Triples and Gauge Theories

Def: A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of:

- an involutive unital algebra $\mathcal{A}$, faithfully represented as operators on a Hilbert space $\mathcal{H}$, $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$;
- a separable Hilbert space $\mathcal{H}$;
- a self-adjoint operator $D : \mathcal{H} \to \mathcal{H}$, with dense domain, such that the resolvent $(i - D)^{-1}$ is a compact operator and the commutators $[D, a]$ are bounded for each $a \in \mathcal{A}$

Additional structure can be added, such as $\mathbb{Z}/2$-grading $\gamma : \mathcal{H} \to \mathcal{H}$ or a real structure, that is, an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$ such that:

$$\underbrace{[a, Jb^* J^{-1}] = 0 \quad \forall a, b \in \mathcal{A}}_{\text{commutation relation}} \qquad \text{and} \qquad \underbrace{[[D, a], Jb^* J^{-1}] = 0, \quad \forall a, b \in \mathcal{A}}_{\text{first-order condition}}$$

➡ Having the spacetime, we need to add the dynamics.

Spectral Triples and Gauge Theories

Def: A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of:

- an involutive unital algebra $\mathcal{A}$, faithfully represented as operators on a Hilbert space $\mathcal{H}$, $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$;
- a separable Hilbert space $\mathcal{H}$;
- a self-adjoint operator $D : \mathcal{H} \to \mathcal{H}$, with dense domain, such that the resolvent $(i - D)^{-1}$ is a compact operator and the commutators $[D, a]$ are bounded for each $a \in \mathcal{A}$

Additional structure can be added, such as $\mathbb{Z}/2$-grading $\gamma : \mathcal{H} \to \mathcal{H}$ or a real structure, that is, an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$ such that:

$$[a, Jb^* J^{-1}] = 0 \quad \forall a, b \in \mathcal{A} \qquad \text{and} \qquad [[D, a], Jb^* J^{-1}] = 0, \quad \forall a, b \in \mathcal{A}$$

commutation relation — first-order condition

➡ Having the spacetime, we need to add the dynamics.

Spectral action: $\quad S[\varphi] := Tr\left(f\left(\frac{D+\varphi}{\Lambda}\right)\right)$

- $\varphi \in \left\{\varphi = \sum_j a_j[D, b_j] : \varphi^* = \varphi, a_i, b_i \in \mathcal{A}\right\}$ inner fluctuations of the operator D
- Λ fixes the energy scale
- f is a test function, plays a role only via its momenta $\quad f_0 := f(0), \quad f_k := \int_0^{+\infty} f(\nu)\nu^{k-1} d\nu$

Spectral Triples and Gauge Theories [2]

Spectral Triple $(\mathcal{A}, \mathcal{H}, D)$ ➡ Gauge Theory $(X_0, S_0, \mathcal{G})$

- $\mathcal{A}$ = unital *-alg., $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$
- $\mathcal{H}$ = Hilbert space
- $D : \mathcal{H} \to \mathcal{H}$ = self-adj. operator

- $X_0 = \{\varphi = \sum_j a_j [D, b_j] : \varphi^* = \varphi\} \rightsquigarrow$ configuration space
- $S_0[D + \varphi] = Tr(f((D + \varphi)/\Lambda)) \rightsquigarrow$ action functional
- $\mathcal{G} = \mathcal{U}(\mathcal{A}) \rightsquigarrow$ gauge group

Spectral Triples and Gauge Theories [2]

Spectral Triple $(\mathcal{A}, \mathcal{H}, D)$ ➡ Gauge Theory $(X_0, S_0, \mathcal{G})$

- $\mathcal{A}$ = unital *-alg., $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$
- $\mathcal{H}$ = Hilbert space
- $D : \mathcal{H} \to \mathcal{H}$ = self-adj. operator

- $X_0 = \{\varphi = \sum_j a_j[D, b_j] : \varphi^* = \varphi\} \rightsquigarrow$ configuration space
- $S_0[D + \varphi] = Tr(f((D + \varphi)/\Lambda)) \rightsquigarrow$ action functional
- $\mathcal{G} = \mathcal{U}(\mathcal{A}) \rightsquigarrow$ gauge group

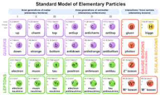
Chamseddine, Connes, Marcolli [2007]

$M \times F$

M = compact Riem. spin manifold with canonical spectral triple

$(\mathcal{C}^\infty(M), L^2(M, S), D_M, J_M, \gamma_M)$

F= finite noncomm. space with finite real spectral triple



$(\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \mathbb{C}^{96}, D_F, J_F, \gamma_F)$

Gauge group:
$U(1) \times SU(2) \times SU(3)$

96 particles

Spectral Triples and Gauge Theories [2]

Spectral Triple $(\mathcal{A}, \mathcal{H}, D)$ ➡ **Gauge Theory** $(X_0, S_0, \mathcal{G})$

- $\mathcal{A}$ = unital *-alg., $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$
- $\mathcal{H}$ = Hilbert space
- $D : \mathcal{H} \to \mathcal{H}$ = self-adj. operator

- $X_0 = \{\varphi = \sum_j a_j[D, b_j] : \varphi^* = \varphi\} \rightsquigarrow$ configuration space
- $S_0[D + \varphi] = Tr(f((D + \varphi)/\Lambda)) \rightsquigarrow$ action functional
- $\mathcal{G} = \mathcal{U}(\mathcal{A}) \rightsquigarrow$ gauge group

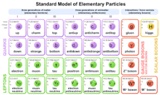
Chamseddine, Connes, Marcolli [2007]

$$M \times F$$

M = compact Riem. spin manifold with canonical spectral triple

$$(\mathcal{C}^\infty(M), L^2(M, S), D_M, J_M, \gamma_M)$$

F= finite noncomm. space with finite real spectral triple



$$(\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \mathbb{C}^{96}, D_F, J_F, \gamma_F)$$

Gauge group: $U(1) \times SU(2) \times SU(3)$

96 particles

The full Lagrangian of the Standard Model can be obtained as heat kernel expansion of the spectral and the fermionic actions

$$S_0(A, \psi) = \underbrace{Tr(f(D_{SM}/\Lambda))}_{\text{spectral action}} + \underbrace{\langle J_{SM}(\psi), D_{SM}\psi \rangle}_{\text{fermionic action}}$$

Towards the quantization of the spectral action

- How can we quantize our theory?

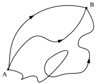
Towards the quantization of the spectral action

- How can we quantize our theory?

$S_0 := Tr(f(D/\Lambda))$

$$Z = \int_{X_0} e^{\frac{i}{\hbar} S_0} [d\mu]$$

$X_0 = \Omega^1_D(\mathcal{A})_{s.a.}$



Path integral

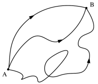
Towards the quantization of the spectral action

How can we quantize our theory?

$S_0 := Tr(f(D/\Lambda))$

$$Z = \int_{X_0} e^{\frac{i}{\hbar} S_0} [d\mu]$$

$X_0 = \Omega^1_D(\mathcal{A})_{s.a.}$

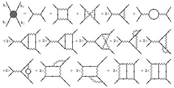


Path integral

Perturbative QFT: define the integral by implementing the stationary phase approximation

$$\underset{\hbar \to 0}{\backsim} \sum_{x_0 \in \{\text{crit. pts } S_0\}} e^{\frac{i}{\hbar} S_0(x_0)} \, |\det S_0''(x_0)|^{-\frac{1}{2}} \; e^{\frac{\pi i}{4} sign(S_0''(x_0))} (2\pi\hbar)^{\frac{\dim X_0}{2}} \sum_\Gamma \frac{\hbar^{-\chi(\Gamma)}}{|Aut(\Gamma)|} \Phi_\Gamma$$

where Γ is a Feynman diagram, with Euler characteristic $\chi(\Gamma)$, and weight Φ_Γ.



Feynman diagrams

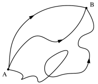
Towards the quantization of the spectral action

How can we quantize our theory?

$S_0 := Tr(f(D/\Lambda))$

$$\boxed{Z = \int_{X_0} e^{\frac{i}{\hbar} S_0} [d\mu]}$$

$X_0 = \Omega^1_D(\mathcal{A})_{s.a.}$

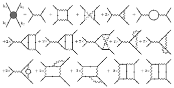


Path integral

Perturbative QFT: define the integral by implementing the stationary phase approximation

$$\underset{\hbar\to 0}{\sim} \sum_{x_0 \in \{\text{crit. pts } S_0\}} e^{\frac{i}{\hbar} S_0(x_0)} \left|\det S_0''(x_0)\right|^{-\frac{1}{2}} e^{\frac{\pi i}{4} sign(S_0''(x_0))} (2\pi\hbar)^{\frac{\dim X_0}{2}} \sum_\Gamma \frac{\hbar^{-\chi(\Gamma)}}{|Aut(\Gamma)|} \Phi_\Gamma$$

where Γ is a Feynman diagram, with Euler characteristic $\chi(\Gamma)$, and weight Φ_Γ.



Feynman diagrams

⚠ To apply it the critical points of S_0 have to be isolated and regular

BUT: for a gauge invariant action functional, critical points appear in orbits

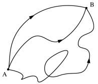
Towards the quantization of the spectral action

How can we quantize our theory?

$$S_0 := Tr(f(D/\Lambda))$$

$$\boxed{Z = \int_{X_0} e^{\frac{i}{\hbar}S_0}[d\mu]}$$

$$X_0 = \Omega^1_D(\mathcal{A})_{s.a.}$$

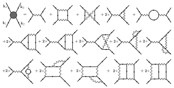


Path integral

Perturbative QFT: define the integral by implementing the stationary phase approximation

$$\underset{\hbar\to 0}{\sim} \sum_{x_0\in\{\text{crit. pts } S_0\}} e^{\frac{i}{\hbar}S_0(x_0)} \,|\det S_0''(x_0)|^{-\frac{1}{2}}\; e^{\frac{\pi i}{4} sign(S_0''(x_0))}(2\pi\hbar)^{\frac{\dim X_0}{2}} \sum_\Gamma \frac{\hbar^{-\chi(\Gamma)}}{|Aut(\Gamma)|}\Phi_\Gamma$$

where Γ is a Feynman diagram, with Euler characteristic $\chi(\Gamma)$, and weight Φ_Γ.



Feynman diagrams

To apply it the critical points of S_0 have to be isolated and regular
BUT: for a gauge invariant action functional, critical points appear in orbits

How to eliminate
these redundancies?

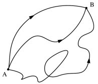
Towards the quantization of the spectral action

How can we quantize our theory?

$S_0 := Tr(f(D/\Lambda))$

$$Z = \int_{X_0} e^{\frac{i}{\hbar}S_0}[d\mu]$$

$X_0 = \Omega^1_D(\mathcal{A})_{s.a.}$

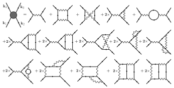


Path integral

Perturbative QFT: define the integral by implementing the stationary phase approximation

$$\underset{\hbar\to 0}{\backsim} \sum_{x_0 \in \{\text{crit. pts } S_0\}} e^{\frac{i}{\hbar}S_0(x_0)} \, |\det S_0''(x_0)|^{-\frac{1}{2}} \; e^{\frac{\pi i}{4} sign(S_0''(x_0))} (2\pi\hbar)^{\frac{\dim X_0}{2}} \sum_\Gamma \frac{\hbar^{-\chi(\Gamma)}}{|Aut(\Gamma)|}\Phi_\Gamma$$

where Γ is a Feynman diagram, with Euler characteristic $\chi(\Gamma)$, and weight Φ_Γ.



Feynman diagrams

To apply it the critical points of S_0 have to be isolated and regular

BUT: for a gauge invariant action functional, critical points appear in orbits

How to eliminate these redundancies? take the quotient w.r.t. the action of the group

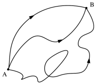
Towards the quantization of the spectral action

How can we quantize our theory?

$S_0 := Tr(f(D/\Lambda))$

$$Z = \int_{X_0} e^{\frac{i}{\hbar} S_0} [d\mu]$$

$X_0 = \Omega^1_D(\mathcal{A})_{s.a.}$

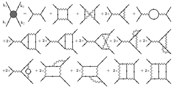


Path integral

Perturbative QFT: define the integral by implementing the stationary phase approximation

$$\underset{\hbar \to 0}{\sim} \sum_{x_0 \in \{\text{crit. pts } S_0\}} e^{\frac{i}{\hbar} S_0(x_0)} \left|\det S_0''(x_0)\right|^{-\frac{1}{2}} e^{\frac{\pi i}{4} sign(S_0''(x_0))} (2\pi\hbar)^{\frac{\dim X_0}{2}} \sum_\Gamma \frac{\hbar^{-\chi(\Gamma)}}{|Aut(\Gamma)|} \Phi_\Gamma$$

where Γ is a Feynman diagram, with Euler characteristic $\chi(\Gamma)$, and weight Φ_Γ.



Feynman diagrams

To apply it the critical points of S_0 have to be isolated and regular

BUT: for a gauge invariant action functional, critical points appear in orbits

How to eliminate these redundancies?

→ take the quotient w.r.t. the action of the group

→ add extra auxiliary variables $\rightsquigarrow$ ghost fields ✓

The BV construction: the key idea

Def. A ghost field φ is characterized by:

$$\textit{ghost degree}\text{: } \deg(\varphi) \in \mathbb{Z} \quad \& \quad \textit{parity}\text{: } \epsilon(\varphi) \in \{0,1\}, \quad \deg(\varphi) \equiv \epsilon(\varphi) \ \textit{mod} \ \mathbb{Z}/\mathbb{Z}2$$

For a ghost φ, its antighost φ^* has

$$\deg(\varphi^*) = -\deg(\varphi) - 1 \quad \& \quad \epsilon(\varphi^*) \equiv \epsilon(\varphi) + 1 \quad (\textit{mod} \ \mathbb{Z}/2\mathbb{Z})$$

The BV construction: the key idea

Def. A ghost field φ is characterized by:

$$\textit{ghost degree}: \deg(\varphi) \in \mathbb{Z} \quad \& \quad \textit{parity}: \epsilon(\varphi) \in \{0,1\}, \quad \deg(\varphi) \equiv \epsilon(\varphi) \;\; \textit{mod } \mathbb{Z}/\mathbb{Z}2$$

For a ghost φ, its antighost φ^* has

$$\deg(\varphi^*) = -\deg(\varphi) - 1 \quad \& \quad \epsilon(\varphi^*) \equiv \epsilon(\varphi) + 1 \quad (\textit{mod } \mathbb{Z}/2\mathbb{Z})$$

Key idea: [B.V.] The integral $(*)$ is invariant under the change of Lagrangian submanifold $\mathcal{L}$ in the homotopy class of $[X_0] \subset X_t$ and of action S_q in the quantum BV cohomology class of S_0

$$(*) \quad \int_{X_0} e^{\frac{i}{\hbar} S_0} \, [d\mu] \underset{BV}{\cong} \int_{[\mathcal{L}] \subset X_t} e^{\frac{i}{\hbar} S_q} \, d\mu_{BV}$$

The BV construction: the key idea

Def. A ghost field φ is characterized by:

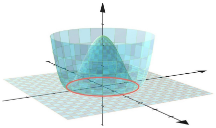
$$\textit{ghost degree}: \deg(\varphi) \in \mathbb{Z} \quad \& \quad \textit{parity}: \epsilon(\varphi) \in \{0,1\}, \quad \deg(\varphi) \equiv \epsilon(\varphi) \ \textit{mod}\ \mathbb{Z}/\mathbb{Z}2$$

For a ghost φ, its antighost φ^* has

$$\deg(\varphi^*) = -\deg(\varphi) - 1 \quad \& \quad \epsilon(\varphi^*) \equiv \epsilon(\varphi) + 1 \quad (\textit{mod}\ \mathbb{Z}/2\mathbb{Z})$$

Key idea: [B.V.] The integral $(*)$ is invariant under the change of Lagrangian submanifold $\mathcal{L}$ in the homotopy class of $[X_0] \subset X_t$ and of action S_q in the quantum BV cohomology class of S_0

$$(*) \quad \int_{X_0} e^{\frac{i}{\hbar} S_0}\, [d\mu] \underset{BV}{\cong} \int_{[\mathcal{L}]\subset X_t} e^{\frac{i}{\hbar} S_q}\, d\mu_{BV}$$



The BV construction: the key idea

Def. A ghost field φ is characterized by:

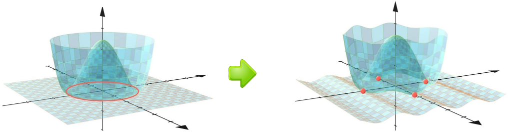
$$\textit{ghost degree}\text{: } \deg(\varphi) \in \mathbb{Z} \quad \& \quad \textit{parity}\text{: } \epsilon(\varphi) \in \{0,1\}, \quad \deg(\varphi) \equiv \epsilon(\varphi) \ \ mod\ \mathbb{Z}/\mathbb{Z}2$$

For a ghost φ, its antighost φ^* has

$$\deg(\varphi^*) = -\deg(\varphi) - 1 \quad \& \quad \epsilon(\varphi^*) \equiv \epsilon(\varphi) + 1 \quad (mod\ \mathbb{Z}/2\mathbb{Z})$$

Key idea: [B.V.] The integral $(*)$ is invariant under the change of Lagrangian submanifold $\mathcal{L}$ in the homotopy class of $[X_0] \subset X_t$ and of action S_q in the quantum BV cohomology class of S_0

$$(*) \qquad \int_{X_0} e^{\frac{i}{\hbar} S_0}\, [d\mu] \underset{BV}{\cong} \int_{[\mathcal{L}] \subset X_t} e^{\frac{i}{\hbar} S_q}\, d\mu_{BV}$$



The BV construction: the key idea

Def. A ghost field φ is characterized by:

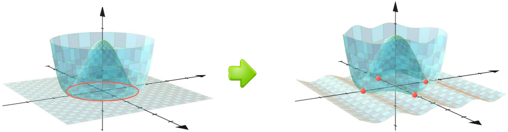
$$\textit{ghost degree}\text{: } \deg(\varphi) \in \mathbb{Z} \quad \& \quad \textit{parity}\text{: } \epsilon(\varphi) \in \{0,1\}, \quad \deg(\varphi) \equiv \epsilon(\varphi) \ \ \textit{mod}\ \mathbb{Z}/\mathbb{Z}2$$

For a ghost φ, its antighost φ^* has

$$\deg(\varphi^*) = -\deg(\varphi) - 1 \quad \& \quad \epsilon(\varphi^*) \equiv \epsilon(\varphi) + 1 \quad (\textit{mod}\ \mathbb{Z}/2\mathbb{Z})$$

Key idea: [B.V.] The integral $(*)$ is invariant under the change of Lagrangian submanifold $\mathcal{L}$ in the homotopy class of $[X_0] \subset X_t$ and of action S_q in the quantum BV cohomology class of S_0

$$(*) \quad \int_{X_0} e^{\frac{i}{\hbar} S_0}\, [d\mu] \underset{BV}{\cong} \int_{[\mathcal{L}] \subset X_t} e^{\frac{i}{\hbar} S_q}\, d\mu_{BV}$$



The goal: To find
- $\mathcal{L}$ Lagrangian $\subset X_t$ ghost sector &
- $S_q \in \mathcal{C}^\infty(X_t)[[\hbar]]$, sol. quant. master eq. s.t. $S_q|_{\mathcal{L}}$ has *isolated* and *regular* critical points.

The classical/quantum BV construction

$$(X_0, S_0) \xrightarrow{\text{BV extension}} (\widetilde{X}, \widetilde{S})$$

- $\widetilde{X} = \bigoplus_{i\in\mathbb{Z}}[\widetilde{X}]^i$, $\mathbb{Z}$-graded super-vect. sp., $\widetilde{X} = \mathcal{F} \oplus \mathcal{F}^*[1]$, $[\widetilde{X}]^0 = X_0$
  Symmetry between the ghost sector $\mathcal{F}$ and the anti-field content $\mathcal{F}^*[1]$

  $\mathcal{F}$ =graded locally free $\mathcal{O}_{X_0}$-mod. with hom. comp. of finite rank

- $\widetilde{S} \in [\mathcal{O}_{\widetilde{X}}]^0$, s.t. $\widetilde{S}|_{X_0} = S_0$ & $\{\widetilde{S}, \widetilde{S}\} = 0$

  1-degree Poisson strut. on $\mathcal{O}_{\tilde{X}}$ $\{\ ,\ \} : \mathcal{O}^n_{\tilde{X}} \times \mathcal{O}^m_{\tilde{X}} \to \mathcal{O}^{n+m+1}_{\tilde{X}}$
  $\{\varphi_i^*, \varphi_j\} = \delta_{ij}$

The classical/quantum BV construction

$$(X_0, S_0) \xrightarrow{\text{BV extension}} (\widetilde{X}, \widetilde{S})$$

- $\widetilde{X} = \bigoplus_{i\in\mathbb{Z}}[\widetilde{X}]^i$, $\mathbb{Z}$-graded super-vect. sp., $\widetilde{X} = \mathcal{F} \oplus \mathcal{F}^*[1]$, $[\widetilde{X}]^0 = X_0$
  Symmetry between the ghost sector $\mathcal{F}$ and the anti-field content $\mathcal{F}^*[1]$

  $\mathcal{F}$ =graded locally free $\mathcal{O}_{X_0}$-mod. with hom. comp. of finite rank

- $\widetilde{S} \in [\mathcal{O}_{\widetilde{X}}]^0$, s.t. $\widetilde{S}|_{X_0} = S_0$ & $\{\widetilde{S}, \widetilde{S}\} = 0$

  1-degree Poisson strut. on $\mathcal{O}_{\tilde{X}}$ $\{\ ,\ \} : \mathcal{O}^n_{\tilde{X}} \times \mathcal{O}^m_{\tilde{X}} \to \mathcal{O}^{n+m+1}_{\tilde{X}}$
  $\{\varphi_i^*, \varphi_j\} = \delta_{ij}$

Each BV-extended th. induces a BV-complex $\rightsquigarrow$ cohomological approach to the study of gauge symmetries.

$$H^0_{BV}(\widetilde{X}, d_{\tilde{S}}) = \{\text{classical observables}\}$$

The classical/quantum BV construction

$$(X_0, S_0) \xrightarrow{\text{BV extension}} (\widetilde{X}, \widetilde{S})$$

- $\widetilde{X} = \bigoplus_{i\in\mathbb{Z}}[\widetilde{X}]^i$, $\mathbb{Z}$-graded super-vect. sp., $\widetilde{X} = \mathcal{F} \oplus \mathcal{F}^*[1]$, $[\widetilde{X}]^0 = X_0$
  Symmetry between the ghost sector $\mathcal{F}$ and the anti-field content $\mathcal{F}^*[1]$

  $\mathcal{F}$ =graded locally free $\mathcal{O}_{X_0}$-mod. with hom. comp. of finite rank

- $\widetilde{S} \in [\mathcal{O}_{\widetilde{X}}]^0$, s.t. $\widetilde{S}|_{X_0} = S_0$ & $\{\widetilde{S}, \widetilde{S}\} = 0$

  1-degree Poisson strut. on $\mathcal{O}_{\tilde{X}}$ $\{\ ,\ \} : \mathcal{O}^n_{\tilde{X}} \times \mathcal{O}^m_{\tilde{X}} \to \mathcal{O}^{n+m+1}_{\tilde{X}}$
  $\{\varphi_i^*, \varphi_j\} = \delta_{ij}$

Each BV-extended th. induces a BV-complex $\rightsquigarrow$ <u>cohomological approach</u> to the study of gauge symmetries.

$$H^0_{BV}(\widetilde{X}, d_{\tilde{S}}) = \{\text{classical observables}\}$$

The classical/quantum BV construction

$$\begin{tikzcd}
 & \underset{\text{BV complex}}{\mathcal{C}^\bullet_{BV}(\widetilde{X}, d_{\tilde{S}/S_q})} \arrow[r, phantom, "\cong"] & \underset{\text{total complex}}{\mathcal{C}^\bullet_{BV}(X_t, d_{S_t/S_{q,t}})} & \underset{\text{BRST complex}}{\mathcal{C}^\bullet_{BRST}(X_t, d_{S_t/S_{q,t}})|_{\mathcal{L}}} \\
\underset{\text{initial gauge theory}}{(X_0, S_0)} \arrow[r, "+\text{ gh./anti-gh.}"] & \underset{\text{BV-extended th.}}{(\widetilde{X}, \widetilde{S}/S_q)} \arrow[u] \arrow[r, "+\text{ aux. flds}"] & \underset{\text{total th}}{(X_t, S_t/S_{q,t})} \arrow[u] \arrow[r, "\text{gauge-}", "\text{fixing}"'] & \underset{\text{gauge-fixed th.}}{(X_t, S_t/S_{q,t})|_{\mathcal{L}}} \arrow[u]
\end{tikzcd}$$

The classical/quantum BV construction

$$(X_0, S_0) \xrightarrow{\text{BV extension}} (\widetilde{X}, \widetilde{S})$$

- $\widetilde{X} = \bigoplus_{i\in\mathbb{Z}}[\widetilde{X}]^i$, $\mathbb{Z}$-graded super-vect. sp., $\widetilde{X} = \mathcal{F} \oplus \mathcal{F}^*[1]$, $[\widetilde{X}]^0 = X_0$
  Symmetry between the ghost sector $\mathcal{F}$ and the anti-field content $\mathcal{F}^*[1]$

$\mathcal{F}$ =graded locally free $\mathcal{O}_{X_0}$-mod. with hom. comp. of finite rank

- $\widetilde{S} \in [\mathcal{O}_{\widetilde{X}}]^0$, s.t. $\widetilde{S}|_{X_0} = S_0$ & $\{\widetilde{S}, \widetilde{S}\} = 0$

1-degree Poisson strut. on $\mathcal{O}_{\tilde{X}}$ $\{\ ,\ \} : \mathcal{O}^n_{\tilde{X}} \times \mathcal{O}^m_{\tilde{X}} \to \mathcal{O}^{n+m+1}_{\tilde{X}}$
$\{\varphi_i^*, \varphi_j\} = \delta_{ij}$

➡ Each BV-extended th. induces a BV-complex $\rightsquigarrow$ cohomological approach to the study of gauge symmetries.

$$H^0_{BV}(\widetilde{X}, d_{\tilde{S}}) = \{\text{classical observables}\}$$

The classical/quantum BV construction

$$\begin{array}{ccccccc}
 & & \underset{\text{BV complex}}{\mathcal{C}^\bullet_{BV}(\widetilde{X}, d_{\tilde{S}/S_q})} & \cong & \underset{\text{total complex}}{\mathcal{C}^\bullet_{BV}(X_t, d_{S_t/S_{q,t}})} & & \underset{\text{BRST complex}}{\mathcal{C}^\bullet_{BRST}(X_t, d_{S_t/S_{q,t}})|_{\mathcal{L}}} \\
 & & \uparrow & & \uparrow & & \uparrow \\
\underset{\text{initial gauge theory}}{(X_0, S_0)} & \xrightarrow{+\text{ gh./anti-gh.}} & \underset{\text{BV-extended th.}}{(\widetilde{X}, \widetilde{S}/S_q)} & \xrightarrow{+\text{ aux. flds}} & \underset{\text{total th}}{(X_t, S_t/S_{q,t})} & \xrightarrow[\text{fixing}]{\text{gauge-}} & \underset{\text{gauge-fixed th.}}{(X_t, S_t/S_{q,t})|_{\mathcal{L}}}
\end{array}$$

$$\begin{array}{ccc}
\underset{\text{[Fredenhagen, Rejzner]}}{\text{functional analysis}} & & \underset{\text{[Costello, Gwilliam, Haugseng]}}{\text{homotopy theory}} \\
\nwarrow & & \nearrow \\
 & \text{BV formalism} & \\
\swarrow & & \searrow \\
\underset{\text{[Cattaneo, Mnev, Reshetikhin, Wernli]}}{\text{differential geometry}} & & \underset{\text{[Felder, Kazhdan, Schlank]}}{\text{algebraic geometry}}
\end{array}$$

Can we describe the BV construction in terms of spectral triples?

Let's first focus on the finite case: we want to study the BV construction for gauge theories induced by

$$(M_n(\mathbb{C}), \mathbb{C}^n, D, f)$$

Can we describe the BV construction in terms of spectral triples?

Let's first focus on the finite case: we want to study the BV construction for gauge theories induced by

$$(M_n(\mathbb{C}), \mathbb{C}^n, D, f)$$

Questions and goals:

- Can the BV construction be described in terms of spectral triples?
- Can the BRST cohomology be related to other (better understood) cohomological theories?

Can we describe the BV construction in terms of spectral triples?

Let's first focus on the finite case: we want to study the BV construction for gauge theories induced by

$$(M_n(\mathbb{C}), \mathbb{C}^n, D, f)$$

Questions and goals:

- Can the BV construction be described in terms of spectral triples?
- Can the BRST cohomology be related to other (better understood) cohomological theories?

$$\begin{tikzcd}
\begin{array}{c}(M_n(\mathbb{C}), \mathbb{C}^n, D, f)\\ \text{initial spectral triple}\end{array} \arrow[r] \arrow[dd, leftrightarrow] & \begin{array}{c}?\\ \text{BV spectral triple}\end{array} \arrow[r] \arrow[d] & \begin{array}{c}?\\ \text{total spectral triple}\end{array} \arrow[r] \arrow[d] & \begin{array}{c}?\\ \text{gauge-fixing fermion}\end{array} \arrow[d] \\
 & \begin{array}{c}\mathcal{C}^\bullet_{BV}(\tilde{X}, d_{\tilde{S}}) \cong ?\\ \text{BV complex} \cong \text{??}\end{array} & \begin{array}{c}\mathcal{C}^\bullet_{BV}(X_t, d_{S_t}) \cong ?\\ \text{BV tot complex} \cong \text{??}\end{array} & \begin{array}{c}\mathcal{C}^\bullet_{BRST}(X_t, d_{S_t})|_\Psi \cong ?\\ \text{BRST complex} \cong \text{??}\end{array} \\
\begin{array}{c}(X_0, S_0)\\ \text{initial gauge theory}\end{array} \arrow[r] & \begin{array}{c}(\tilde{X}, \tilde{S})\\ \text{extended theory}\end{array} \arrow[r] \arrow[u] & \begin{array}{c}(X_t, S_t)\\ \text{total theory}\end{array} \arrow[r] \arrow[u] & \begin{array}{c}(X_t, S_t)|_\Psi\\ \text{gauge-fixed theory}\end{array} \arrow[u]
\end{tikzcd}$$

noncommutative geometry

BV construction

Can we describe the BV construction in terms of spectral triples?

Let's first focus on the finite case: we want to study the BV construction for gauge theories induced by

$$(M_n(\mathbb{C}), \mathbb{C}^n, D, f)$$

Reached goals:

- Established the noncom. geometrical BV construction for finite spectral triple over $M_n(\mathbb{C})$.
- Discovered the relation existing between BV/BRST cohomology and Hochschild cohomology

$$\begin{tikzcd}
\begin{array}{c}(M_n(\mathbb{C}), \mathbb{C}^n, D, f)\\ \text{initial spectral triple}\end{array} \arrow[r] \arrow[dd, leftrightarrow] & \begin{array}{c}(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})\\ \text{BV spectral triple}\end{array} \arrow[r] \arrow[d] & \begin{array}{c}(\mathcal{A}_t, \mathcal{H}_t, D_t, J_t)\\ \text{total spectral triple}\end{array} \arrow[r] \arrow[d] & \begin{array}{c}\Psi \in \mathcal{F}^{-1}_{\mathcal{O}_{X_0}}(\mathcal{H}_{BV,f} \oplus \mathcal{H}_{aux,f})\\ \text{gauge-fixing fermion}\end{array} \arrow[d] & \text{NCG} \\
 & \begin{array}{c}\mathcal{C}^\bullet_H(\mathcal{B}, \mathcal{M}) \cong \mathcal{C}^\bullet_{BV}(\widetilde{X}, d_{\tilde{S}})\\ \text{BV/ Hochschild complex}\end{array} & \begin{array}{c}\mathcal{C}^\bullet_H(\mathcal{B}_t, \mathcal{M}) \cong \mathcal{C}^\bullet_{BV}(X_t, d_{S_t})\\ \text{BV tot / Hochschild complex}\end{array} & \begin{array}{c}\mathcal{C}^\bullet_H(\mathcal{B}_\Psi, \mathcal{M}) \cong \mathcal{C}^\bullet_{BRST}(X_t, d_{S_t})|_\Psi\\ \text{BRST/ Hochschild complex}\end{array} & \\
\begin{array}{c}(X_0, S_0)\\ \text{initial gauge theory}\end{array} \arrow[r] & \begin{array}{c}(\widetilde{X}, \widetilde{S})\\ \text{extended theory}\end{array} \arrow[r] \arrow[u] & \begin{array}{c}(X_t, S_t)\\ \text{total theory}\end{array} \arrow[r] \arrow[u] & \begin{array}{c}(X_t, S_t)|_\Psi\\ \text{gauge-fixed theory}\end{array} \arrow[u] & \text{BV}
\end{tikzcd}$$

Can we describe the BV construction in terms of spectral triples?

Let's first focus on the finite case: we want to study the BV construction for gauge theories induced by

$$(M_n(\mathbb{C}), \mathbb{C}^n, D, f)$$

Reached goals:

- Established the noncom. geometrical BV construction for finite spectral triple over $M_n(\mathbb{C})$.
- Discovered the relation existing between BV/BRST cohomology and Hochschild cohomology

$$\begin{tikzcd}
\underset{\text{initial spectral triple}}{(M_n(\mathbb{C}), \mathbb{C}^n, D, f)} \arrow[r] \arrow[dd, leftrightarrow] & \underset{\text{BV spectral triple}}{(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})} \arrow[r] \arrow[d] & \underset{\text{total spectral triple}}{(\mathcal{A}_t, \mathcal{H}_t, D_t, J_t)} \arrow[r] \arrow[d] & \underset{\text{gauge-fixing fermion}}{\Psi \in \mathcal{F}^{-1}_{\mathcal{O}_{X_0}}(\mathcal{H}_{BV,f} \oplus \mathcal{H}_{aux,f})} \arrow[d] & \text{NCG} \\
 & \underset{\text{BV/ Hochschild complex}}{\mathcal{C}^\bullet_H(\mathcal{B}, \mathcal{M}) \cong \mathcal{C}^\bullet_{BV}(\widetilde{X}, d_{\tilde{S}})} & \underset{\text{BV tot / Hochschild complex}}{\mathcal{C}^\bullet_H(\mathcal{B}_t, \mathcal{M}) \cong \mathcal{C}^\bullet_{BV}(X_t, d_{S_t})} & \underset{\text{BRST/ Hochschild complex}}{\mathcal{C}^\bullet_H(\mathcal{B}_\Psi, \mathcal{M}) \cong \mathcal{C}^\bullet_{BRST}(X_t, d_{S_t})|_\Psi} & \\
\underset{\text{initial gauge theory}}{(X_0, S_0)} \arrow[r] & \underset{\text{extended theory}}{(\widetilde{X}, \widetilde{S})} \arrow[r] \arrow[u] & \underset{\text{total theory}}{(X_t, S_t)} \arrow[r] \arrow[u] & \underset{\text{gauge-fixed theory}}{(X_t, S_t)|_\Psi} \arrow[u] & \text{BV}
\end{tikzcd}$$

What if we consider spectral triples over $\oplus_{i=1}^r M_{n_i}(\mathbb{C})$?

The general finite case, almost...

Let $(\mathcal{A}, \mathcal{H}, D)$ be a finite spectral triple, with

$$\mathcal{A} \simeq \bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C}) \qquad \pi : A \to B(\mathcal{H}) \qquad D = (D_{ij})_{i,j}$$

$$\mathcal{H} \simeq \bigoplus_{i=1}^{r} \mathbb{C}^{n_i} \otimes V_{n_i}, \longrightarrow \boxed{\begin{array}{l} V_{n_i} = \text{multiplicity space} \\ \dim(V_{n_i}) = \text{mult. of dim } n_i \text{ repr.} \end{array}} \qquad D_{ij} : \mathcal{H}_{n_i} \to \mathcal{H}_{n_j}.$$

The general finite case, almost...

Let $(\mathcal{A}, \mathcal{H}, D)$ be a finite spectral triple, with

$\mathcal{A} \simeq \bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C})$ $\quad \pi : A \to B(\mathcal{H})$ $\quad D = (D_{ij})_{i,j}$

$\mathcal{H} \simeq \bigoplus_{i=1}^{r} \mathbb{C}^{n_i} \otimes V_{n_i}, \longrightarrow$ $\boxed{\begin{array}{l} V_{n_i} = \text{multiplicity space} \\ \dim(V_{n_i}) = \text{mult. of dim } n_i \text{ repr.} \end{array}}$ $\quad D_{ij} : \mathcal{H}_{n_i} \to \mathcal{H}_{n_j}.$

Spectral Triple $(\mathcal{A}, \mathcal{H}, D)$	⇒	Gauge Theory $(X_0, S_0, \mathcal{G})$
▸ $\mathcal{A}$ = unital *-alg., $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$		▸ $X_0 = \{\varphi = \sum_j a_j[D, b_j] : \varphi^* = \varphi\} \rightsquigarrow$ configuration space
▸ $\mathcal{H}$ = Hilbert space		▸ $S_0[D + \varphi] = Tr(f((D + \varphi)/\Lambda)) \rightsquigarrow$ action functional
▸ $D : \mathcal{H} \to \mathcal{H}$ = self-adj. operator		▸ $\mathcal{G} = \mathcal{U}(\mathcal{A}) \rightsquigarrow$ gauge group

The general finite case, almost...

Let $(\mathcal{A},\mathcal{H},D)$ be a finite spectral triple, with

$$\mathcal{A} \simeq \bigoplus_{i=1}^r M_{n_i}(\mathbb{C}) \qquad \pi : A \to B(\mathcal{H}) \qquad \mathcal{H} \simeq \bigoplus_{i=1}^r \mathbb{C}^{n_i} \otimes V_{n_i}, \longrightarrow \boxed{\begin{array}{l} V_{n_i} = \text{multiplicity space} \\ \dim(V_{n_i}) = \text{mult. of dim } n_i \text{ repr.}\end{array}} \qquad D = (D_{ij})_{i,j} \quad D_{ij} : \mathcal{H}_{n_i} \to \mathcal{H}_{n_j}.$$

Spectral Triple $(\mathcal{A},\mathcal{H},D)$	Gauge Theory $(X_0, S_0, \mathcal{G})$
$\mathcal{A}$ = unital *-alg., $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$	$X_0 = \{\varphi = \sum_j a_j[D,b_j] : \varphi^* = \varphi\} \rightsquigarrow$ configuration space
$\mathcal{H}$ = Hilbert space	$S_0[D+\varphi] = Tr(f((D+\varphi)/\Lambda)) \rightsquigarrow$ action functional
$D : \mathcal{H} \to \mathcal{H}$ = self-adj. operator	$\mathcal{G} = \mathcal{U}(\mathcal{A}) \rightsquigarrow$ gauge group

Prop: Let $(\bigoplus_{i=1}^r M_{n_i}(\mathbb{C}), \bigoplus_{i=1}^r \mathbb{C}^{n_i}_{n_i}, (D_{ij})_{i,j})$ be multiplicity-1 with $n_i \neq n_j$ for $i \neq j$. Then the induced gauge th. is: [R.I.]

- $X_0 = \left[\bigoplus_{i,j} \Omega^1_D(\mathcal{A})_{ij}\right]_{\text{s.a.}}$ where $\Omega^1_D(\mathcal{A})_{ij} = \text{Hom}(\mathbb{C}^{n_j},\mathbb{C}^{n_i})$ for every pair (i,j) such that $D_{ij} \neq 0$
- $S_0(\varphi) = \text{Tr}\big(f(D+\varphi)\big)$ for any polynomial function $f \in \mathbb{R}[x]$.
- the effective gauge group is $\mathcal{G}_{\text{eff}} := \mathcal{U}(\mathcal{A})/Z(\mathcal{U}(\mathcal{A}))$, with $\text{Lie}(\mathcal{G}_{\text{eff}}) \simeq \bigoplus_{i=1}^r \mathfrak{su}(n_i)$.

$\Rightarrow$ $\mathrm{U}(n_i)$-sectors coupled by the nonzero off-diagonal blocks of the Dirac operator. IF $r = 1$ ✓

BV irreducibility for finite spectral triples

What is the structure of the *gauge symmetries* of these theories?

BV irreducibility for finite spectral triples

What is the structure of the *gauge symmetries* of these theories?

IF there exist non-trivial relations among the generators of the infinitesimal gauge symmetries ➡ redundancies characterize reducible gauge theories ➡ this requires the introduction of higher ghost fields

BV irreducibility for finite spectral triples

What is the structure of the *gauge symmetries* of these theories?

IF there exist non-trivial relations among the generators of the infinitesimal gauge symmetries	⇒ redundancies characterize reducible gauge theories	⇒ this requires the introduction of higher ghost fields

Step 1: to rewrite the condition of being BV-irreducible within the language of spectral triples

Prop: Given $(\mathcal{A}, \mathcal{H}, D)$ with induced gauge theory $(X_0, S_0, \mathcal{G})$ and effective Lie algebra $\mathfrak{g} = (\mathcal{G}_{\text{eff}})$.
[R.I.] Then:

The theory is BV-irreducible $\Leftrightarrow$ The infinitesimal gauge action $\rho : \mathfrak{g} \longrightarrow \mathrm{End}(X_0)$ is faithful $\Leftrightarrow$ $\mathrm{Comm}_{\mathfrak{g}}^{D}(X_0) := \{\xi \in \mathfrak{g} : [\xi, D + \varphi] = 0, \forall \varphi \in X_0\} = \{0\}$.

BV irreducibility for finite spectral triples

What is the structure of the *gauge symmetries* of these theories?

IF there exist non-trivial relations among the generators of the infinitesimal gauge symmetries ➡	redundancies characterize reducible gauge theories ➡	this requires the introduction of higher ghost fields

Step 1: to rewrite the condition of being BV-irreducible within the language of spectral triples

Prop: Given $(\mathcal{A}, \mathcal{H}, D)$ with induced gauge theory $(X_0, S_0, \mathcal{G})$ and effective Lie algebra $\mathfrak{g} = (\mathcal{G}_{\text{eff}})$.
[R.I.] Then:

The theory is BV-irreducible $\Leftrightarrow$ The infinitesimal gauge action $\rho : \mathfrak{g} \longrightarrow \mathrm{End}(X_0)$ is faithful $\Leftrightarrow$ $\mathrm{Comm}^D_{\mathfrak{g}}(X_0) := \{\xi \in \mathfrak{g} : [\xi, D + \varphi] = 0, \forall \varphi \in X_0\} = \{0\}$.

The $U(n)$-model: let's consider $(M_n(\mathbb{C}), \mathbb{C}^n, D)$. The induced gauge theory is: $X_0(D) = [M_n(\mathbb{C})]_{\text{s.a.}}$ $\mathfrak{g} = \mathfrak{su}(n)$
To test if the theory is BV-irreducible, let $\xi \in \mathfrak{su}(n)$.

$$\xi \in \mathrm{Comm}^D_{\mathfrak{g}}(X_0) \Rightarrow \xi \text{ commutes with every self-adj. matrix} \Rightarrow \xi \in Z(M_n(\mathbb{C})) = \lambda \cdot Id$$

. But $\mathrm{Tr}(\xi) = 0 \Rightarrow \xi = 0$. All non-trivial $U(n)$-gauge theories are BV-irreducible

BV irreducibility for finite spectral triples [2]

- Let's study the BV-irreducibility of multiplicity 1 finite spectral triple.

$$\mathcal{A} \simeq \bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C}), \text{ with } n_i \neq n_j \qquad \mathcal{H} \cong \bigoplus_{i=1}^{r} \mathbb{C}^{n_i},$$

$\mathcal{D}$ = affine space of admissible Dirac operators on $\mathcal{H}$ completing $(\mathcal{A}, \mathcal{H})$ to a finite spectral triple.

BV irreducibility for finite spectral triples [2]

- Let's study the BV-irreducibility of multiplicity 1 finite spectral triple.

$\mathcal{A} \simeq \bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C})$, with $n_i \neq n_j$ $\mathcal{H} \cong \bigoplus_{i=1}^{r} \mathbb{C}^{n_i}$,

$\mathcal{D}$ = affine space of admissible Dirac operators on $\mathcal{H}$ completing $(\mathcal{A}, \mathcal{H})$ to a finite spectral triple.

Theorem [R.I.]: Let $D \in \mathcal{D}$ and define $X_0(D) := [\Omega^1_D(\mathcal{A})]_{\mathrm{s.a.}}$ to be the corresponding space of self-adjoint inner fluctuations. Then the subset of Dirac operators inducing an irreducible gauge theory, that is,

$$\mathcal{D}_{\mathrm{irr}} := \left\{ D \in \mathcal{D} : \mathrm{Comm}^D_{\mathfrak{g}}(X_0) = \{\xi \in \mathfrak{g} : [\xi, D + \varphi] = 0, \forall \varphi \in X_0\} = 0 \right\}$$

is Zariski open in $\mathcal{D}$.

BV irreducibility for finite spectral triples [2]

- Let's study the BV-irreducibility of multiplicity 1 finite spectral triple.

$$\mathcal{A} \simeq \bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C}), \text{ with } n_i \neq n_j \qquad \mathcal{H} \cong \bigoplus_{i=1}^{r} \mathbb{C}^{n_i},$$

$\mathcal{D}$ = affine space of admissible Dirac operators on $\mathcal{H}$ completing $(\mathcal{A}, \mathcal{H})$ to a finite spectral triple.

Theorem: [R.I.] Let $D \in \mathcal{D}$ and define $X_0(D) := [\Omega^1_D(\mathcal{A})]_{\mathrm{s.a.}}$ to be the corresponding space of self-adjoint inner fluctuations. Then the subset of Dirac operators inducing an irreducible gauge theory, that is,

$$\mathcal{D}_{\mathrm{irr}} := \left\{D \in \mathcal{D} : \mathrm{Comm}^D_{\mathfrak{g}}(X_0) = \{\xi \in \mathfrak{g} : [\xi, D + \varphi] = 0, \forall \varphi \in X_0\} = 0\right\}$$

is Zariski open in $\mathcal{D}$.

For a dense subspace of admissible Dirac operator we can introduce only ghost fields of degree 1.

Goal: given a finite spectral triple, to determine a method to associate to it a BV-spectral triple, containing all the relevant information about the ghost field sector and the *extended action*

BV irreducibility for finite spectral triples [2]

- Let's study the BV-irreducibility of multiplicity 1 finite spectral triple.

$\mathcal{A} \simeq \bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C})$, with $n_i \neq n_j$ $\qquad \mathcal{H} \cong \bigoplus_{i=1}^{r} \mathbb{C}^{n_i}$,

$\mathcal{D}$ = affine space of admissible Dirac operators on $\mathcal{H}$ completing $(\mathcal{A}, \mathcal{H})$ to a finite spectral triple.

Theorem [R.I.]: Let $D \in \mathcal{D}$ and define $X_0(D) := [\Omega^1_D(\mathcal{A})]_{\mathrm{s.a.}}$ to be the corresponding space of self-adjoint inner fluctuations. Then the subset of Dirac operators inducing an irreducible gauge theory, that is,

$$\mathcal{D}_{\mathrm{irr}} := \left\{D \in \mathcal{D} : \mathrm{Comm}^D_{\mathfrak{g}}(X_0) = \{\xi \in \mathfrak{g} : [\xi, D + \varphi] = 0, \forall \varphi \in X_0\} = 0\right\}$$

is Zariski open in $\mathcal{D}$.

For a dense subspace of admissible Dirac operator we can introduce only ghost fields of degree 1.

Goal: given a finite spectral triple, to determine a method to associate to it a BV-spectral triple, containing all the relevant information about the *ghost field sector* and the *extended action*

- How to extract from the original spectral triple all the relevant information about the gauge theory and its symmetries?

The notion of BV spectral triple

The Plan:

$$\boxed{\begin{array}{c}\textbf{Initial finite}\\ \textbf{spectral triple}\\ (\mathcal{A}_0, \mathcal{H}_0, D_0).\end{array}} \xrightarrow{\text{extract}} \boxed{\begin{array}{l}\quad\textbf{Gauge-theoretic input}\\ X_0 = [\Omega^1_{D_0}(\mathcal{A}_0)]_{\mathrm{s.a.}},\\ \mathfrak{g} = \mathrm{Lie}(\mathcal{U}(\mathcal{A}_0)/Z(\mathcal{U}(\mathcal{A}_0)))\,,\\ \rho_\xi(\varphi) = [\xi, \varphi].\end{array}} \xrightarrow{\text{construct}} \boxed{\begin{array}{c}\textbf{BV spectral triple}\\ (\mathcal{A}_{\mathrm{BV}}, \mathcal{H}_{\mathrm{BV}}, D_{\mathrm{BV}}, J_{\mathrm{BV}}).\end{array}}$$

The notion of BV spectral triple

The Plan:

$$\boxed{\begin{array}{c}\textbf{Initial finite}\\ \textbf{spectral triple}\\ (\mathcal{A}_0, \mathcal{H}_0, D_0).\end{array}} \xrightarrow{\text{extract}} \boxed{\begin{array}{l}\qquad\textbf{Gauge-theoretic input}\\ X_0 = [\Omega^1_{D_0}(\mathcal{A}_0)]_{\text{s.a.}},\\ \mathfrak{g} = \mathrm{Lie}(\mathcal{U}(\mathcal{A}_0)/Z(\mathcal{U}(\mathcal{A}_0)))\,,\\ \rho_\xi(\varphi) = [\xi, \varphi].\end{array}} \xrightarrow{\text{construct}} \boxed{\begin{array}{c}\textbf{BV spectral triple}\\ (\mathcal{A}_{\rm BV}, \mathcal{H}_{\rm BV}, D_{\rm BV}, J_{\rm BV}).\end{array}}$$

We introduce the notion of BV-spectral triple.

Def. Let $(\mathcal{A}_0, \mathcal{H}_0, D_0)$ be a finite spectral triple whose induced gauge theory is $(X_0, S_0, \mathcal{G})$. A real finite spectral triple $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}(\chi), J_{,BV})_{\chi\in\mathcal{H}_{BV,f}}$, over a graded parameter space $\mathcal{H}_{BV,f} \subseteq \mathcal{H}_{BV}$ is a BV spectral triple associated to $(\mathcal{A}_0, \mathcal{H}_0, D_0)$ if there exists a $\mathbb{Z}_{>0}$-graded real vector space $\mathcal{Q}_f$ s.t.

$$\mathcal{H}_{BV,f} \simeq \left(\mathcal{Q}_f^*[1] \oplus X_0^*[1]\right) \oplus \left(X_0 \oplus \mathcal{Q}_f\right),$$

and $(\widetilde{X}, \widetilde{S})$ defines a BV extension of the gauge theory $(X_0, S_0, \mathcal{G})$, where

$$\widetilde{X} \simeq \mathcal{H}_{BV,f} \qquad \text{and} \qquad \widetilde{S} := S_0 + \tfrac{1}{2} S_{\rm ferm}, \quad S_{\rm ferm}(\chi) := \tfrac{1}{2}\langle J(\chi), D_{BV}(\chi)\chi\rangle, \qquad \chi \in \mathcal{H}_{BV,f},$$

where $D_{BV}(\chi)$ denotes the field-dependent Dirac operator.

How to construct a BV-spectral triple?

For a multiplicity-1 finite spectral triple:

$\mathcal{A}_{BV} = \mathcal{A}_0 = \oplus_{i=1}^{r} M_{n_i}(\mathbb{C})$ ➡ one can prove, a posteriori, that this algebra is the maximal $*$-algebra completing $(\mathcal{H}_{BV}, D_{BV}(\chi), J_{BV})_{\chi \in \mathcal{H}_{BV,f}}$ to a spectral triple

The algebra: stays unchanged as it describes the physical field-content of the theory.

How to construct a BV-spectral triple?

For a multiplicity-1 finite spectral triple:

$\mathcal{A}_{BV} = \mathcal{A}_0 = \oplus_{i=1}^{r} M_{n_i}(\mathbb{C})$ ➡ one can prove, a posteriori, that this algebra is the maximal $*$-algebra completing $(\mathcal{H}_{BV}, D_{BV}(\chi), J_{BV})_{\chi \in \mathcal{H}_{BV,f}}$ to a spectral triple

The algebra: stays unchanged as it describes the physical field-content of the theory.

$\mathcal{H}_{BV} = \mathcal{Q}^*[1] \oplus \mathcal{Q}$ where $\mathcal{Q} := (X_0)_{\mathbb{C}}^{[0]} \oplus (\mathfrak{g}_{\mathbb{C}})^{[1]}$ ➡ The grading corresponds to the ghost degree in the ghost sector

$$\mathcal{H}_0 \cong \bigoplus_{i=1}^{r} \mathbb{C}^{n_i} \xrightarrow{\text{+ ghost/anti-ghost fields}} \mathcal{H}_{BV} = \boxed{\mathfrak{g}^*_{\mathbb{C}}[-2] \oplus X^*_{0,\mathbb{C}}[-1]} \oplus \boxed{X_{0,\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}[1]} \subseteq \left[\bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C})\right]^{\oplus 4}$$

How to construct a BV-spectral triple?

For a multiplicity-1 finite spectral triple:

$\mathcal{A}_{BV} = \mathcal{A}_0 = \oplus_{i=1}^r M_{n_i}(\mathbb{C})$ ➡ one can prove, a posteriori, that this algebra is the maximal $*$-algebra completing $(\mathcal{H}_{BV}, D_{BV}(\chi), J_{BV})_{\chi \in \mathcal{H}_{BV,f}}$ to a spectral triple

The algebra: stays unchanged as it describes the physical field-content of the theory.

$\mathcal{H}_{BV} = \mathcal{Q}^*[1] \oplus \mathcal{Q}$ where $\mathcal{Q} := (X_0)_{\mathbb{C}}^{[0]} \oplus (\mathfrak{g}_{\mathbb{C}})^{[1]}$ ➡ The grading corresponds to the ghost degree in the ghost sector

$$\mathcal{H}_0 \cong \bigoplus_{i=1}^r \mathbb{C}^{n_i} \xrightarrow{+\text{ ghost/anti-ghost fields}} \mathcal{H}_{BV} = \boxed{\mathfrak{g}_{\mathbb{C}}^*[-2] \oplus X_{0,\mathbb{C}}^*[-1]} \oplus \boxed{X_{0,\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}[1]} \subseteq \Big[\bigoplus_{i=1}^r M_{n_i}(\mathbb{C})\Big]^{\oplus 4}$$

where

$$\mathcal{H}_{BV,f} = \underbrace{\mathfrak{g}^*[-2]}_{\text{ghost antifields}} \oplus \underbrace{X_0^*[-1]}_{\text{field antifields}} \oplus \underbrace{X_0}_{\text{fields}} \oplus \underbrace{\mathfrak{g}[1]}_{\text{ghosts}}.$$

➡ fully determined by
$X_0 = [\Omega^1_{D_0}(\mathcal{A}_0)]_{\text{s.a.}}$
$\mathfrak{g} := \mathfrak{u}(\mathcal{A}_0)/\mathcal{Z}(\mathfrak{u}(\mathcal{A}_0))$

$\mathcal{H}_{BV,f} := (\mathcal{H}_{BV})^{J_{BV}}$ ➡ It selects the real graded subspace on which the fermionic functional is evaluated

The Hilbert space: describes the ghost sector of the BV-extended theory.

How to construct a BV-spectral triple? [2]

$$D_{BV}(\chi) = \begin{pmatrix} 0 & 0 & 0 & \mathrm{T}(C) \\ 0 & 0 & \mathrm{R}(C) & 0 \\ 0 & \mathrm{R}(C)^\dagger & 0 & 0 \\ \mathrm{T}(C)^\dagger & 0 & 0 & 0 \end{pmatrix}.$$

$$\mathrm{R}(C) := \kappa_{X_0} \circ \tilde{\rho}_C : X_{0,\mathbb{C}} \longrightarrow X^*_{0,\mathbb{C}}[-1] \qquad \mathrm{T}(C) := \kappa_{\mathfrak{g}} \circ \tilde{\sigma}_C : \mathfrak{g}_{\mathbb{C}}[1] \longrightarrow \mathfrak{g}^*_{\mathbb{C}}[-2].$$

$\kappa_{X_0} : X_{0,\mathbb{C}} \xrightarrow{\sim} X^*_{0,\mathbb{C}}[-1]$

Duality fields/antifields

$\rho : \mathfrak{g} \longrightarrow \mathrm{End}(X_0)$,

$\rho_\xi(\varphi) := [\xi, \varphi]$,

$\kappa_{\mathfrak{g}} : \mathfrak{g}_{\mathbb{C}}[2] \xrightarrow{\sim} \mathfrak{g}^*_{\mathbb{C}}[-2]$.

Duality ghost fields/ ghost antifields

$\tilde{\sigma}_C : \mathfrak{g}_{\mathbb{C}}[1] \to \mathfrak{g}_{\mathbb{C}}[2]$

$\tilde{\sigma}_C(C') := \frac{1}{2} \sum_{a,b,c} c^a c'^b f_{ab}{}^c \tau_c$.

How to construct a BV-spectral triple? [2]

$$D_{BV}(\chi) = \begin{pmatrix} 0 & 0 & 0 & \mathrm{T}(C) \\ 0 & 0 & \mathrm{R}(C) & 0 \\ 0 & \mathrm{R}(C)^\dagger & 0 & 0 \\ \mathrm{T}(C)^\dagger & 0 & 0 & 0 \end{pmatrix}.$$

$$\mathrm{R}(C) := \kappa_{X_0} \circ \widetilde{\rho}_C : X_{0,\mathbb{C}} \longrightarrow X^*_{0,\mathbb{C}}[-1] \qquad \mathrm{T}(C) := \kappa_{\mathfrak{g}} \circ \widetilde{\sigma}_C : \mathfrak{g}_{\mathbb{C}}[1] \longrightarrow \mathfrak{g}^*_{\mathbb{C}}[-2].$$

$\kappa_{X_0} : X_{0,\mathbb{C}} \xrightarrow{\sim} X^*_{0,\mathbb{C}}[-1]$

Duality fields/antifields

$\rho : \mathfrak{g} \longrightarrow \mathrm{End}(X_0),$

$\rho_\xi(\varphi) := [\xi, \varphi],$

$\kappa_{\mathfrak{g}} : \mathfrak{g}_{\mathbb{C}}[2] \xrightarrow{\sim} \mathfrak{g}^*_{\mathbb{C}}[-2].$

Duality ghost fields/ ghost antifields

$\widetilde{\sigma}_C : \mathfrak{g}_{\mathbb{C}}[1] \to \mathfrak{g}_{\mathbb{C}}[2]$

$\widetilde{\sigma}_C(C') := \frac{1}{2} \sum_{a,b,c} c^a c'^b f_{ab}{}^c \, \tau_c.$

- D_{BV} is completely determined by the two gauge-theoretic data extracted from the initial spectral triple:
  - The infinitesimal action $\rho : \mathfrak{g} \rightharpoonup X_0$ determines the part of the BV functional coupling the fields in X_0 with the ghosts in $\mathfrak{g}$
  - The Lie bracket $[-,-]_{\mathfrak{g}}$ of $\mathfrak{g}$ describes the self-interaction of the ghost sector.

How to construct a BV-spectral triple? [2]

$$D_{BV}(\chi) = \begin{pmatrix} 0 & 0 & 0 & \mathrm{T}(C) \\ 0 & 0 & \mathrm{R}(C) & 0 \\ 0 & \mathrm{R}(C)^\dagger & 0 & 0 \\ \mathrm{T}(C)^\dagger & 0 & 0 & 0 \end{pmatrix}.$$

$$\mathrm{R}(C) := \kappa_{X_0} \circ \tilde{\rho}_C : X_{0,\mathbb{C}} \longrightarrow X^*_{0,\mathbb{C}}[-1]$$

$\kappa_{X_0} : X_{0,\mathbb{C}} \xrightarrow{\sim} X^*_{0,\mathbb{C}}[-1]$

Duality fields/antifields

$\rho : \mathfrak{g} \longrightarrow \mathrm{End}(X_0)$,

$\rho_\xi(\varphi) := [\xi, \varphi]$,

$$\mathrm{T}(C) := \kappa_{\mathfrak{g}} \circ \tilde{\sigma}_C : \mathfrak{g}_{\mathbb{C}}[1] \longrightarrow \mathfrak{g}^*_{\mathbb{C}}[-2].$$

$\kappa_{\mathfrak{g}} : \mathfrak{g}_{\mathbb{C}}[2] \xrightarrow{\sim} \mathfrak{g}^*_{\mathbb{C}}[-2]$.

Duality ghost fields/ ghost antifields

$\tilde{\sigma}_C : \mathfrak{g}_{\mathbb{C}}[1] \rightarrow \mathfrak{g}_{\mathbb{C}}[2]$

$\tilde{\sigma}_C(C') := \frac{1}{2} \sum_{a,b,c} c^a c'^b f_{ab}{}^c \tau_c$.

- D_{BV} is completely determined by the two gauge-theoretic data extracted from the initial spectral triple:
  - The infinitesimal action $\rho : \mathfrak{g} \rightharpoonup X_0$ determines the part of the BV functional coupling the fields in X_0 with the ghosts in $\mathfrak{g}$
  - The Lie bracket $[-,-]_{\mathfrak{g}}$ of $\mathfrak{g}$ describes the self-interaction of the ghost sector.

The operator D_{BV} determines the BV-action $S_{BV} := \tilde{S} - S_0$ as induced fermionic action.

How to construct a BV-spectral triple? [2]

$$D_{BV}(\chi) = \begin{pmatrix} 0 & 0 & 0 & \mathrm{T}(C) \\ 0 & 0 & \mathrm{R}(C) & 0 \\ 0 & \mathrm{R}(C)^\dagger & 0 & 0 \\ \mathrm{T}(C)^\dagger & 0 & 0 & 0 \end{pmatrix}.$$

$$\mathrm{R}(C) := \kappa_{X_0} \circ \widetilde{\rho}_C : X_{0,\mathbb{C}} \longrightarrow X^*_{0,\mathbb{C}}[-1] \qquad \mathrm{T}(C) := \kappa_{\mathfrak{g}} \circ \widetilde{\sigma}_C : \mathfrak{g}_{\mathbb{C}}[1] \longrightarrow \mathfrak{g}^*_{\mathbb{C}}[-2].$$

$\kappa_{X_0} : X_{0,\mathbb{C}} \xrightarrow{\sim} X^*_{0,\mathbb{C}}[-1]$

Duality fields/antifields

$\rho : \mathfrak{g} \longrightarrow \mathrm{End}(X_0),$

$\rho_\xi(\varphi) := [\xi, \varphi],$

$\kappa_{\mathfrak{g}} : \mathfrak{g}_{\mathbb{C}}[2] \xrightarrow{\sim} \mathfrak{g}^*_{\mathbb{C}}[-2].$

Duality ghost fields/ ghost antifields

$\widetilde{\sigma}_C : \mathfrak{g}_{\mathbb{C}}[1] \to \mathfrak{g}_{\mathbb{C}}[2]$

$\widetilde{\sigma}_C(C') := \frac{1}{2} \sum_{a,b,c} c^a c'^b f_{ab}{}^c \tau_c.$

- D_{BV} is completely determined by the two gauge-theoretic data extracted from the initial spectral triple:
  - The infinitesimal action $\rho : \mathfrak{g} \curvearrowright X_0$ determines the part of the BV functional coupling the fields in X_0 with the ghosts in $\mathfrak{g}$
  - The Lie bracket $[-,-]_{\mathfrak{g}}$ of $\mathfrak{g}$ describes the self-interaction of the ghost sector.

The operator D_{BV} determines the BV-action $S_{BV} := \widetilde{S} - S_0$ as induced fermionic action.

The real structure: $J_{BV} : \mathcal{H}_{BV} \to \mathcal{H}_{BV}$ with $J_{BV}(\varphi) = \varphi^\dagger$

Where we are and where we are going

Noncommutative geometry

$$\begin{tikzcd}
\begin{array}{c}(\oplus_i M_{n_i}(\mathbb{C}), \oplus_i \mathbb{C}^{n_i}, D)\\ \text{initial spectral triple}\end{array} \arrow[r, red] \arrow[dd, leftrightarrow] & \begin{array}{c}(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})\\ \text{BV spectral triple}\end{array} \arrow[r, red] \arrow[d] & \begin{array}{c}(\mathcal{A}_t, \mathcal{H}_t, D_t, J_t)\\ \text{total spectral triple}\end{array} \arrow[r, red] \arrow[d] & \begin{array}{c}\Psi \in \mathcal{F}^{-1}_{\mathcal{O}_{X_0}}(\mathcal{H}_{BV,f} \oplus \mathcal{H}_{aux,f})\\ \text{gauge-fixing fermion}\end{array} \arrow[d] \\
 & \begin{array}{c}\mathcal{C}^\bullet_H(\mathcal{B}, \mathcal{M}) \cong \mathcal{C}^\bullet_{BV}(\widetilde{X}, d_{\tilde{S}})\\ \text{BV/ Hochschild complex}\end{array} & \begin{array}{c}\mathcal{C}^\bullet_H(\mathcal{B}_t, \mathcal{M}) \cong \mathcal{C}^\bullet_{BV}(X_t, d_{S_t})\\ \text{BV tot / Hochschild complex}\end{array} & \begin{array}{c}\mathcal{C}^\bullet_H(\mathcal{B}_\Psi, \mathcal{M}) \cong \mathcal{C}^\bullet_{BRST}(X_t, d_{S_t})|_\Psi\\ \text{BRST/ Hochschild complex}\end{array} \\
\begin{array}{c}(X_0, S_0)\\ \text{initial gauge theory}\end{array} \arrow[r] & \begin{array}{c}(\widetilde{X}, \widetilde{S})\\ \text{extended theory}\end{array} \arrow[r] \arrow[u] & \begin{array}{c}(X_t, S_t)\\ \text{total theory}\end{array} \arrow[r] \arrow[u] & \begin{array}{c}(X_t, S_t)|_\Psi\\ \text{gauge-fixed theory}\end{array} \arrow[u]
\end{tikzcd}$$

BV construction & BRST cohomology

Reached goals:

- Given a noncommutative geometric interpretation of the notion of BV-irrebucibility
- Established the (noncom.) geometrical BV construction for finite spectral triple
- Discovered the relation existing between BV/BRST cohomology and Hochschild cohomology

Where we are and where we are going

Noncommutative geometry

$$\begin{tikzcd}
\underset{\text{initial spectral triple}}{(\oplus_i M_{n_i}(\mathbb{C}), \oplus_i \mathbb{C}^{n_i}, D)} \arrow[r] \arrow[dd, leftrightarrow] & \underset{\text{BV spectral triple}}{(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})} \arrow[r] \arrow[d] & \underset{\text{total spectral triple}}{(\mathcal{A}_t, \mathcal{H}_t, D_t, J_t)} \arrow[r] \arrow[d] & \underset{\text{gauge-fixing fermion}}{\Psi \in \mathcal{F}^{-1}_{\mathcal{O}_{X_0}}(\mathcal{H}_{BV,f} \oplus \mathcal{H}_{aux,f})} \arrow[d] \\
 & \underset{\text{BV/ Hochschild complex}}{\mathcal{C}^\bullet_H(\mathcal{B}, \mathcal{M}) \cong \mathcal{C}^\bullet_{BV}(\widetilde{X}, d_{\tilde{S}})} & \underset{\text{BV tot / Hochschild complex}}{\mathcal{C}^\bullet_H(\mathcal{B}_t, \mathcal{M}) \cong \mathcal{C}^\bullet_{BV}(X_t, d_{S_t})} & \underset{\text{BRST/ Hochschild complex}}{\mathcal{C}^\bullet_H(\mathcal{B}_\Psi, \mathcal{M}) \cong \mathcal{C}^\bullet_{BRST}(X_t, d_{S_t})|_\Psi} \\
\underset{\text{initial gauge theory}}{(X_0, S_0)} \arrow[r] & \underset{\text{extended theory}}{(\widetilde{X}, \widetilde{S})} \arrow[r] \arrow[u] & \underset{\text{total theory}}{(X_t, S_t)} \arrow[r] \arrow[u] & \underset{\text{gauge-fixed theory}}{(X_t, S_t)|_\Psi} \arrow[u]
\end{tikzcd}$$

BV construction & BRST cohomology

Reached goals:

- Given a noncommutative geometric interpretation of the notion of BV-irrebucibility
- Established the (noncom.) geometrical BV construction for finite spectral triple
- Discovered the relation existing between BV/BRST cohomology and Hochschild cohomology

What is next:

Study BV-reducible theories and going beyond the finite case