Quantum Chiral Superfields

Rita Fioresi

University of Bologna

One geometric thread

Minkowski space as a big cell in Grassmannian → Supergeometry and chirality → Quantum coordinate superalgebras → Differential calculus

Guiding questions

1) How much of classical spacetime geometry survives when functions no longer commute?

2) Can we do a meaningful differential geometry on superspace and quantum superspace?

The classical geometric model

Minkowski space sits inside conformal space

Conformal space

$$\mathrm{Gr}(2,4) = \mathrm{SL}_4(\mathbb{C})/Q, \quad Q = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}.$$

It compactifies complex Minkowski space and removes the singularities of special conformal transformations.

The big cell

$$M_{\mathbb{C}} \simeq \mathbb{C}^4 \longleftrightarrow \mathrm{span} \begin{pmatrix} I_2 \\ A \end{pmatrix}, \quad A \in M_2(\mathbb{C}).$$

The open condition is the invertibility of one Plücker coordinate.

The affine spacetime chart is selected inverting the $(12) \times (12)$ minor.

The Poincaré action is already encoded

For a lower parabolic element and a big-cell representative,

$$\begin{pmatrix} L & 0 \\ TL & R \end{pmatrix} \begin{pmatrix} I_2 \\ A \end{pmatrix} = \begin{pmatrix} L \\ TL + RA \end{pmatrix} \sim \begin{pmatrix} I_2 \\ T + RAL^{-1} \end{pmatrix}.$$

Hence the induced action of the Poincarè on the Minkowski coordinates in A is

$$A \longmapsto T + RAL^{-1}.$$

For the real form, $R = (L^\dagger)^{-1}$ and $A = A^\dagger$.

Writing $A = x^\mu \sigma_\mu$ gives

$$\det A = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2.$$

The metric is recovered from the determinant!

Superspace and chirality

Supersymmetry replaces the Grassmannian by a flag

The complex conformal superspace is

$$\mathrm{Fl}(2|0, 2|N; 4|N).$$

Its big cell consists of compatible even and mixed subspaces.

The inclusion condition produces the supertwistor relation

$$B = A - \theta\bar{\theta}^{,t}.$$

With

$$C = \tfrac{1}{2}(A + B),$$

the coordinates become $(C, \theta, \bar{\theta})$.

Geometric meaning

The incidence relation is the compatibility condition defining the flag.

Chiral superspace remembers only half the odd directions

A chiral superfield satisfies

$$\bar{D}_{\dot{\alpha}i}\Phi = 0, \qquad i = 1, \ldots, N.$$

In chiral coordinates it depends on

$$\Phi = \Phi(y, \theta), \qquad y^\mu = x^\mu + \text{bilinear}(\theta, \bar{\theta}).$$

Geometrically:

$$\text{chiral superspace} \subset \mathrm{Gr}(2|0;4|N),$$
$$\text{antichiral superspace} \subset \mathrm{Gr}(2|N;4|N).$$

They form the two projections whose compatibility reconstructs the flag.

Penrose Transform: $\mathrm{Gr}(2|0;4|N) \longrightarrow \mathrm{Fl}(2|0,2|N;4|N) \longleftarrow \mathrm{Gr}(2|N;4|N)$

$N = 2$: what changes and what does not

The geometric mechanism is unchanged

- homogeneous superspaces;
- big cells and supertwistor relations;
- Poincaré supergroup action;
- chiral and antichiral projections.

The odd sector doubles

$$\theta^{\alpha i}, \quad \bar{\theta}_i^{\dot{\alpha}}, \qquad i = 1, 2,$$

The chiral superfields remain naturally attached to a super Grassmannian big cell.

This uniformity is precisely what makes the construction suitable for quantization.

Quantization

Quantization changes the language

Classical geometry

$$X \rightsquigarrow \mathcal{O}(X),$$
$$G \curvearrowright X \rightsquigarrow \mathcal{O}(X) \longrightarrow \mathcal{O}(G) \otimes \mathcal{O}(X).$$

Quantum geometry

$$X_q \rightsquigarrow \mathcal{O}_q(X),$$
$$G_q \curvearrowright X_q \rightsquigarrow \mathcal{O}_q(X) \longrightarrow \mathcal{O}_q(G) \otimes \mathcal{O}_q(X).$$

The coordinate superalgebra is now given by generators and relations.

Principle

Symmetry survives as a coaction of a Hopf superalgebra.

The quantum big cell is a localization

Start from the Manin quantum supergroup $\mathcal{O}_q(\mathrm{SL}(4|N))$.

1. The quantum super Grassmannian is generated by quantum minors.
2. Choose the Plücker coordinate defining the classical big cell.
3. Localize by powers of d^{12}_{12}, quantum determinant defining the big cell.
4. Read off the affine generators t_{ij} and the odd generators τ_{ij}.

$$\mathcal{O}_q(\text{chiral Minkowski}) \subset \mathcal{O}_q(\text{super Poincaré}).$$

Localization is the quantum counterpart of restricting to an affine chart.

Quantum Minkowski Superspace

Euristically: Minkowski superspace is a big cell in superflag:

$$\begin{pmatrix} g_{11} & g_{12} & 0 & 0 & 0 \\ g_{21} & g_{22} & 0 & 0 & 0 \\ g_{31} & g_{32} & g_{33} & g_{34} & g_{35} \\ g_{41} & g_{42} & g_{43} & g_{44} & g_{45} \\ g_{51} & g_{52} & g_{53} & g_{54} & g_{55} \end{pmatrix} = \begin{pmatrix} x & 0 & 0 \\ tx & y & y\eta \\ \tau x & d\xi & d \end{pmatrix} = \begin{pmatrix} x & 0 & 0 \\ tx & y & y\eta \\ d\tau & d\xi & d \end{pmatrix}$$

Interpret these as quantum Manin supermatrices i.e.

$$g_{11} g_{12} = q^{-1} g_{12} g_{11}, \dots$$

Coordinate functions do not commute!

Quantum Minkowski Superspace: N=1

$$x = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \qquad y = \begin{pmatrix} g_{33} & g_{34} \\ g_{43} & g_{44} \end{pmatrix}$$

$$t = \begin{pmatrix} -d_{23}d_{12}^{-1} & d_{13}d_{12}^{-1} \\ -d_{24}d_{12}^{-1} & d_{14}d_{12}^{-1} \end{pmatrix}, \qquad d = g_{55},$$

$$\tau = (g_{55}^{-1}g_{51}, g_{55}^{-1}g_{52}), \qquad \eta = \left(d_{34}^{34}{}^{-1}g_{35} \quad d_{34}^{34}{}^{-1}g_{45}\right),$$

$$\xi = \left(g_{55}^{-1}g_{53}, g_{55}^{-1}g_{54}\right),$$

Relations replace ordinary coordinates

The even generators satisfy quantum-matrix relations, schematically

$$t_{i1}t_{i2} = q\, t_{i2}t_{i1}, \qquad t_{1j}t_{2j} = q^{-1}t_{2j}t_{1j}, \ldots$$

The odd generators satisfy braided anticommutation rules,

$$\tau_a\tau_b = -q^{\varepsilon_{ab}}\tau_b\tau_a, \qquad \tau_a^2 = 0,$$

and braid nontrivially with the t_{ij}.

Classical limit

At $q = 1$ one recovers the commutative coordinate superalgebra of the chiral Minkowski big cell.

Quantum symmetry remains explicit

The coproduct of the quantum supergroup restricts to a coaction

$$\Delta_L : \mathcal{O}_q(M^{\mathrm{ch}}) \longrightarrow \mathcal{O}_q(P) \otimes \mathcal{O}_q(M^{\mathrm{ch}}).$$

where

$$\mathcal{O}_q(P) = \mathbb{C}\langle x, y, t, d, \tau, \eta, \xi \rangle / \mathrm{Rel.}, \qquad \mathcal{O}_q(M^{\mathrm{ch}}) = \mathbb{C}\langle t, \tau \rangle / \mathrm{Rel.}$$

The left coaction reads as:

$$\begin{aligned} t &\longmapsto \quad yS(x) \otimes t + t \otimes 1 + y\eta S(x) \otimes \tau \\ \tau &\longmapsto \quad (d \otimes 1)(\tau \otimes 1 + \xi \otimes t + 1 \otimes \tilde{\tau})(S(x) \otimes 1) \end{aligned}$$

The quantum space and its quantum symmetry are constructed together.

Why differential calculus?

Coordinates alone do not yet give geometry

The algebra $A = \mathcal{O}_q(M^{\mathrm{ch}})$ tells us which functions exist and how they multiply.

To discuss fields and geometry we also need

$$d : A \longrightarrow \Omega^1(A), \qquad d(ab) = (da)b + a\,db,$$

and an exterior algebra

$$\Omega^\bullet(A), \qquad d^2 = 0.$$

Covariance The differential calculus must respect the quantum-group coaction.

Braiding Interchanging forms or vector fields is controlled by the braiding, not by the ordinary flip.

Braided Cartan geometry

For a braided-commutative H-module algebra A, one can formulate:

forms	braided derivations	connections
$\Omega^1(A)$	$\mathrm{Der}_H(A)$	∇

and recover the Cartan language

$$T = d\vartheta + \omega \wedge \vartheta, \qquad R = d\omega + \omega \wedge \omega,$$

with all tensor exchanges interpreted through the braiding.

Key observation

In the triangular setting, forms and vector-field approaches to torsion and curvature agree; Cartan equations, Bianchi identities, and a Levi–Civita theory are available.

Warning: Manin deformations are not triangular.

Curvature and torsion in the braided setting

Let $(H, \mathcal{R})$ be a triangular Hopf algebra, A be an $\mathcal{R}$-commutative H-module algebra:

$$ab = (\overline{\mathcal{R}}^{\alpha} \triangleright b)(\overline{\mathcal{R}}_{\alpha} \triangleright a), \qquad \mathcal{R}^{-1} = \overline{\mathcal{R}}^{\alpha} \otimes \overline{\mathcal{R}}_{\alpha}.$$

A **left connection** on a finitely generated projective A-module Γ is a map

$$\nabla : \Gamma \longrightarrow \Omega^1(A) \otimes_A \Gamma, \qquad \nabla(as) = da \otimes_A s + a\,\nabla s.$$

Its extension d_∇ to $\Omega^\bullet(A) \otimes_A \Gamma$ defines the curvature, $d_\nabla^2 = d_\nabla \circ d_\nabla$, giving:

$$R_\nabla(u, v, s) := -\iota_u \iota_v\, d_\nabla^2(s), \qquad \boxed{R_\nabla(u, v, s) = \Big(\nabla_u \nabla_v - \nabla_{\overline{\mathcal{R}}^{\alpha} \triangleright v} \nabla_{\overline{\mathcal{R}}_{\alpha} \triangleright u} - \nabla_{[u,v]}\Big)\, s}$$

When $\Gamma = \mathfrak{X}(A)$, define the torsion $T_\nabla(u, v) := -\iota_u \iota_v\, d_\nabla(I)$. Hence

$$\boxed{T_\nabla(u, v) = \nabla_u v - \nabla_{\overline{\mathcal{R}}^{\alpha} \triangleright v}(\overline{\mathcal{R}}_{\alpha} \triangleright u) - [u, v]., \quad I = \omega^i \otimes_A e_i \in \Omega^1(A) \otimes_A \mathfrak{X}(A)}$$

P. Aschieri, *Bianchi identities in noncommutative geometry*, arXiv:2608.24536, §4.

A bridge between NCG language and quantum super Minkowski

What is already available

- a covariant quantum chiral coordinate superalgebra;
- quantum Poincaré coactions;
- general frameworks for covariant and braided differential calculi.

What remains to be checked

- the appropriate calculus on the chosen quantum superspace;
- compatibility with the super and q braidings;
- metrics, connections, and curvature in this specific model.

Key Question: What is the correct metric to choose on the quantum Minkowski superspace to get the Levi-Civita connection and a curved quantum spacetime?

Outlook

Conclusions: From quantum superfields to quantum geometry and back!

Quantum homogeneous superspace

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Covariant differential calculus

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Quantum connections and curvature

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Noncommutative gauge theory and supergravity

THANK YOU

Selected references

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