A thermodynamical study of finite spectral triples

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joint work with

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Introduction

Connes's characterisation of riemannian geometry in terms of spectral data:

$$\overbrace{\mathcal{M}}^{\text{Riemannian, compact, spin, manifold}} \Longleftrightarrow \underbrace{\left(C^\infty(\mathcal{M}),\ L^2(\mathcal{M}, S),\ \partial\!\!\!/ = -i\gamma^\mu \nabla_\mu\right)}_{\text{spectral triple}}$$

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In particular, identifying a point z of $\mathcal{M}$ with the pure state of $C^\infty(\mathcal{M})$

$$\delta_z : f \mapsto f(z),$$

the geodesic distance is retrieved as

$$d(\delta_x, \delta_y) = \sup_{f \in C^\infty(\mathcal{M})} \{ \delta_x(f) - \delta_y(f), \quad ||[\partial\!\!\!/, f]|| \leq 1 \}.$$

Connes, Lott (1992)

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with a suitably chosen finite dimensional algebra $\mathcal{A}_F$ and matrix D_F, one retrieves the bosonic part of the lagrangian of the Standard Model as the asymptotic expansion $\Lambda \to \infty$ of the spectral action

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where $D_A = D + A + JAJ^{-1}$ is the covariant Dirac operator defined by a generalised 1-form

$$A = A^* \in \left\{\sum_i a_i[D, b_i],\ a_i, b_i \in \mathcal{A}\right\} \quad \text{with } J \text{ the "real structure"},$$

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and the test function f is a smooth approximation of the characteristic function of the interval $[0, 1]$.

Chamseddine, Connes, Marcolli, Suijlekom (1996-2015)

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By studying the most simple finite dimensional spectral triple (the two-point space), we will show ho to define other thermodynamical quantities by suitable test functions in the spectral action.

1. **Entropy from the spectral action**
   C^*-dynamical system and KMS state
   The result of Chamseddine, Connes and v. Suijlekom

2. **The two-point space**
   Entropy & distance
   Hidden parameter

3. **Two-level fermionic system**
   Grand canonical entropy
   Entropy of the KMS state
   Average energy and mean occupation number

I. Entropy from the spectral action

C^*-dynamical system: a (unital) C^*-algebra $\mathcal{C}$ together with a 1-parameter group of automorphism $\sigma_t \in \mathrm{Aut}(\mathcal{C})$, $t \in \mathbb{R}$.

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A state φ (linear, positive map $\mathcal{C} \to \mathbb{C}$ of norm 1) satisfies the *KMS* condition at inverse temperature $\beta > 0$ if there exists a function $F_{a,b}(z)$ holomorphic on

$$I_\beta = \{z \in \mathbb{C},\ 0 < \mathrm{Im}(z) < \beta\},$$

continuous on the boundary ∂I_β and bounded, such that

$$F_{a,b}(t) = \varphi(a\,\sigma_t(b)) \quad \text{and} \quad F_{a,b}(t + i\beta) = \varphi(\sigma_t(b)\,a) \quad \forall t \in \mathbb{R}.$$

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Finite dimensional case: for $\mathcal{C} = M_n(\mathbb{C})$, one has $H = H^* \in M_n(\mathbb{C})$ such that

$$\sigma_t(A) = e^{itH}\,A\,e^{-itH} \quad \forall A \in M_n(\mathbb{C}),\ t \in \mathbb{R}.$$

For any $\beta > 0$, there exists a unique *KMS* state φ_β on $(\mathcal{C}, \sigma_t)$:

$$\varphi_\beta(\cdot) = \mathrm{Tr}(\rho_\beta\,\cdot) \quad \text{for} \ \ \rho_\beta = Ze^{-\beta H} \ \text{with}\ Z = \frac{1}{\mathrm{Tr}(e^{-\beta H})}.$$

Any density matrix ρ on an Hilbert space $\mathcal{H}$ (i.e. positive operator with trace 1) defines a state

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Example — Chamseddine, Connes, V. Suijlekom

For $x > 0$, the partition of the unit interval by two intervals α, β of ratio x is

$$\alpha = \frac{1}{x+1}, \quad \beta = \frac{x}{x+1}.$$

The corresponding density matrix

$$\rho = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

defines a state with entropy

$$\mathcal{E}(x) := \log(x+1) - \frac{x \log x}{x+1}.$$

Proposition

Chamseddine, Connes, V. Suijlekom (2018)

Let $\mathcal{H}$ be a complex Hilbert space, D a selfadjoint operator on $\mathcal{H}$ with compact resolvent. Let

- $\mathcal{C} = \mathbb{C}l(\mathcal{H}_{\mathbb{R}})$ be the complex Clifford algebra of the underlying real Hilbert space $\mathcal{H}_{\mathbb{R}}$,
- σ_t the 1-parameter group of automorphisms associated to $e^{itD} \in \mathrm{Aut}(\mathcal{H}_R)$,

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then for any $\beta > 0$ there exists a unique KMS state φ_β on the C^*-dynamical system $(\mathcal{C}, \sigma_t)$.

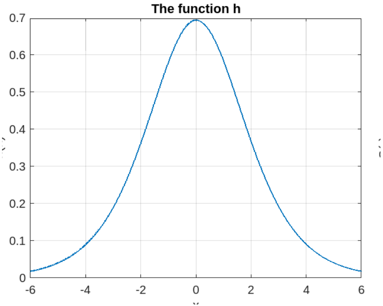
Moreover, if the operator $e^{-\beta|D|}$ is trace-class, the entropy of φ_β is equal to the spectral action,

$$S(\varphi_\beta) = \mathrm{Tr}\left(h(\beta D)\right)$$

for the test function

$$h(x) = \mathcal{E}(e^{-x}).$$

The momenta of the function h are given by specific value of Riemann ζ function.



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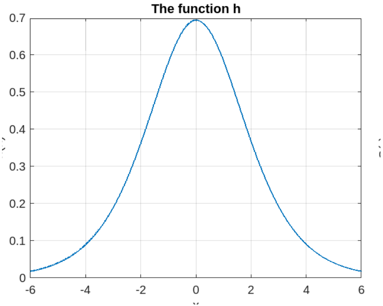

Given any spectral triple $(\mathcal{A}, \mathcal{H}, D)$ such that $e^{-\beta|D|}$ is trace-class, the spectral action for the test function h measures a "intrinsic" entropy associated with the noncommutative geometry.

Paves the way to a thermodynamical study of spectral triples.

II. The two-point space

Suijlekom (2025)

$$\mathcal{A} = \mathbb{C}^2, \quad \mathcal{H} = \mathbb{C}^2, \quad D = \begin{pmatrix} 0 & m \\ \bar{m} & 0 \end{pmatrix}$$

with $m \in \mathbb{C}^*$ and $\mathcal{A}$ acts on $\mathcal{H}$ as

$$\pi(z_1, z_2) = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \quad \text{for} \quad (z_1, z_2) \in \mathbb{C}^2.$$

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The two pure states of $\mathcal{A}$ are

$$\delta_1(z_1, z_2) = z_1, \quad \delta_1(z_1, z_2) = z_2,$$

at distance

$$d(\delta_1, \delta_2) = \frac{1}{|m|} =: r$$

from one another.

The entropy depends only on the distance

$$S(r) = \mathrm{Tr}\,(h\,(\beta D)) = 2h(\frac{\beta}{r}) = 2\left(\log(1 + e^{-\frac{\beta}{r}}) + \frac{\beta}{r}\frac{1}{1 + e^{\frac{\beta}{r}}}\right).$$

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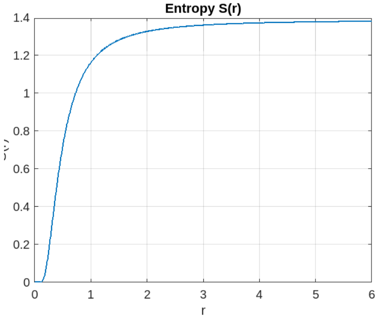

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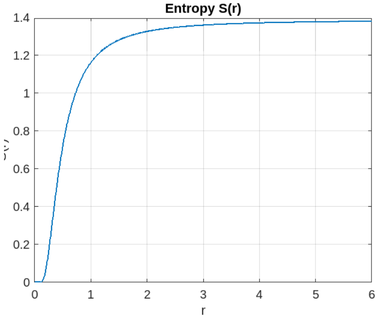

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- Maximal (yet finite) disorder when the points are far from one another. Minimal disorder when the two points coincide.

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$$D = \begin{pmatrix} 0 & m \\ \bar{m} & 0 \end{pmatrix} \quad \text{with} \quad D' = \begin{pmatrix} \lambda_1 & m \\ \bar{m} & \lambda_2 \end{pmatrix} \quad \text{with } \lambda_{i=1,2} \in \mathbb{C},$$

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the distance is unchanged but the entropy

$$S(\delta, t) = h\left(\beta(t+\delta)\right) + h\left(\beta(t-\delta)\right)$$

now depends on two parameters

$$\delta = \frac{1}{2}(D_+ - D_-), \quad t = \frac{1}{2}(D_+ + D_-)$$

where the eigenvalues of D are

$$D_\pm = \frac{1}{2}\left((\lambda_1 + \lambda_2) \pm \sqrt{(\lambda_1 - \lambda_2)^2 + 4|m|^2}\right)$$

(Martinetti, Priou, 2025)

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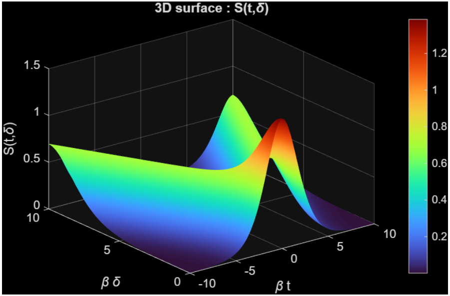
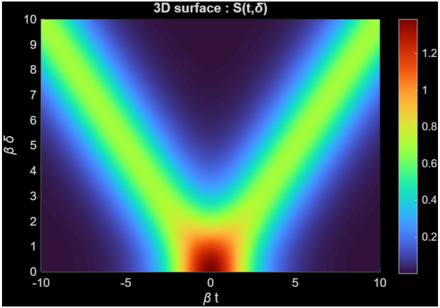
- When the diagonal is zero, then $t = 0$ and $\delta = |m| = \frac{1}{r}$, so that

$$S(\delta, 0) = 2h(\frac{\beta}{r}) = S(r).$$





- $S(t,\delta)$ reaches its maximum $2\log 2$ in the unique point $t=\delta=0$.

- When one of the two parameters t,δ is fixed, then the entropy tends to zero as the other parameter tends to infinity.

- On the diagonal $t=\pm\delta$, one has

$$S(\pm\delta,\delta)=h(2\beta\,\delta)+\log 2,$$

which tends to $\log 2$ (that is half of the maximum) when $|t|\to\infty$.

When the diagonal of D is not a multiple of the identity, then

$$\delta \neq |m|$$

and the entropy no longer depends on the distance.

One may force this dependence by viewing $S(\delta, t)$ as a function of $|m|$.

But there remains two independent variables $(\lambda_1 + \lambda_2)$ and $(\lambda_1 - \lambda_2)$.

What is the physical interpretation of δ and t ?

III. Fermionic two-levels system

The grand partition function of a gas of non-interacting fermions with non degenerate energy level, at inverse temperature β and chemical potential μ is

$$\Xi = \prod_{(\lambda)} \xi_\lambda$$

where (λ) denotes the set of all possible individual states for each particle, and

$$\xi_\lambda = \sum_{n_\lambda} \exp\left[- \beta n_\lambda (\varepsilon_\lambda - \mu) \right] = 1 + e^{-\beta(\varepsilon_\lambda - \mu)}$$

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Proposition Priou, PM 2026

For a two-level system $\varepsilon_\lambda = \pm\varepsilon$ with $\varepsilon > 0$, the grand canonical entropy concides with the entropy of the two-point space,

$$S_\mu(\varepsilon) = S(t, \delta) \quad \text{for } \epsilon = \delta, \qquad \mu = -t.$$

- The difference δ of the eigenvalues of the Dirac operator D is an energy,
- The trace of the Dirac operator is a chemical potential.

Other quantities to compute besides the entropy: average energy, mean occupation number etc.

To do so, let us make the explicit computation of the entropy of Connes and al for the two-point space.

Given a (complex) Hilbert space $\mathcal{H}$, the fermionic Fock space is the completion of

$$\bigoplus_{n\geq 0} \mathcal{F}(\mathcal{H}^n) \quad \text{where} \quad \mathcal{F}(\mathcal{H}^n) = P_-(\mathcal{H}^n)$$

is the antisymmetrisation of $\mathcal{H}^n$, obtained by applications of the projection P_-

$$P_-(f_1 \otimes f_2 \otimes \ldots \otimes f_n) = \frac{1}{n!} \sum_{\pi \in S_n} (-1)^{|\pi|} f_{\pi(1)} \otimes f_{\pi(2)} \ldots \otimes f_{\pi(n)} \qquad f_i \in \mathcal{H}.$$

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A selfadjoint operator $\mathcal{O}$ on $\mathcal{H}$ defines an operator $d\Gamma(\mathcal{O})$ on the Fock space:

$$d\Gamma(\mathcal{O})\, P_-(f_1 \otimes f_2 \otimes \ldots \otimes f_n) = P_- \left(\sum_{p=1}^{n} f_1 \otimes \ldots \mathcal{O} f_p \otimes \ldots \otimes f_n \right)$$

while $d\Gamma(\mathcal{O})$ acts as zero on $H^0 \simeq \mathbb{C}$.

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For $\mathcal{H} = \mathbb{C}^2 = \mathrm{Span}\,\{e_1, e_2\}$, a basis of the Fock space is

$$1 \text{ (for } \mathcal{H}^0), \quad e_1,\ e_2 \text{ (for } \mathcal{H}^1), \quad e_{12} := \frac{1}{2}(e_1 \otimes e_2 - e_2 \otimes e_1) \text{ (for } \mathcal{H}^2).$$

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Then

$$d\Gamma(\mathcal{O})\, 1 = 0, \quad d\Gamma(\mathcal{O}) e_i = \mathcal{O} e_i \quad \text{for } i = 1, 2,$$

$$d\Gamma(\mathcal{O}) e_{12} = \mathcal{O} e_1 \otimes e_2 + e_1 \otimes \mathcal{O} e_2 - e_2 \otimes \mathcal{O} e_1 - \mathcal{O} e_2 \otimes e_1.$$

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$$e^{-\beta d\Gamma(D)} e_{12} = P_-(e^{-\beta D} e_1 \otimes e^{-\beta D} e_2) = e^{-\beta D_+} e^{-\beta D_-} e_{12}.$$

Thus

$$Z = 1 + \mathrm{Tr}\left(\exp(-\beta D)\right) + e^{-\beta D_+} e^{-\beta D_-} = \left(1 + e^{-\beta D_+}\right)\left(1 + e^{-\beta D_-}\right),$$

while

$$\rho = \mathrm{diag}\left(\frac{1}{Z}, \quad \frac{1}{Z} e^{-\beta D_+}, \quad \frac{1}{Z} e^{-\beta D_-}, \quad \frac{1}{Z} e^{-\beta D_+} e^{-\beta D_-} \right).$$

The KMS state φ_β has density matrix

$$\rho = \frac{1}{Z} e^{-\beta d\Gamma(D)}, \qquad \text{with} \qquad Z = \mathrm{Tr}(e^{-\beta d\Gamma(D)}).$$

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Hence

$$\rho \log \rho = \begin{pmatrix} -\frac{\log Z}{Z} & & & \\ & (-\log Z - \beta D_+)\frac{e^{-\beta D_+}}{Z} & & \\ & & (-\log Z - \beta D_-)\frac{e^{-\beta D_-}}{Z} & \\ & & & (-\log Z - \beta(D_+ + D_-))\frac{e^{-(\beta D_+ + D_-)}}{Z} \end{pmatrix}$$

By taking the trace, one gets

$$
\begin{aligned}
\mathrm{Tr}\left(-\rho\log\rho\right) &= \frac{\log Z}{Z}(1+e^{-\beta D_+}+e^{-\beta D_-}+e^{-\beta D_+}e^{-\beta D_-}) \\
&+\frac{1}{Z}\left(\beta D_+ e^{-\beta D_+}+\beta D_- e^{-\beta D_-}+\beta(D_+ + D_-)e^{-\beta(D_+ + D_-)}\right), \\
&= \log Z - \beta\frac{\partial}{\partial\beta}(\log Z), \\
&= h(\beta D_+) + h(\beta D_-), \\
&= \mathrm{Tr}\left(h(\beta D)\right).
\end{aligned}
$$

- This is the result of Conne and al., as expected !

Lemma

In the two-point space, the mean value of $d\Gamma(D)$ in the KMS-state φ_β is

$$< d\Gamma(D) >= \frac{D_+}{1+e^{-\beta D_+}} + \frac{D_-}{1+e^{-\beta D_-}}.$$

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$$\langle d\Gamma(D)\rangle = \frac{\epsilon}{1+e^{\beta(\epsilon-\mu)}} + \frac{-\epsilon}{1+e^{\beta(-\epsilon-\mu)}} - \mu\left(\frac{1}{1+e^{\beta(\epsilon-\mu)}} + \frac{1}{1+e^{\beta(-\epsilon-\mu)}}\right) = \langle E\rangle - \mu\langle N\rangle$$

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Definition

The average energy and mean occupation number of a spectral triple are

$$\langle E\rangle = \frac{1}{\beta}\mathrm{Tr}(u_{\beta\mu}(\beta D)) \quad \text{for} \quad u_\mu(x) := \frac{x+\mu}{1+e^x},$$

$$\langle N\rangle = \mathrm{Tr}(v(\beta D)) \quad \text{for} \quad v(x) := \frac{1}{1+e^x} \quad \text{where } \mu = -\frac{1}{2}\mathrm{Tr}D.$$

Conclusion

Starting from the (canonical) entropy of Connes and al., one identifies the trace of the Dirac operator as the chemical potential.

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