Quantization, Modular Theory and Non-commutative Space

Paolo Bertozzini
(independent researcher - Bangkok)

Workshop on "Noncommutative and Generalised Geometries and their Physical Applications"

EISA - Corfu Summer Institute - Greece - 15 Septemebr 2026

Thanks to the organizers of this "Corfu Summer Institute" workshop for the opportunity to present these ideas based on

Modular Quantization and Non-commutative Space-time

PB, R.Conti (version 8+4 - 01.09.2026 - manuscript available on request)

paolo.th@gmail.com

This talk is inspired by a long-term ongoing collaboration with:

- Roberto Conti (Sapienza Università di Roma)
- Matti Raasakka (Aalto University - Helsinki)

and some related previous and (currently) developing work with:

- Wicharn Lewkeeratiyutkul (Chulalongkorn University)
- Natee Pitiwan (Satit Chula Bangkok)
- Chatchai Puttirungroj (Sukhothai Thammatirat University)
- Paratat Bejrakarbum (Thammasat University)
- Thitikorn Suwannapeng (TAIST, Kasetsart University)

Abstract

- A structural link between quantum theory and a "thermal-doubled" *non-commutative Born geometry* is uncovered via Tomita-Takesaki *first-quantized modular theory* for complexified and polarized ortho-symplectic spaces.
- Implications for the *modular algebraic quantum gravity* program (arXiv:1007.4094v1) hint at further deep modifications of the notions of space and geometry (*thermal quantum space-time hypothesis*) waiting to be further explored.
- Speculative suggestions in the direction of Boyle-Turok *CPT-mirror cosmologies* will be mentioned, among others.

- **Strategy for Modular Algebraic Quantum Gravity**

(★ denotes slides with original material)
(★ denotes slides with heretical conjectures)

Is there "geometry" at the fundamental quantum level?
Yes . . . modular non-commutative geometry!

Modular Algebraic Quantum Theory (arXiv:1007.4094)

We aim to revert the implication

$\boxed{\text{classical geometry} \Rightarrow \text{local equilibrium states}}$

and obtain **geometry from states**:

$\boxed{\text{quantum geometry} \Leftarrow \text{local equilibrium states}}$

classical space-time geometry $\xrightarrow[\text{motion}]{\text{locality free geodesic}}$ classical dynamics $\xrightarrow[\text{KMS-property}]{\text{Katz Gallavotti}}$ classical (local) equilibrium states

classical space-time geometry $\rightsquigarrow$ modular quantization $\rightsquigarrow$ quantum space-time geometry

classical (local) equilibrium states $\rightsquigarrow$ algebraic (Mauro Weyl) quantization $\rightsquigarrow$ quantum (local) equilibrium states

quantum space-time geometry $\xleftarrow[\text{modular transport (holonomy)}]{\text{modular localization (descent) polarization}}$ quantum dynamics $\xleftarrow[\text{modular theory}]{\text{Tomita-Takesaki}}$ quantum (local) equilibrium states

- **Conjecture**: even when starting with a "classical geometry", via modular theory, a new "quantum geometry" is produced.

(Some) Fundamental Physical Principles in MAQT ★

- *Modular localization*: quantum subsystems are determined by states via a KMS-equilibrium condition
- *Jacobi-Einstein generalized equivalence principle*: all interactions are subsumed in a background geometry and dynamics is free subject only to the geometric constraints

- **Geometric origin of modular theory**: (first-quantized) Tomita-Takesaki modular theory originates in complexification of ortho-symplectic spaces and encodes free dynamics contraints on quantized phase-spaces
- **Modular CPT symmetry**: quantum systems are always in CPT-symmetric thermal double pairs
- **Modular Born geometry**: that is induced by a CPT symmetry J, a modular duality L, a Born reciprocity Φ, a modular $\mathbb{C}$-polarization Γ, a (choice of) modular Born $\mathbb{R}$-polarization Λ.

- **Tomita-Takesaki Modular Theory**

(the missing ingredient for “quantization”)

Modular Theory - von Neumann Algebras (Tomita-Takesaki)

For a von Neumann algebra $\mathcal{M} = \mathcal{M}'' \subset \mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, for every vector state $\phi_\xi(T) := \langle \xi \mid T(\xi) \rangle$, with $\xi \in \mathcal{H}$ that is:

$$\textit{cyclic}: \quad \overline{(\mathcal{M}\xi)} = \mathcal{H}, \qquad \textit{separating}: \quad A\xi = 0 \Rightarrow A = 0,$$

the $\mathbb{R}$-subspace $\{\, T(\xi) \mid T = T^*, \ T \in \mathcal{M}\} \subset \mathcal{H}$ is standard and

- its modular one-parameter unitary group

$$t \mapsto \Delta_\xi^{it} = \exp(itK_\xi) \in \mathcal{B}(\mathcal{H}),$$

- its modular conjugation antilinear isometry $\mathcal{H} \xrightarrow{J_\xi} \mathcal{H}$, satisfy:

$$\boxed{\sigma_t^{\phi}(\mathcal{M}) := \Delta_\xi^{it} \mathcal{M} \Delta_\xi^{-it} = \mathcal{M}, \quad \forall t \in \mathbb{R}, \qquad J_\xi \mathcal{M} J_\xi = \mathcal{M}',}$$

where the **commutant** $\mathcal{M}'$ of $\mathcal{M}$ is defined by:

$$\mathcal{M}' := \{A' \in \mathcal{B}(\mathcal{H}) \mid [A', A]_- = 0, \forall A \in \mathcal{M}\}.$$

Notice that $\mathcal{M}'$ is a von Neumann algebra and $\mathcal{M}'' = \mathcal{M}$.

Modular Theory - First Quantized Level

(originated in M.Rieffel, A.van Daele - see for example R.Longo lecture notes)

Any complex Hilbert space $\mathcal{H}_\mathbb{C}$ with a **standard real subspace** $V_\mathbb{R}$:

$$\mathcal{H}_\mathbb{C} = \overline{V_\mathbb{R} \oplus i \cdot V_\mathbb{R}}, \qquad V_\mathbb{R} \cap i \cdot V_\mathbb{R} = \{0_\mathcal{H}\},$$

canonically determines the following data:

- a complex unitary structure $T_V : \zeta \mapsto i \cdot \xi$, for $\xi \in \mathcal{H}_\mathbb{R}$,
- a densely defined Tomita conjugation $S_V : v \oplus (i \cdot w) \mapsto v \oplus (-i \cdot w), \quad x, y \in V_\mathbb{R}$ with polar Hilbert decomposition $\overline{S}_V = J_V \circ \Delta_V^{\frac{1}{2}}$,
- a unitary one-parameter group $t \mapsto \Delta_V^{it} = \exp(itK_V)$ on $\mathcal{H}_\mathbb{C}$, leaving $V_\mathbb{R}$ stable: $\Delta_V^{it}(V_\mathbb{R}) = V_\mathbb{R}$, for all $t \in \mathbb{R}$,
- an anti-unitary operator J_V mapping $V_\mathbb{R}$ on its symplectic complement $V' := \{h \in \mathcal{H}_\mathbb{C} \mid \operatorname{Im}\langle h \mid v\rangle_\mathcal{H} = 0, v \in V_\mathbb{R}\}$.

The physical nature of the modular ortho-symplectic dynamics $t \mapsto \Delta_V^{it}$ and of the symmetry J_V is still mysteriously unknown!

AQFT, Free Scalar Fields and Reeh-Schlieder Theorem

(H.Araki 1 2, R.Wald, C.Bär, N.Ginoux, F.Pfäffle)

globally hyperbolic Lorentzian submanifold $(\mathcal{O}, g)$	$\Rightarrow$	symplectic $\mathbb{R}$-space $(\Gamma_{\mathcal{O}}, \omega)$ of local solutions of Klein-Gordon-equation $\Box_g \phi = m\phi$	$\Rightarrow$	complexified Weyl-algebra $\mathcal{A}(\mathcal{O}) := \mathbb{W}\mathrm{y}(\Gamma_{\mathcal{O}}, \omega)$

- the functor $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ is an algebraic quantum field theory
- the complexification $\Gamma_{\mathcal{O}}^{\mathbb{C}}$ of $(\Gamma_{\mathcal{O}}, \omega)$ is a Kreĭn space
- a choice of Kähler $\mathbb{C}$-polarization (equivalently a quasi-free state on $\mathbb{W}\mathrm{y}(\Gamma_{\mathcal{O}}, \omega)$) selects a one-particle Hilbert space $\mathcal{H}^+$.

H.Reeh-S.Schlieder: whenever the open region $\mathcal{O} \neq \varnothing$ has a non-trivial causal complement $\mathcal{O}' \neq \varnothing$, the Bosonic Fock space $\mathcal{H} := \mathcal{F}_-(\mathcal{H}^+)$ of $\mathcal{H}^+$ is a standard Hilbert space for the von Neumann algebra $\mathcal{A}(\mathcal{O})''$ generated by $\mathcal{A}(\mathcal{O})$, with vacuum $\xi_o \in \mathcal{H}$ that is cyclic for both $\mathcal{A}(\mathcal{O})$ and its commutant $\mathcal{A}(\mathcal{O})'$.

$\Rightarrow$ there is a mysterious modular dynamics on each $\mathcal{A}(\mathcal{O})''$

- **Heretical conjecture**: the modular dynamics is a "Coriolis' effect" of the non-commutative geometry of space-time.

- **Modular Quantization**

"quantization of phase-space entails, via modular theory, also a quantization of configuration/momentum spaces"

External Ortho-symplectic Complexifications ★

- by **ortho-symplectic $\mathbb{R}$-space** [1] $(V_{\mathbb{R}}, g_V, \omega_V)$ we mean a $\mathbb{R}$-vector space $V_{\mathbb{R}}$ with an inner product g_V and an anti-symmetric bilinear map ω_V such that $\|\omega_V\| < 1$.
- the **external complexification** of $(V_{\mathbb{R}}, g_V, \omega_V)$ is given by the Hilbert completion $\mathcal{H}_{\mathbb{C}}$ of $V_{\mathbb{R}} \oplus V_{\mathbb{R}}$, with $\mathbb{C}$-action

$$(\alpha + i\beta) \cdot (v \oplus w) := (\alpha v - \beta w) \oplus (\alpha w + \beta v),$$

and the positive sesquilinear form $h(v_1 \oplus w_1, v_2 \oplus w_2) :=$

$$\begin{pmatrix} +g_V(v_1, v_2) & +g_V(w_1, w_2) \\ -\omega_V(v_1, w_2) & +\omega_V(w_1, v_2) \end{pmatrix} + i \cdot \begin{pmatrix} +g_V(v_1, w_2) & -g_V(w_1, v_2) \\ +\omega_V(v_1, v_2) & +\omega_V(w_1, w_2) \end{pmatrix}.$$

- there is a categorical equivalence:

$$\boxed{\text{complexifications of ortho-symplectic spaces}} \leftrightarrow \boxed{\text{standard } \mathbb{R}\text{-subspaces of } \mathbb{C}\text{-Hilbert spaces}}$$

[1] Notice that in the Kähler case $\omega_V(v, w) = g_V(i \cdot v, w)$ and so $\|\omega_V\| = 1$.

Gauge Modular Flow of a Riemannian Manifold ★

Riemannian manifold (X, g_X)	$\Longrightarrow$	modular gauge groups $\sigma_q^{g_X,\gamma}$, $\gamma \in]0,1[$, $q \in X$

- A Riemannian classical geometry (X, g_X), for $0 < \gamma < 1$, has canonically associated (first-quantized) gauge modular flows $\sigma^{g_X,\gamma}$ on the **complexified** $\mathcal{H}_X := (TX \oplus T^*X) \otimes_{\mathbb{R}} \mathbb{C}$ **of its Hitchin-Gualtieri generalized tangent bundle** $TM|_X$:
  - $TM|_X \simeq TX \oplus T^*X$ is a Kähler ortho-symplectic bundle with simplectic structure $\omega_V(x_1 \oplus y_1^*, x_2 \oplus y_2^*) := y_2^*(x_1) - y_1^*(x_2)$, inner product $g_V := g_X \oplus g_{X^*}$ with $g_{X^*}(x_1^*, x_2^*) := g_X(x_1, x_2)$,
  - $TM|_X$ is a standard $\mathbb{R}$-subbundle of the Hilbert bundle $\mathcal{H}_X$ with inner product $h^g + i\gamma h^\omega$ where h^g and h^ω are the complexified sesquilinear forms induced by g_V and ω_V.
- **What is the physical/geometrical meaning of**

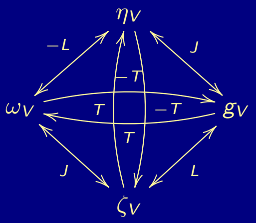
$$t \mapsto \sigma_{q,t}^{g_X,\gamma} := \Delta^{it}_{T_qX \oplus T_q^*X} \quad \textbf{and} \quad J^{g_X,\gamma}_{T_qX \oplus T_q^*X^*} \ ?$$

Real Polarization, Duality, Born Reciprocity (M.Born)

- When $M = T^*X$ we have a global $\mathbb{R}$-**polarization** of M and, given a metric (X, g_X), the Ehresmann Levi-Civita connection on T^*X induces a canonical decomposition $T_mM \simeq T_{\pi(m)}X \oplus T^*_{\pi(m)}X; \quad R : x \oplus y^* \mapsto x \oplus (-y^*)$.
- The space TM decomposes as direct sum of Riesz-dual Lagrangian spaces $TM \simeq TX \oplus T^*X$ with $\mathbb{R}$-polarization R.
- The symplectic form ω induces (modulo sign) a canonical **Born symmetry** $TM \xrightarrow{B} TM$ (given by $(x, y^*) \mapsto (-y, x^*)$), where $x \mapsto x^*$ denotes the **Riesz duality isomorphism**).
- $\omega(Bv, Bw) = \omega(v, w) \Rightarrow B \in \mathrm{Sp}(TM, \omega)$ furthermore:
$$B^2 = -\,\mathrm{Id}, \quad F := B \circ R = -R \circ B,$$
almost split-quaternionic structure: generators Id, B, R, F.

Classical Born Geometry and Split-quaternions $\not{\mathbb{H}}$ ($\star$)

(L.Freidel, R.Leigh, D.Minic, C.Brody, E.-M.Graefe, J.Naudts, J.Zhang)



$$\not{\mathbb{H}} := \left\{ \alpha I + \beta T + \gamma J + \delta L \;\middle|\; \alpha, \beta, \gamma, \delta \in \mathbb{R} \right\},$$

$$T^2 = -I, \quad J^2 = +I, \quad L^2 = +I,$$

$$[T, J]_+ = [L, T]_+ = [J, L]_+ = 0, \quad L \circ T \circ J = I,$$

$$\boxed{\not{\mathbb{H}}\text{-C*-module} : I \cdot g_V + T \cdot \omega_V + J \cdot \eta_V + L \cdot \zeta_V} \; (\star)$$

$$\begin{array}{lll}
g_V(+Tv, w) = \omega_V(v, w), & g_V(+Jv, w) = \eta_V(v, w), & g_V(+Lv, w) = \zeta_V(v, w), \\
\omega_V(-Tv, w) = g_V(v, w), & \omega_V(+Jv, w) = \zeta_V(v, w), & \omega_V(-Lv, w) = \eta_V(v, w), \\
\eta_V(-Tv, w) = \zeta_V(v, w), & \eta_V(+Jv, w) = g_V(v, w), & \eta_V(-Lv, w) = \omega_V(v, w), \\
\zeta_V(+Tv, w) = \eta_V(v, w), & \zeta_V(+Jv, w) = \omega_V(v, w), & \zeta_V(+Lv, w) = g_V(v, w).
\end{array}$$

- Born reciprocity of polarized phase-space $V = X \oplus X^*$ of (X, g_X):

$$x \oplus y^* \xrightarrow{T:=B} (-y) \oplus x^*, x \oplus y^* \xrightarrow{J:=R} x \oplus (-y^*), x \oplus y^* \xrightarrow{L:=F} y \oplus x^*,$$

- standard $\mathbb{R}$-subspaces $V_{\mathbb{R}}$ of Hilbert spaces $\overline{V_{\mathbb{R}} \oplus V_{\mathbb{R}}}$ with unitary $\mathbb{C}$-structure T_V and anti-unitary $\mathbb{R}$-structures $J_V, L_V := T_V \circ J_V$.

Modular Born Geometries and Hypercomplex Actions ★

Given an ortho-symplectic space (V, g_V, ω_V)

- its complexification has a canonical modular flux preserving V:

$$\Delta_V^{it} = \exp(iK_V t), \quad K_V = \Gamma_V \circ \Theta_V, \quad \Theta_V := |K_V|,$$

- $\Phi_V := -T_V \cdot \Gamma_V = B \oplus B$ is a **modular Born symmetry** that coincides with the phase of the polar decomposition of the complexification of the polarizers[2] P, Q of (V, g_V, ω_V):

$$\omega_V(v, w) = g_V(Pv, w), \quad g_V(v, w) = \omega_V(v, Qw),$$
$$P = B \circ |P|, \quad Q = B \circ |Q|, \quad |Q| = |P|^{-1},$$

- the operators J_V, Φ_V, T_V generate, on $\mathbb{C} \otimes_{\mathbb{R}} V$, the action of the hypercomplex $\mathbb{R}$-algebra $\mathbb{M}_2(\mathbb{C})_{\mathbb{R}} \simeq \mathbb{C}_{\mathbb{R}} \otimes_{\mathbb{R}} \not{\mathbb{H}}$ of the q-bit, a Φ_V-complexification of the split-quaternions of T_V, J_V, L_V.
- The thermal-double $V_{\mathbb{R}} \oplus V'_{\mathbb{R}}$ has a *quantized* Born geometry.

[2] We further assume here (and for now on) the invertibility of P.

Modular CPT-symmetry and Modular $\mathbb{C}$-polarization ★

$$V^\perp := \{v \oplus w \mid \forall z, \operatorname{Re} h(v \oplus w, z \oplus 0) = 0\},$$
$$V' = J_V V \simeq \overline{V}, \quad V^\perp = L_V V \simeq V^*,$$
$$(V')^\perp = (V^\perp)' = T_V V =: iV,$$

$$\begin{tikzcd}
V' \arrow[rr, leftrightarrow, dotted, "J_V"] \arrow[dd, leftrightarrow, dotted, "L_V"'] \arrow[ddrr, leftrightarrow, dotted] & & V \arrow[dd, leftrightarrow, dotted, "L_V"] \arrow[ddll, leftrightarrow, dotted] \\
 & T_V & \\
iV \arrow[rr, leftrightarrow, dotted, "J_V"'] & & V^\perp
\end{tikzcd}$$

- the modular conjugation J_V is a **modular CPT-symmetry**,
- the phase Γ_V of K_V is a **modular $\mathbb{C}$-polarization**: $\Gamma_V^2 = I_V$,
- the Γ_V-eigenspaces $\Gamma_V^\pm$ are modularly Δ_V^{it}-stable, and describe a pair of "complexified" CPT-mirror copies of (V, g_V, ω_V):

$$\Gamma_V^+ = \{v \oplus Bv \mid v \in V\},$$
$$\Gamma_V^- = \{Bv \oplus v \mid v \in V\},$$

$$\begin{tikzcd}
\Gamma_V^+ \arrow[rr, leftrightarrow, dotted, "J_V"] \arrow[dd, leftrightarrow, dotted, "L_V"'] \arrow[ddrr, leftrightarrow, dotted] & & \Gamma_V^- \arrow[dd, leftrightarrow, dotted, "L_V"] \arrow[ddll, leftrightarrow, dotted] \\
 & T_V & \\
\Gamma_V^- \arrow[rr, leftrightarrow, dotted, "J_V"'] & & \Gamma_V^+
\end{tikzcd}$$

- conjecture: L_V is a duality operator $L_V(\Gamma_V^-) = \Gamma_V^+ \simeq (\Gamma_V^-)^*$.

Internal Complexifications and Mauro-Weyl Quantization★

A $\mathbb{C}$-structure B on an ortho-symplectic space (V, g_V, ω_V),
induces an **internal complexification** $\mathcal{K}_{\mathbb{C}}^{B}$ with inner product:

$$\frac{1}{4}\begin{pmatrix} +g_V(v,w) & +\omega_V(v,Bw) \\ -\omega_V(Bv,w) & +g_V(Bv,Bw) \end{pmatrix} + \frac{i}{4}\cdot\begin{pmatrix} +\omega_V(v,w) & -g_V(v,Bw) \\ +g_V(Bv,w) & +\omega_V(Bv,Bw) \end{pmatrix}$$

that for ortho-symplectic $\mathbb{C}$-structures takes the form:

$$\frac{1}{2}\cdot\Big(g_V(v,w)+\omega_V(v,Bw)\Big)+\frac{i}{2}\cdot\Big(\omega_V(v,w)+g_V(Bv,w)\Big),$$

that in the Kählerian case reproduces: $g_V(v,w)+i\cdot\omega_V(v,w)$.

- $\Gamma_V^{\mp}$ is the internal $(\pm B)$-complexification of $(V, \frac{1}{2}g_V, \frac{1}{2}\omega_V)$.
- $v \mapsto v \oplus \pm Bv$ is an algebraic D.Mauro first-quantization map.
- Weyl second-quantization produces a CPT-thermal-double

$$\mathbb{W}\mathrm{y}(\Gamma_V^{+}) \otimes_{\mathbb{C}} \mathbb{W}\mathrm{y}(\Gamma_V^{-}), \qquad \mathbb{W}\mathrm{y}(\Gamma_V^{+}) \simeq \mathbb{W}\mathrm{y}(\Gamma_V^{-})'$$

$$\mathbb{W}\mathrm{y}(\Gamma_V^{+}) \xleftrightarrow{\ \mathbb{W}\mathrm{y}(J_V)\ } \mathbb{W}\mathrm{y}(\Gamma_V^{-})\ .$$

Modular Flux and Modular Inverse Temperature Θ_V ★

$\boxed{T_V, S_V \text{ generate } \not{\mathbb{H}}} \quad \Leftrightarrow \quad \boxed{V \subset \mathcal{H}_{\mathbb{C}} \text{ standard subspace}}$

$\boxed{S_V \text{ not anti-unitary}} \quad \Leftrightarrow \quad \boxed{\text{non-trivial modular flux}}$

- Kähler spaces have divergent modular flux (semi-Hilbert $\mathcal{H}_{\mathbb{C}}$)
- for ortho-symplectic spaces ($\Rightarrow$ non-Kählerian), we have:

$$\boxed{\textbf{non-trivial modular flux}} \Leftrightarrow \boxed{\begin{array}{c} iV \neq V^{\perp} \\ \Updownarrow \\ \mathrm{Re}\, h|_{V \times iV} \neq 0 \end{array}} \Leftrightarrow \boxed{\begin{array}{c} V \neq V' \\ \Updownarrow \\ \mathrm{Im}\, h|_{V} \neq 0 \end{array}} \Leftrightarrow \boxed{\begin{array}{c} \textbf{non-commutative} \\ V \subset \mathbb{W}\mathrm{y}(V) \end{array}}$$

$$\Theta_V := |K_V| = \begin{bmatrix} \hat{\theta} & 0 \\ 0 & \hat{\theta} \end{bmatrix}, \quad \hat{\theta} := \log\left(\frac{\mathrm{Id}_V + |P|}{\mathrm{Id}_V - |P|}\right), \quad \Delta_V^{it} = \exp(-t \cdot \Phi_V \circ \Theta_V).$$

- Θ_V invertible $\Leftrightarrow$ P invertible $\Leftrightarrow$ *fully* non-trivial modular flux.

Modular Born $\mathbb{R}$-polarizations and NC-spaces ★

- we couldn't find a unique "modular Born $\mathbb{R}$-polarization"!
- hence we take seriously **Born relative localization** proposal (Amelino-Camelia, Freidel, Kowalski-Glickman, Smolin [1], [2]) and
- we define **a modular Born $\mathbb{R}$-polarization** as any operator Λ_V such that $\Lambda_V^2 = I_V$, $[\Lambda_V, \Phi_V]_+ = 0$, $[\Lambda_V, \Gamma_V]_- = 0$.
- each Λ_V produces standard eigen-subspaces in $\Gamma_V^\pm$ of "coordinates" $\Lambda_V^+ \cap \Gamma_V^\pm \subset \Gamma_V^\pm$ and "momenta" $\Lambda_V^- \cap \Gamma_V^\pm \subset \Gamma_V^\pm$
- generically our Born $\mathbb{R}$-polarization Λ_V will have non-trival modular flux [3] and will reduce to a "classical" fully degenerate polarization when the modular flux is trivial ($\Theta_{\Lambda_V^\pm} = 0$).

NC coordinates/momenta spaces $\Lambda_V^\pm$	$\Longleftrightarrow$	Λ_V has non-trivial modular flux: $\Theta_{\Lambda_V^\pm} \neq 0$

[3] Trivial fluxes on some proper subspace can happen if $\Theta_{\Lambda_V^\pm}$ is not invertible.

Modular Quantum Born Geometries ★

A metric g_V that is "diagonal" in the polarization R of a Hitchin-Gualtieri generalized geometry $(V := X \oplus X^*, \omega_V, R, \zeta_V)$

$$\omega_V(x_1 \oplus y_1^*, x_2 \oplus y_2^*) = y_2^*(x_1) - y_1^*(x_2), \quad \zeta_V(x_1 \oplus y_1^*, x_2 \oplus y_2^*) = y_2^*(x_1) + y_1^*(x_2),$$

$$x \oplus y^* \xrightarrow{R} x \oplus (-y^*), \quad g_V(Rv, Rw) = g_V(v, w), \quad v, w \in V,$$

produces a modular deformation of a classical Born geometry of V

$$P = B \circ |P|, \ |P| = \frac{\cosh(\hat{\theta}) - \mathrm{Id}_V}{\sinh(\hat{\theta})} = |Q|^{-1}, \ \hat{\theta} = \Theta_V|_V, \ g_V(v, w) = g_B(|P|v, w).$$

$$\begin{tikzcd}
& \eta_V \arrow[dl, bend left=15, "-F"] \arrow[dr, bend left=30, "+R"] & \\
\omega_V \arrow[ur, bend left=30, "-F"] \arrow[rr, bend left=10, "-B"] \arrow[dr, bend left=15, "+R"] & & g_B \arrow[ul, bend left=15, "+R"] \arrow[ll, bend left=10, "+B"] \arrow[dl, bend left=30, "+F"] \\
& \zeta_V \arrow[ul, bend left=30, "+R"] \arrow[ur, bend left=15, "+F"] &
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
& \eta_V \arrow[dl, bend left=15, "-F"] \arrow[dr, bend left=30, "+R\circ|Q|"] & \\
\omega_V \arrow[ur, bend left=30, "-F"] \arrow[rr, bend left=10, "-Q"] \arrow[dr, bend left=15, "+R"] & & g_V \arrow[ul, bend left=15, "+|P|\circ R"] \arrow[ll, bend left=10, "+P"] \arrow[dl, bend left=30, "+|P|\circ F"] \\
& \zeta_V \arrow[ul, bend left=30, "+R"] \arrow[ur, bend left=15, "+F\circ|Q|"] &
\end{tikzcd}$$

- "off diagonal" metrics g_V induce *Kalb-Ramond deformations* (see Jurčo, Schupp, Vysoky) related to modular theories of the no-more-orthogonal polarizations R, $F \Rightarrow$ (X and X^* are NC).

Tulczyjew Double Bundles and External Born Reciprocity★

(W.M.Tulczyjew, K.Grabowska, J.Grabowski)

Globally, we suggest a **non-linear momentum manifold** $\dot{X} \simeq X$ with a **Riesz dual metric** $\dot{g}_X$ such that $(T\dot{X}, \dot{g}_X) \simeq (T^*X, g_{X^*})$.

$$
\begin{tikzcd}
T(X) \oplus_X T^*(X) \simeq T(T^*X) \arrow[rr, "\text{“Tulczyjew” duality isomorphism } \tau", "T^*\delta_X \circ T^* G^X \circ \tau_X = \tau_{\dot{X}} \circ TG^{\dot{X}} \circ T\delta_X"'] \arrow[d] & & T^*(T\dot{X}) \simeq T^*(\dot{X}) \oplus_{\dot{X}} T^*(\dot{X}) \arrow[d] \\
(T^*(X), g_{X^*}) \arrow[rr, "\text{Riesz duality isometric isomorphism}", "\delta_X"'] \arrow[d] & & (T(\dot{X}), \dot{g}_X) \arrow[d] \\
(X, g_X) \arrow[rr, "\text{non-linear duality diffeomorphism}", "x \mapsto \dot{x}"'] & & (\dot{X}, \dot{g}_X),
\end{tikzcd}
$$

There are dual modular flows $\sigma^{g_X,\gamma}$ and $\sigma^{g_{\dot{X}}\dot{\gamma}}$ acting as a dynamics constraints on the complexified of the Tulczyjew bundles.

Do we have a **global non-linear modular flows** on $X \times \dot{X}$?

CPT-mirror Quantum Born Geometries and $\emptyset$ ★

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\boxed{\begin{array}{c}(\dot{X}', \dot{g}'_X, \dot{\omega}'_X) \quad (\Gamma^+_V)^* \\ \text{CPT-mirror momentum space}\end{array}} \arrow[r, dotted, leftrightarrow, "\text{CPT}", "\text{symmetry}"'] \arrow[d, dotted, leftrightarrow, "\text{Born reciprocity}"] & \boxed{\begin{array}{c}(\dot{X}, \dot{g}_X, \dot{\omega}_X) \quad (\Gamma^-_V)^* \\ \text{momentum space}\end{array}} \arrow[d, dotted, leftrightarrow, "\text{Born reciprocity}"] \\
\boxed{\begin{array}{c}(X', g'_X, \omega'_X) \quad \Gamma^+_V \\ \text{CPT-mirror configuration space}\end{array}} \arrow[r, dotted, leftrightarrow, "\text{CPT}", "\text{symmetry}"'] & \boxed{\begin{array}{c}(X, g_X, \omega_X) \quad \Gamma^-_V \\ \text{configuration space}\end{array}}
\end{tikzcd}
$$

- upon Mauro-Weyl quantization we obtain a quantum version of the classical CPT-Born double above: via $\Gamma^{\pm}$ and $(\Gamma^{\pm})^*$.
- external $\not{\mathbb{H}}$ generated by the CPT symmetry J and duality L
- internal $\not{\mathbb{H}}$ due to Born reciprocity ($\not{\mathbb{H}}$-C*-modules $\Gamma^{\pm}$)
- total symmetry given by a (non-associative?) $\pm$-twisted crossed-product $\not{\mathbb{H}} \ltimes \not{\mathbb{H}}$?

Modular Geometric Pre-quantization and . . . ★

- one might use **modular time** to construct an associated $\mathbb{C}$-line bundle on phase-space and **modular inverse temperature** to produce a connection on such line-bundle,
- as long as a metric is fixed, this provides a canonical geometric pre-quantization of a classical theory,
- this might also suggest the possibility of developing a thermodynamics treatment of individual quantum systems,
- the modular quantum dynamics induces a Weinstein classical modular (harmonic oscillator) dynamics that, under suitable limits, "converges" to the geodesic Liouvillian flow,
- much more remains to be done for a complete understanding of the geometric nature of modular dynamics . . .

- **Heretical Physical Predictions ?**

Tomita CPT, Contravariant NCG, Mirror Cosmology ★

A quite intriguing aspect of modular quantum gravity is that:

- **CPT symmetry** J_ϕ is a built-in feature of local **reducible** standard GNS-representations of a covariant vacuum state ϕ: they are representations of **thermal doubles** $\mathcal{A} \otimes J_\phi \mathcal{A} J_\phi$
- such chiral symmetry seems also a direct byproduct of non-commutativity of space-time: non-commutative vector fields (derivations) and connections split in right/left variants (see for example A.Boroviec, M.van den Bergh).[4]
- L.Boyle and N.Turok have proposed an alternative semi-classical **CPT-symmetric cosmological model**: arXiv:1803.08928 arXiv:1803.08930 arXiv:2109.06204
  Modular Algebraic Quantization can be a quantum-level justification for CPT-mirror (Boyle-Turok) cosmologies.
- **very heretical conjectures** : $\left\{ \begin{array}{l} \text{CMB} \simeq \text{thermal horizon/Unruh effect?} \\ \boxed{\text{anti-matter}} \xleftrightarrow{\text{CPT}} \boxed{\text{matter}} \ ? \end{array} \right\}$

[4]See also PB, R.Conti, C.Puttirungroj "Contravariant Non-commutative Geometry" (work in progress partially based on C.P. PhD thesis).

Relation with GQuEST Interferometry Experiments?

Some theoretical developments, broadly inspired by (variations of) AdS/CFT (celestial) holography, claim that:

- the quantum nature of the geometry induced by modular theory might actually be soon testable by the GQuEST collaboration proposed interferometry experiment based on: [5]

K.Zurek, E.Verlinde et al. arXiv:1902.08207 arXiv:1911.02018 arXiv:2208.01059 arXiv:2205.01799 arXiv:2012.05870 arXiv:2305.11224

The purpose of such experiments is to observe the quantum superposition/fluctuation nature of geometry induced by the vacuum state of (conformal) free quantum scalar field theory on Minkowski spacetime via the modular theory of double cones algebras (seen as conformal equivalent to black holes).

- positive results of GQuEST should give support to our proposed modular algebraic quantum gravity.

[5] I became aware of this very intriguing ongoing experimental attempts at the QG 2023 Nijmegen K.Zurek's talk.

The talk is directly based on this project:

- **Modular Quantization and Non-commutative Space-time**
  PB, R.Conti (version 8+4 - 01.09.2026 - manuscript available on request)

paolo.th@gmail.com

. . . with its (possibly) forthcoming works:

- **Complexification, Quantization and Modular Theory**
  PB, R.Conti, C.Puttirungroj
- **Algebraic Formalism for Covariant Quantum Theory**
  PB, T.Suwannapeng
- **The Modular Gauge of a Riemannian Manifold**
  PB, R.Conti
- **Contravariant Non-commutative Geometry**
  PB, R.Conti, C.Puttirungroj
- **Non-commutative Gel'fand-Naĭmark Duality**
  PB, R.Conti, N.Pitiwan
- **Modular Algebraic Quantum Gravity**
  PB, R.Conti, M.Raasakka

. . . and is broadly inspired by these previous papers:

- M.Raasakka (2019) **Spacetime Granularity from Finite-dimensionality of Local Observable Algebras** arXiv:1908.09293
- PB, R.Conti, N.Pitiwan (2019) **Discrete Non-commutative Gel'fand-Naĭmark Duality** East West Journal of Mathematics 21(2)
- M.Raasakka (2017) **Local Lorentz Covariance in Finite-dimensional Local Quantum Physics** arXiv:1705.06711
- M.Raasakka (2016) **Spacetime-Free Approach to Quantum Theory and Effective Spacetime Structure** arXiv:1605.03942
- PB (2014) **Categorical Operator Algebraic Foundations of Relational Quantum Theory** arXiv:1412.7256
- PB, R.Conti, W.Lewkeeratiyutkul (2011) **A Horizontal Categorification of Gel'fand Duality** Advances in Mathematics 226(1):584-607
- PB, R.Conti, W.Lewkeeratiyutkul (2010) **Modular Theory, Non-commutative Geometry and Quantum Gravity** arXiv:1007.4094v1
- PB, R.Conti, W.Lewkeeratiutkul (2008) **Non-commutative Geometry, Categories and Quantum Physics** arXiv:0801.2826

Thank You for Your Kind Attention!

This is a pre-AI-era work and no AI/LLMs have been used for the conception, exploration, elaboration and writing of this material.

This file has been realized using beamer in the TEX-live distribution and LATEXila editor of Ubuntu Linux 24.04 on Lenovo X270.

- **Modular Algebraic Quantum Gravity and Thermal Space-Time**

Basic Ideology of Modular Algebraic Quantum Gravity ★

Modular Algebraic Quantum Gravity is based on these four points:

- *space-time (NC) geometry is induced by (covariant) states*

$$\boxed{\textit{covariant states} \quad (\mathcal{A}, \phi)} \Rightarrow \boxed{\textit{NC-space-time geometry}}$$

- *modular theory takes the role of dynamical constraint*

$$\boxed{\textit{phase-space Einstein equation}} \simeq \boxed{\textit{Tomita-Takesaki theory}}$$

- *quantum geometry is spectrally reconstructed (spaceoids)*

$$\boxed{\begin{array}{c}\textit{nc } C^*\textit{-algebras} \\ \textit{spectral triples}\end{array}} \xleftrightarrow[\textit{duality}]{\textit{Gel'fand-Naĭmark}} \boxed{\begin{array}{c}\textit{nc spaceoids} \\ ??\end{array}}$$

- *covariance is enforced by categorical principles*

$$\boxed{\begin{array}{c}\textit{covariant transport} \\ + \\ \textit{descent/localization}\end{array}} \simeq \boxed{\begin{array}{c}\textit{modular holonomy} \\ + \\ \textit{modular localization}\end{array}}$$

Proposed Roadmap of MAQG ★

Starting from a pair $(\mathscr{A}, \phi)$ algebra/state, define:

1. via **modular localization** a net of "local" subsystems

$$(\mathscr{A}, \phi) \longmapsto \mathscr{A}^{\phi} := \{\mathcal{A} \subset \mathscr{A} \mid \phi|_{\mathcal{A}} \text{ KMS}\}$$

2. a **spectral modular Liouvillian nc-geometries** on subsystems

$$\mathscr{A}^{\phi} \longmapsto \mathscr{M}^{\phi}_{\mathscr{A}} := \{(\mathcal{A}_{\phi}, \mathcal{H}_{\phi}, K_{\phi}, J_{\phi}) \mid \mathcal{A} \in \mathscr{A}^{\phi}\}$$

3. a **modular holonomy groupoid**, from geometries in $\mathscr{M}^{\phi}_{\mathscr{A}}$,

$$\mathscr{M}^{\phi}_{\mathscr{A}} \longmapsto \mathscr{H}^{\phi}_{\mathscr{A}}$$

   inducing **modular fluctuations** of modular generators K_{ϕ}

4. **nc-phase-space** $\Rightarrow$ **nc-configuration-space** (via a choice of **Born polarization/factorization**) $\Rightarrow$ **thermal NC-space-time**
   (notions of global nc-geometry are needed in the last implication)