Conformal Renormalisation of 8D Einstein Gravity

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Based on the paper **2511.01368** [JHEP 02 (2026) 101] with Davide **Rovere** and on the paper **2607.03679** [hep-th] with Giorgos **Anastasiou**, Ignacio **Araya**, Rodrigo **Olea,** and D. Rovere

FREEDOM OF RESEARCH

Workshop on Quantum Gravity and Strings, Corfu, *07/09/2026*

ACADÉMIE ROYALE DE BELGIQUE 1772

Goals of the talk

- Explain the emergence of conformal (or Weyl) symmetry at the level of the *renormalised, on-shell Einstein-Hilbert action* with a negative cosmological constant on even dimensional spaces. This quantity is central in AdS/CFT.

- Explain the relation between the program called Topological Regularisation developed by Rodrigo Olea and his collaborators [O. Miskovic, G. Anastasiou, I. Araya, I. Papadimitriou, …], and the Holographic Renormalisation scheme developed earlier by [Henningson and Skenderis] and [de Haro, Solodukhin and Skenderis], based on the Fefferman-Graham machinery.

Conformal Gravity and Einstein Gravity in 4D

- Weyl gravity:

$$I_{CG}[g_{\mu\nu}] = -\alpha \int_{M_4} d^4x\sqrt{-g}\, W^{\alpha}{}_{\beta\mu\nu} W^{\beta}_{\alpha\rho\sigma} g^{\mu\rho} g^{\nu\sigma} = \frac{\alpha}{3!}\int_{M_4} d^4x\sqrt{-g}\, W_{\mu_1\mu_2}{}^{\nu_1\nu_2} W_{\mu_3\mu_4}{}^{\nu_3\nu_4}\, \delta^{[\mu_1}_{\nu_1} \ldots \delta^{\mu_4]}_{\nu_4}$$

Weyl tensor $$W^{\alpha\beta}{}_{\mu\nu} = R^{\alpha\beta}{}_{\mu\nu} - (S^{\alpha}{}_{\mu}\,\delta^{\beta}_{\nu} - S^{\beta}{}_{\mu}\,\delta^{\alpha}_{\nu} - S^{\alpha}{}_{\nu}\,\delta^{\beta}_{\mu} + S^{\beta}{}_{\nu}\,\delta^{\alpha}_{\mu})$$

Schouten tensor (4D) $$S^{\alpha}_{\mu} = \frac{1}{2}(R^{\alpha}_{\mu} - \frac{1}{6}R\delta^{\alpha}_{\mu})$$

  - Renormalisable [K. Stelle, 1977]
  - … but non-unitary

- Field equations: $$\frac{1}{\sqrt{-g}}\frac{\delta I_{CG}}{\delta g^{\mu\nu}} = 0 \quad \Leftrightarrow \quad B_{\mu\nu} = 0$$

Bach tensor $$(-4)B_{\mu\nu} = 4\,\nabla^{\alpha}\nabla^{\beta}W_{\alpha\mu\nu\beta} + 2\,R^{\alpha\beta}\,W_{\alpha\mu\nu\beta}$$

Conformal Gravity and Einstein Gravity

- Einstein manifolds are only a subset of the Bach-flat solutions
- Maldacena (2011): solution space of 4D conformal gravity with Newmann boundary condition equivalent to solution space of Einstein gravity with negative cosmological constant. The BC kill the non-Einstein-AdS branch
- Parallel to this, the fact the 4D Einstein theory with CC, renormalised, is on-shell equivalent to 4D Weyl gravity [O. Miskovic and R. Olea, 2009] for

$$\alpha = \frac{\ell^2}{64\pi G}$$

- The boundary counter-terms that render the Einstein-Hilbert action and its variation finite can be summed up in the addition of a single topological invariant in the bulk, the Euler invariant in 4D.

Topological regularisation of Einstein-AdS Gravity

[O. Miskovic and R. Olea, 2009]

- In 4D, adding the Euler term with appropriate coupling:

$$I_{ren}^{(E)} = \frac{1}{16\pi G}\int_M d^4x\sqrt{-g}\left[R - 2\Lambda + \frac{\ell^2}{16}\delta^{[\nu_1\nu_2\nu_3\nu_4]}_{[\mu_1\mu_2\mu_3\mu_4]}R^{\mu_1\mu_2}_{\nu_1\nu_2}R^{\mu_3\mu_4}_{\nu_3\nu_4}\right]$$

is equivalent to performing holographic renormalisation [S. De Haro, S.N.Solodukhin, K. Skenderis, 2001], as far as boundary terms are concerned

- The Gauss-Bonnet term can be written as a surface term that depends on both intrinsic and extrinsic curvatures of the boundary:

$$I_{ren}^{(E)} = \frac{1}{16\pi G}\int_M d^4x\sqrt{-g}\,(R - 2\Lambda) + \frac{\ell^2}{16\pi G}\int_{\partial M} d^3x\sqrt{-h}\,\delta^{[j_1j_2j_3]}_{[i_1i_2i_3]}K^{i_1}_{j_1}\left(\frac{1}{2}\mathcal{R}^{i_2i_3}_{j_2j_3} - \frac{1}{3}K^{i_2}_{j_2}K^{i_3}_{j_3}\right)$$

up to the addition of the Euler characteristic of the manifold, $\chi(M)$

Topological regularisation of 4D Einstein-AdS Gravity

[O. Miskovic and R. Olea, 2009]

- The renormalised action features a remarkable property:

$$I^{(E)}_{ren} = \frac{\ell^2}{256\pi G}\int_M d^4x\sqrt{-g}\,\Omega_{\mu_1\mu_2}{}^{\nu_1\nu_2}\,\Omega_{\mu_3\mu_4}{}^{\nu_3\nu_4}\,\delta^{[\mu_1}_{\nu_1}\ldots\delta^{\mu_4]}_{\nu_4}$$

where

$$\Omega_{\mu\nu}{}^{\alpha\beta} := R_{\mu\nu}{}^{\alpha\beta} + \frac{1}{\ell^2}\,(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu)$$

- When evaluated on an Einstein-AdS manifold for which $R_{\mu\nu}|_E = -\frac{3}{\ell^2}\,g^{(E)}_{\mu\nu}$

one has the equality $\Omega_{\mu\nu}{}^{\alpha\beta}\Big|_E = W_{\mu\nu}{}^{\alpha\beta}$

One recovers the action of 4D conformal gravity !

- The renormalised volume of a 4D manifold [Graham, Anderson, Albin et al.]

$$\frac{1}{32\pi^2}\int_{M_4} d^4x\sqrt{g}\,W^{\mu\nu\rho\sigma}\,W_{\mu\nu\rho\sigma} = \chi[M_4] - \frac{3}{4\pi^2}\,\mathrm{Vol}_{\mathrm{ren}}[M_4]$$

the value of the finite universal part in the asympt. expansion of Vol[M]

Topological regularisation of 6D Einstein-AdS Gravity

[G. Anastasiou, I. Araya, and R. Olea, 2021]

- Now there is a two-parameter family of Weyl-invariant models. Only one choice of parameters give field equations that admit Einstein-AdS space, yielding the Lu, Pang and Pope action

$$I_1 = W_{\alpha\beta\mu\nu} W^{\alpha\rho\lambda\nu} W_{\rho}{}^{\beta\mu}{}_{\lambda}$$
$$I_2 = W_{\mu\nu\alpha\beta} W^{\alpha\beta\rho\lambda} W_{\rho\lambda}{}^{\mu\nu}$$
$$I_3 = W_{\mu\rho\sigma\lambda}\left(\delta^{\mu}_{\nu}\Box + 4R^{\mu}_{\nu} - \frac{6}{5}R\delta^{\mu}_{\nu}\right) W^{\nu\rho\sigma\lambda} + \nabla_{\mu} J^{\mu}$$

take $\quad c_1 I_1 + c_2 I_2 + c_3 I_3 \quad$ with $\quad c_1 = 4c_2 = -12c_3$

the CG action can be rewritten as

$$I_{CG} = \alpha \int_M d^6x \sqrt{-\mathcal{G}} \left(\frac{1}{4!} \delta^{\nu_1 \cdots \nu_6}_{\mu_1 \cdots \mu_6} W^{\mu_1\mu_2}_{\nu_1\nu_2} W^{\mu_3\mu_4}_{\nu_3\nu_4} W^{\mu_5\mu_6}_{\nu_5\nu_6} + \frac{1}{2} \delta^{\nu_1 \cdots \nu_5}_{\mu_1 \cdots \mu_5} W^{\mu_1\mu_2}_{\nu_1\nu_2} W^{\mu_3\mu_4}_{\nu_3\nu_4} S^{\mu_5}_{\nu_5} + 8C^{\mu\nu\lambda}C_{\mu\nu\lambda} \right)$$
$$+ \alpha \int_{\partial M} d^5x \sqrt{-h}\, n_{\mu} \left(8W^{\mu\kappa\lambda\nu} C_{\kappa\lambda\nu} - W^{\kappa\lambda}_{\nu\sigma} \nabla^{\mu} W^{\nu\sigma}_{\kappa\lambda} \right),$$

Note that, when D>4, Topological Regularisation does not reproduce Holographic Renormalisation, B.T. missing

8D Topological Regularisation

Topological regularisation of 8D Einstein-AdS Gravity

- The Einstein-Hilbert Lagrangian with negative C.C. and fine-tuned Euler term added:

$$R + \frac{42}{\ell^2} + \frac{\ell^6}{4\,6!\,\sqrt{-g}}\,\mathcal{E}_8 = \frac{\ell^6}{80}\left(\frac{1}{\ell^4}\,\Phi_{(2)} - \frac{1}{36\,\ell^2}\,\Phi_{(3)} + \frac{1}{576}\,\Phi_{(4)}\right)$$

where

$$\mathcal{E}_{2n} = \frac{\sqrt{-g}}{2^n}\,\delta^{\mu_1\cdots\mu_{2n}}_{\nu_1\cdots\nu_{2n}}\,R^{\nu_1\nu_2}_{\mu_1\mu_2}\cdots R^{\nu_{2n-1}\nu_{2n}}_{\mu_{2n-1}\mu_{2n}}\,;$$

$$\delta^{\mu_1\cdots\mu_{2k}}_{\nu_1\cdots\nu_{2k}} = (2k)!\,\delta^{[\mu_1}_{\nu_1}\ldots\delta^{\mu_{2k}]}_{\nu_{2k}}\,,$$

and

$$\Phi_{(k)} := \delta^{\mu_1\cdots\mu_{2k}}_{\nu_1\cdots\nu_{2k}}\,\Omega_{\mu_1\mu_2}{}^{\nu_1\nu_2}\cdots\,\Omega_{\mu_{2k-1}\mu_{2k}}{}^{\nu_{2k-1}\nu_{2k}}$$

- This partially-regularised action ensures a vanishing Noether-Wald charge for the AdS background.

$$Q[\xi] = 2\int_{\Sigma_\infty} E^{\mu\nu}_{\lambda\sigma}\nabla^\lambda\xi^\sigma\,\mathrm{d}\Sigma_{\mu\nu}$$

Topological regularisation of 8D Einstein-AdS Gravity

- The gravity action for AAdS spaces contains divergent pieces when the boundary is not conformal flat, (the boundary $r \to \infty$ of Schwarzschild-AdS black hole is conformally flat, but there are solutions whose boundary geometry is not conformally flat). Therefore, Topological Regularisation seems at odds with thermodynamics of gravitational instantons in dimensions greater than four.

- New counter-terms are needed, which should be expressed in terms of the boundary Weyl tensor and its derivatives

- Evaluating the partial action on Einstein-AdS space gives

$$I[g_{\mu\nu}]|_E = \frac{\kappa\ell^6}{80}\int_{\mathcal{M}_8} d^8x\sqrt{-g}\left(\frac{1}{\ell^4}\,\Phi_{(2)}\big|_E - \frac{1}{36\ell^2}\,\Phi_{(3)}\big|_E + \frac{1}{576}\,\Phi_{(4)}\big|_E\right)$$

Topological regularisation of 8D Einstein-AdS Gravity

- One introduces the Fefferman-Graham expansion of the metric

$$ds^2 = \frac{\ell^2}{z^2}\left(dz^2 + \bar{g}_{ij}(x,z)dx^i dx^j\right)$$

where $$\bar{g}_{ij}(x,z) = g_{(0)ij} + \frac{z^2}{\ell^2} g_{(2)ij} + \frac{z^4}{\ell^4} g_{(4)ij} + \dots .$$

- Various components of the bulk Weyl tensor, asymptotically:

$$\begin{aligned}
W_{jz}{}^{iz} &= \frac{z^4}{3\ell^2}\mathcal{B}^i_{(0)j} + \mathcal{O}\left(z^6\right) , \\
W_{km}{}^{ij} &= \frac{z^2}{\ell^2}\mathcal{W}^{ij}_{(0)km} - \frac{z^4}{\ell^2}\left[\frac{2}{3}\mathcal{B}^{[i}_{(0)[k}\delta^{j]}{}_{m]} + 2\,\mathcal{S}^{[i}_{(0)n}\,\mathcal{W}^{j]n}_{(0)}{}_{km} - \mathcal{S}^{n}_{(0)[k}\,\mathcal{W}^{ij}_{(0)m]n} + \mathcal{D}^{[i}_{(0)}\mathcal{C}^{j]}_{(0)km}\right] + \mathcal{O}\left(z^6\right) \\
W_{jk}{}^{iz} &= -\frac{z^3}{\ell^2}\mathcal{C}^i_{(0)jk} + \mathcal{O}\left(z^5\right) , \qquad (2.9)
\end{aligned}$$

- The divergent part of the topologically-regularised action reads …

Topological regularisation of 8D Einstein-AdS Gravity

- The divergent part of the topologically-renormalised action, near the conformal boundary, reads

$$\boxed{\begin{aligned} I_{\mathrm{div}}[g_{(0)ij}] = \frac{\kappa\ell^6}{80}\int\limits_{\partial\mathcal{M}_8} d^7x\sqrt{-g_{(0)}}\Big[&\frac{4}{3z^3}\mathcal{W}^{ij}_{(0)ms}\mathcal{W}^{ms}_{(0)ij} - \frac{1}{2z}\Big(\delta^{i_1\ldots i_5}_{j_1\ldots j_5}\mathcal{W}^{j_1j_2}_{(0)i_1i_2}\mathcal{W}^{j_3j_4}_{(0)i_3i_4}\mathcal{S}^{j_5}_{(0)i_5} \\ &+\frac{1}{18}\delta^{i_1\ldots i_6}_{j_1\ldots j_6}\mathcal{W}^{j_1j_2}_{(0)i_1i_2}\mathcal{W}^{j_3j_4}_{(0)i_3i_4}\mathcal{W}^{j_5j_6}_{(0)i_5i_6} + 32\,\mathcal{C}^{ijm}_{(0)}\mathcal{C}_{(0)ijm}\Big)\Big]\end{aligned}}$$

- This exhibits the departure from Holographic Renormalisation

$\longrightarrow$ What counter-terms should be added?
How to cure this divergence?

Comparison with Holographic Renormalisation

- The mismatch between the two schemes vanishes for conformally flat conformal boundaries, like Schwarzschild-AdS black hole, where the topologically regularised action agrees with Holographic Renormalisation. However, the topologically regularised action is not adequate for conformal boundaries such as gravitational instantons and Taub-NUT-AdS, that are not conformally-flat.

Comparing the two regularisation schemes

- At the level of the action, the discrepancy between the two schemes:

$$\int_{\partial M_8} d^7x \left(\mathcal{L}_{ct}^{(4)} + \mathcal{L}_{ct}^{(6)} - \mathcal{L}_{K}^{(4)} - \mathcal{L}_{K}^{(6)}\right) = -I_{\rm div}$$

up to total derivatives at the conformal boundary. These are the divergences of topological regularisation, since by construction HR gives a finite result:

$$\boxed{\begin{aligned} I_{\rm div}[g_{(0)ij}] = \frac{\kappa\ell^6}{80}\int_{\partial\mathcal{M}_8} d^7x\sqrt{-g_{(0)}}\Bigg[&\frac{4}{3z^3}\mathcal{W}^{ij}_{(0)ms}\mathcal{W}^{ms}_{(0)ij} - \frac{1}{2z}\Big(\delta^{i_1\ldots i_5}_{j_1\ldots j_5}\mathcal{W}^{j_1j_2}_{(0)i_1i_2}\mathcal{W}^{j_3j_4}_{(0)i_3i_4}\mathcal{S}^{j_5}_{(0)i_5}\\ &+\frac{1}{18}\delta^{i_1\ldots i_6}_{j_1\ldots j_6}\mathcal{W}^{j_1j_2}_{(0)i_1i_2}\mathcal{W}^{j_3j_4}_{(0)i_3i_4}\mathcal{W}^{j_5j_6}_{(0)i_5i_6} + 32\,\mathcal{C}^{ijm}_{(0)}\mathcal{C}_{(0)ijm}\Big)\Bigg]\end{aligned}}$$

- Independent proof of the well-know fact that Holographic Renormalisation does the job. Now we want to correct Topological Regularisation by adding boundary terms, this time in the form of bulk-covariant total derivatives, that cure the divergence and therefore *express Holographic Renormalisation in a bulk-covariant fashion*, which is new.

Comparing the two regularisation schemes

- What terms to be added to the action in order to cancel the divergences ?
- Requirements: (i) the boundary terms should vanish for AdS, as the free energy of the gravitational vacuum should remain zero. This restrict the total derivative terms to be constructed out of the bulk Weyl tensor and its bulk-covariant derivatives; (ii) we also assume that the total derivative terms can be written in terms of the antisymmetrised quantities $\Phi_{(2)}$ and $\Phi_{(3)}$

$$\Phi_{(2)}\big|_E = 4\, W_{\alpha\beta\gamma\delta}\, W^{\alpha\beta\gamma\delta} \qquad \Phi_{(3)}\big|_E = 32\,(2\, W_{\alpha\beta\gamma\delta}\, W^{\alpha\lambda\eta\beta}\, W_{\lambda}{}^{\gamma\delta}{}_{\eta} + W_{\alpha\beta\gamma\delta}\, W^{\gamma\delta\lambda\eta}\, W_{\lambda\eta}{}^{\alpha\beta})$$

- Our choice of three boundary terms: $\Box^2\Phi_{(2)}, \quad \Box\Phi_{(3)}, \quad \frac{1}{\ell^2}\Box\Phi_{(2)}$
- We find a unique triplet of constants that cancel the divergences of TG, so that our expression for the renormalised Einstein-Hilbert action is

$$\boxed{\begin{aligned} I_{\mathrm{ren}}[g_{\mu\nu}]\big|_E = \frac{\kappa\ell^6}{80}\int_{\mathcal{M}_8} d^8x\,\sqrt{-g}\,\Big[&\frac{1}{\ell^4}\,\Phi_{(2)}\big|_E - \frac{1}{36\ell^2}\,\Phi_{(3)}\big|_E + \frac{1}{576}\,\Phi_{(4)}\big|_E \\ &+ \frac{1}{72}\,\Box^2\Phi_{(2)}\big|_E - \frac{1}{216}\,\Box\,\Phi_{(3)}\big|_E + \frac{1}{4\ell^2}\,\Box\,\Phi_{(2)}\big|_E\Big]\end{aligned}}$$

8D Weyl gravity with an Einstein-AdS sector

8D Conformal Gravity with Einstein sector

- The 8D Weyl-invariant gravity theory whose field equations admit an Einstein-AdS spacetime is unique up to the addition of two boundary conformal invariants N. Boulanger and D. Rovere, *8D conformal gravity with Einstein sector, and its relation to the Q-curvature*, *JHEP* **02** (2026) 101, [2511.01368].

$$I_{\mathrm{BR}}[g_{\mu\nu}] = \int_{\mathcal{M}_8} \mathrm{d}^8x\,\sqrt{-g}\,\mathcal{L}_8$$

with

$$\mathcal{L}_8 = -\tfrac{1}{4}\,(I_1 - \tfrac{23}{25}\,I_2 - \tfrac{21}{25}\,I_3 - 39\,I_6 + \tfrac{2}{5}\,I_7 - \tfrac{7}{2}\,I_8 + \tfrac{124}{5}\,I_9 + \tfrac{9}{20}\,I_{10} - 4\,I_{11} - \tfrac{568}{5}\,I_{12})$$

constructed from the basis of 12 locally Weyl-invariant, parity-invariant scalar densities found in [N.B. and J. Erdmenger, 2004], seven of which are “trivial”

$$\begin{aligned}
I_6 &= W_{\alpha\beta}{}^{\nu\sigma}\,W^{\alpha\beta\gamma\delta}W_{\gamma\nu}{}^{\rho\mu}\,W_{\delta\sigma\rho\mu}\;,\;\; I_7 = W_{\alpha}{}^{\nu}{}_{\gamma}{}^{\sigma}\,W^{\alpha\beta\gamma\delta}\,W_{\beta}{}^{\rho}{}_{\delta}{}^{\mu}\,W_{\nu\rho\sigma\mu}\;,\\
I_8 &= W_{\alpha\beta}{}^{\nu\sigma}\,W^{\alpha\beta\gamma\delta}\,W_{\gamma\delta}{}^{\rho\mu}\,W_{\nu\rho\sigma\mu}\;,\;\; I_9 = W_{\alpha\beta\gamma}{}^{\nu}\,W^{\alpha\beta\gamma\delta}\,W_{\delta}{}^{\sigma\rho\mu}\,W_{\nu\rho\sigma\mu}\;,\\
I_{10} &= W_{\alpha\beta\gamma\delta}\,W^{\alpha\beta\gamma\delta}\,W_{\nu\rho\sigma\mu}\,W^{\nu\sigma\rho\mu}\;,\;\; I_{11} = W_{\alpha}{}^{\nu}{}_{\gamma}{}^{\sigma}\,W^{\alpha\beta\gamma\delta}\,W_{\beta}{}^{\rho}{}_{\sigma}{}^{\mu}\,W_{\delta\mu\nu\rho}\;,\\
I_{12} &= W_{\alpha\gamma}{}^{\nu\sigma}\,W^{\alpha\beta\gamma\delta}\,W_{\beta}{}^{\rho}{}_{\nu}{}^{\mu}\,W_{\delta\mu\sigma\rho}\;.
\end{aligned}$$

8D Conformal Gravity with Einstein sector

- five of which involve the covariant derivatives of the Weyl tensor, among which three are not expressible as total derivative:

$$
\begin{aligned}
I_1 = {} & W_{\rho\gamma\mu\sigma}\, W^{\rho\gamma\mu\sigma,\alpha}{}_{\alpha\beta}{}^{\beta} + \tfrac{48}{25}\, W^{\beta}{}_{\gamma\mu\alpha,\beta} W^{\rho\gamma\mu\alpha}{}_{\rho\nu}{}^{\nu} \\
& + 2\, W_{\mu\beta\gamma\nu,\alpha}\, W^{\mu\beta\gamma\nu,\alpha\rho}{}_{\rho} + \tfrac{42}{125}\, W_{\gamma\alpha\beta\mu}{}^{\beta\alpha}\, W^{\gamma\nu\rho\mu}{}_{\rho\nu} \\
& + \tfrac{9}{10}\, W_{\alpha\mu\nu\beta,\gamma}{}^{\gamma}\, W^{\alpha\mu\nu\beta,\rho}{}_{\rho} + \tfrac{3}{5}\, W_{\nu\gamma\mu\rho,\beta\alpha}\, W^{\nu\gamma\mu\rho,\beta\alpha} \\
& + \tfrac{96}{125}\, W^{\gamma}{}_{\mu\nu\beta,\gamma\alpha}\, W^{\rho\mu\nu\beta}{}_{\rho}{}^{\alpha} + \tfrac{74}{25}\, W_{\beta}{}^{\alpha\gamma\mu}\, W_{\nu\alpha\gamma\mu}\, W^{\beta}{}_{\rho\sigma}{}^{\nu,\sigma\rho} \\
& + \tfrac{208}{5}\, W_{\mu\beta\gamma\alpha}\, W_{\sigma}{}^{\nu\rho\alpha}\, W^{\mu}{}_{\nu\rho}{}^{\sigma,\gamma\beta} - 8\, W_{\alpha}{}^{\gamma}{}_{\beta}{}^{\mu}\, W^{\alpha}{}_{\nu}{}^{\beta}{}_{\rho}\, W^{\nu}{}_{\gamma}{}^{\rho}{}_{\mu,\sigma}{}^{\sigma} \\
& + \tfrac{16}{5}\, W_{\alpha\gamma\mu\rho}\, W_{\beta\nu}{}^{\alpha\gamma}\, W^{\beta\nu\mu\rho}{}_{\sigma}{}^{\sigma} - \tfrac{144}{25}\, W^{\gamma}{}_{\alpha}{}^{\mu}{}_{\beta}\, W_{\rho\gamma}{}^{\nu}{}_{\mu}{}^{\rho}\, W_{\sigma}{}^{\alpha}{}_{\nu}{}^{\beta,\sigma} \\
& + \tfrac{104}{5}\, W_{\alpha}{}^{\gamma}{}_{\beta}{}^{\mu}\, W^{\beta}{}_{\mu}{}^{\sigma\nu,\alpha}\, W_{\rho\gamma\sigma\nu}{}^{\rho} - \tfrac{88}{25}\, W_{\alpha\beta\gamma\mu}\, W_{\rho\nu}{}^{\alpha\beta,\rho}\, W_{\sigma}{}^{\nu\gamma\mu,\sigma}\ , \\
I_2 = {} & W_{\beta}{}^{\alpha\gamma\mu}\, W_{\nu\alpha\gamma\mu}\, W^{\beta}{}_{\rho\sigma}{}^{\nu,\sigma\rho} + 5\, W_{\alpha\gamma\mu\rho}\, W_{\beta\nu}{}^{\alpha\gamma}\, W^{\beta\nu\mu\rho}{}_{\sigma}{}^{\sigma} \\
& + 5\, W_{\alpha\beta\gamma\mu}\, W^{\alpha\beta\rho\sigma}{}_{\nu}\, W^{\gamma\mu}{}_{\rho\sigma}{}^{\nu} + \tfrac{12}{5}\, W_{\alpha\beta\gamma\mu}\, W_{\rho\nu}{}^{\alpha\beta,\rho}\, W_{\sigma}{}^{\nu\gamma\mu,\sigma}\ , \\
I_3 = {} & W_{\beta}{}^{\alpha\gamma\mu}\, W_{\nu\alpha\gamma\mu}\, W^{\beta}{}_{\rho\sigma}{}^{\nu,\sigma\rho} - 20\, W_{\alpha}{}^{\gamma}{}_{\beta}{}^{\mu}\, W^{\alpha}{}_{\nu}{}^{\beta}{}_{\rho}\, W^{\nu}{}_{\gamma}{}^{\rho}{}_{\mu,\sigma}{}^{\sigma} \\
& - \tfrac{48}{5}\, W^{\gamma}{}_{\alpha}{}^{\mu}{}_{\beta}\, W_{\rho\gamma}{}^{\nu}{}_{\mu}{}^{\rho}\, W_{\sigma}{}^{\alpha}{}_{\nu}{}^{\beta,\sigma} - 20\, W^{\alpha}{}_{\mu}{}^{\gamma}{}_{\beta}\, W^{\mu}{}_{\rho}{}^{\beta}{}_{\sigma,\nu}\, W^{\rho}{}_{\alpha}{}^{\sigma}{}_{\gamma}{}^{\nu}\ .
\end{aligned}
$$

the other two being total total derivatives $\sqrt{-g}\ \nabla_{\mu} J^{\mu}_{(i)}(W)\,,\ i = 1, 2$

e.g. $$
\begin{aligned}
J^{\alpha}_{(2)} = {} & W^{\alpha\beta\gamma\delta}\, W_{\gamma}{}^{\epsilon\sigma\rho}\, \nabla_{\rho} W_{\beta\epsilon\delta\sigma} - W^{\alpha\beta\gamma\delta}\, W_{\beta}{}^{\epsilon\sigma\rho}\, \nabla_{\rho} W_{\gamma\delta\epsilon\sigma} \\
& - \tfrac{2}{5}\, W^{\alpha\beta\gamma\delta}\, W_{\beta}{}^{\epsilon}{}_{\gamma}{}^{\sigma}\, \nabla^{\rho} W_{\epsilon\sigma\delta\rho} - \tfrac{2}{5}\, W^{\alpha\beta\gamma\delta}\, W_{\beta}{}^{\epsilon}{}_{\gamma}{}^{\sigma}\, \nabla^{\rho} W_{\delta\sigma\epsilon\rho}\ .
\end{aligned}
$$

Conformal gravity and the Q-curvature

- The last two being total derivatives, they were not considered in the conformal gravity action that admits Einstein manifolds as externals.

- That two of the five nontrivial conformal invariants of [N.B., J. Erdmenger 2004] are expressible as total derivative was noticed only in 2024

  F.-Y. Chen and H. Lu, *Holographic Weyl anomaly in 8D from general higher curvature gravity*, *Phys. Rev. D* **111** (2025) 086032, [2410.16097].

  J. S. Case, A. Khaitan, Y.-J. Lin, A. J. Tyrrell and W. Yuan, *Computing renormalized curvature integrals on Poincaré-Einstein manifolds*, 2404.11319.

- It was verified in [N.B., D. Rovere, 2025] that the 8D action of conformal gravity that admits conformally-Einstein manifolds is proportional to the integrated 8D Q-curvature of Branson

Relation with the renormalised EH action

- On the one hand, consider the most general conformal gravity action that admits an Einstein sector:

$$\mathcal{L}'_8(\alpha,\beta,\gamma) := \alpha\,\mathcal{L}_8 + \beta\,\nabla_\mu\,J^\mu_{(1)} + \gamma\,\nabla_\mu\,J^\mu_{(2)}$$

- On the other hand, consider the topologically regularised EH action with the addition of a linear combination of the three boundary terms considered above:

$$\mathcal{L}_{\mathrm{ren}}(c_1,c_2,c_3) := \tfrac{1}{80}\,\big(\tfrac{1}{\ell^4}\,\Phi_{(2)} - \tfrac{1}{36\,\ell^2}\,\Phi_{(3)} + \tfrac{1}{576}\,\Phi_{(4)} + c_1\,\Box^2\,\Phi_{(2)} + c_2\,\Box\,\Phi_{(3)} + \tfrac{c_3}{\ell^2}\,\Box\,\Phi_{(2)}\big)$$

- Question: Is there a choice of the six parameters $\alpha,\beta,\gamma,c_1,c_2,c_3$ such that the two Lagrangians are equal, when evaluated on-shell

$$\mathcal{L}'_8(\alpha,\beta,\gamma)\big|_E = \mathcal{L}_{\mathrm{ren}}(c_1,c_2,c_3)\big|_E$$

- Answer: there is a unique choice of parameters, that exactly reproduces $I_{\mathrm{ren}}[g_{\mu\nu}]|_E$

$$\alpha = -\tfrac{2}{63}\,,\quad \beta = \tfrac{2}{45}\,,\quad \gamma = -\tfrac{14}{135}\,,\quad c_1 = \tfrac{1}{72}\,,\quad c_2 = -\tfrac{1}{216}\,,\quad c_3 = \tfrac{1}{4}$$

Conclusion: conformal regularisation of EH

- Topological Regularisation (the Kounterterm method) matches Holographic Renormalisation, *only after proper boundary terms are introduced* that render the action finite and are vanishing on a conformally flat boundary

- The final Einstein-Hilbert action with negative cosmological constant, regularised and renormalised, is a *global conformal invariant* that contains Branson's Q-curvature, itself entering Volume renormalisation of even-dimensional manifolds

  where $$\tilde{S}_8[g_{\mu\nu}] = \frac{576\pi^4}{5}\,\chi(M_8) + \frac{3}{20}\int_{M_8} \mathrm{d}^8x\ \mathcal{Q}_8$$

- In principle, the on-shell, renormalised EH action can be computed by HR, a method that works in both even and odd dimensions. A natural question: Is there a bulk-covariant expression of the regularised action, since there are no conformal invariants in odd dimensions? In odd dimension, the renormalised volume is given by boundary data (induced metric at the boundary, extrinsic curvature, etc.)

Extra material

Comparison with Holographic Renormalisation

- We want to express the mismatch in covariant form. For that, first express the Euler density part of the topologically regularised action as

$$\int_{\mathcal{M}_{2n}} d^{2n}x\mathcal{E}_{2n} = (4\pi)^n\, n!\chi\left[\mathcal{M}_{2n}\right] + \int_{\partial\mathcal{M}_{2n}} d^{2n-1}xB_{2n-1}$$

where

$$B_{2n-1} = -2n\delta^{i_1\dots i_{2n-1}}_{j_1\dots j_{2n-1}}K^{j_1}_{i_1}\int_0^1 dt\left[\frac{1}{2}\mathcal{R}^{j_2j_3}_{i_2i_3}(h) - t^2K^{j_2}_{i_2}K^{j_3}_{i_3}\right]\times\cdots\times\left[\frac{1}{2}\mathcal{R}^{j_{2n-1}j_{2n-1}}_{i_{2n-1}i_{2n-1}}(h) - t^2K^{j_{2n-1}}_{i_{2n-1}}K^{j_{2n-1}}_{i_{2n-1}}\right]$$

- On the other hand, the general form of the holographically-renormalised action is [de Haro, Solodukhin, Skenderis]

$$S_{\text{ren}} = \lim_{\epsilon\to 0}\frac{1}{16\pi G_N}\left[\int_{\mathcal{M}_{2n},\,\rho>\epsilon} d^{d+1}x\sqrt{-g}\,(R-2\Lambda) + \int_{\partial\mathcal{M}_{2n},\,\rho=\epsilon} d^dx\,\sqrt{-h}\,2K + \int_{\partial\mathcal{M}_{2n}} d^dx\,\mathcal{L}_{ct,\,\epsilon}\right]$$

Comparison with Holographic Renormalisation

- Therefore, to make contact between the two types of actions, one adds and subtracts the Gibbons-Hawking terms to the topo. regul. action to get

$$I[g_{\mu\nu}]|_E = \frac{1}{16\pi G_N}\int\limits_{\mathcal{M}_{2n}} d^{d+1}x\,\sqrt{-g}\,(R-2\Lambda) + \frac{1}{8\pi G_N}\int\limits_{\partial\mathcal{M}_{2n}} d^dx\,\sqrt{-h}\,K + \int\limits_{\partial\mathcal{M}_{2n}} d^dx\,\mathcal{L}_K\,(h,K,\mathcal{R})$$

where $d+1=2n$ and

$$\mathcal{L}_K\,(h,K,\mathcal{R}) = \alpha_{2n}B_{2n-1} - \frac{1}{8\pi G_N}\int_{\partial M} d^{2n-1}x\sqrt{-h}K$$

or, with $\alpha_{2n} = \frac{1}{16\pi G}\frac{(-1)^n}{n\,(2n-2)!}\,\ell^{2n-2}$

$$\mathcal{L}_K = \frac{\sqrt{-h}}{8\pi G}\frac{(-1)^{n-1}\,\ell^{2n-2}}{(2n-2)!}\delta^{i_1\ldots i_{2n-1}}_{j_1\ldots j_{2n-1}}K^{j_1}_{i_1}\left\{\int_0^1 dt\left[\frac{1}{2}\mathcal{R}^{j_2j_3}_{i_2i_3} - t^2K^{j_2}_{i_2}K^{j_3}_{i_3}\right]\times\right.$$
$$\left.\cdots\times\left[\frac{1}{2}\mathcal{R}^{j_{2n-2}j_{2n}}_{i_{2n-1}i_{2n}} - t^2K^{j_{2n-2}}_{i_{2n-2}}K^{j_{2n}}_{i_{2n}}\right] + \frac{(-1)^n}{\ell^{2n-2}}\delta^{j_2}_{i_2}\cdots\delta^{j_d}_{i_d}\right\}$$

- The extrinsic curvature $K^i_j = \frac{1}{\ell}\,\delta^i_j + \frac{z^2}{\ell}\,\mathcal{S}^i_{(0)j} + \frac{z^4}{2\ell}\left(\frac{1}{d-4}\,\mathcal{B}^i_{(0)j} + \mathcal{S}^i_{(0)k}\mathcal{S}^k_{(0)j}\right) + \mathcal{O}\left(z^6\right)$

Comparison with Holographic Renormalisation

- The subscript and calligraphic symbols indicates when a quantity is computed with a conformal representative of the boundary metric $g_{(0)\,ij}$

- Up to cubic order terms in the boundary curvature, one has

$$\mathcal{L}_K = \mathcal{L}_K^{(0)} + \mathcal{L}_K^{(2)} + \mathcal{L}_K^{(4)} + \mathcal{L}_K^{(6)} + \mathcal{O}\left(\mathcal{R}^4\right) \,,$$

where

$$\mathcal{L}_K^{(0)} = -\frac{\sqrt{-h}}{8\pi G}\frac{d-1}{\ell}\,, \qquad \mathcal{L}_K^{(2)} = -\frac{\ell\sqrt{-h}}{8\pi G}\frac{d-1}{d-2}\mathcal{S}\,,$$

$$\mathcal{L}_K^{(4)} = \frac{\sqrt{-h}}{8\pi G}\frac{\ell^3}{2(d-4)}\left[\delta^{i_1 i_2}_{j_1 j_2}\mathcal{S}^{j_1}_{i_1}\mathcal{S}^{j_2}_{i_2} + \frac{1}{4\,(d-2)}\mathcal{W}^{mn}_{ij}\mathcal{W}^{ij}_{mn}\right]\,,$$

$$\mathcal{L}_K^{(6)} = -\frac{\sqrt{-h}}{8\pi G}\frac{\ell^5}{2(d-2)\,(d-6)}\left[\delta^{i_1 i_2 i_3}_{j_1 j_2 j_3}\mathcal{S}^{j_1}_{i_1}\mathcal{S}^{j_2}_{i_2}\mathcal{S}^{j_3}_{i_3} + \frac{1}{4}\delta^{i_1 i_2 i_3 i_4}_{j_1 j_2 j_3 j_4}\mathcal{W}^{j_1 j_2}_{i_1 i_2}\mathcal{S}^{j_2}_{i_2}\mathcal{S}^{j_3}_{i_3}\right.$$
$$\left. + \frac{1}{16(d-4)}\delta^{i_1 i_2 i_3 i_4 i_5}_{j_1 j_2 j_3 j_4 j_5}\mathcal{W}^{j_1 j_2}_{i_1 i_2}\mathcal{W}^{j_3 j_4}_{i_3 i_4}\mathcal{S}^{j_5}_{i_5} + \frac{1}{192(d-4)}\delta^{i_1 i_2 i_3 i_4 i_5 i_6}_{j_1 j_2 j_3 j_4 j_5 j_6}\mathcal{W}^{j_1 j_2}_{i_1 i_2}\mathcal{W}^{j_3 j_4}_{i_3 i_4}\mathcal{W}^{j_5 j_6}_{i_5 i_6}\right]$$

Comparing the two regularisation schemes

- Similarly, the boundary counter-terms in holographic renormalisation have been computed by several authors [de Haro, Solodukhin, Skendredis], and [Anastasiou, Miskovic, Olea, and Papadimitriou, *Counterterms, Kounterterms, and the variational problem in AdS gravity,* (2020)]

$$8\pi G L_{ct} = -\frac{d-1}{\ell}\sqrt{-h} - \frac{\ell\,(d-1)\sqrt{-h}}{(d-2)}\mathcal{S} + \frac{\ell^3\sqrt{-h}}{2\,(d-4)}\delta^{ik}_{jl}\mathcal{S}^j_i\mathcal{S}^l_k$$
$$-\frac{\ell^5\sqrt{-h}}{(d-2)\,(d-4)\,(d-6)}\left(\mathcal{S}^i_j\mathcal{B}^j_i + \frac{d-4}{2}\delta^{ikm}_{jln}\mathcal{S}^j_i\mathcal{S}^l_k\mathcal{S}^n_m\right) + \mathcal{O}\left(\mathcal{R}^4\right)$$

- The two regularisation schemes agree for the first two terms. However, starting from the quadratic terms in the boundary intrinsic curvature:

$$\mathcal{L}^{(4)}_K = \mathcal{L}^{(4)}_{ct} + \frac{\sqrt{-h}}{8\pi G}\frac{\ell^3}{8\,(d-2)\,(d-4)}\mathcal{W}^{ab}_{ij}\mathcal{W}^{ij}_{ab}\,,$$
$$\mathcal{L}^{(6)}_K = \mathcal{L}^{(6)}_{ct} + \frac{\sqrt{-h}}{8\pi G}\frac{\ell^5}{2\,(d-2)\,(d-4)\,(d-6)}\left(2\mathcal{S}^i_j\mathcal{B}^j_i - \frac{d-4}{4}\delta^{i_1i_2i_3i_4}_{j_1j_2j_3j_4}\mathcal{W}^{j_1j_2}_{i_1i_2}\mathcal{S}^{j_2}_{i_2}\mathcal{S}^{j_3}_{i_3}\right.$$
$$\left.-\frac{1}{16}\delta^{i_1i_2i_3i_4i_5}_{j_1j_2j_3j_4j_5}\mathcal{W}^{j_1j_2}_{i_1i_2}\mathcal{W}^{j_3j_4}_{i_3i_4}\mathcal{S}^{j_5}_{i_5} - \frac{1}{192}\delta^{i_1i_2i_3i_4i_5i_6}_{j_1j_2j_3j_4j_5j_6}\mathcal{W}^{j_1j_2}_{i_1i_2}\mathcal{W}^{j_3j_4}_{i_3i_4}\mathcal{W}^{j_5j_6}_{i_5i_6}\right)\,.$$

Comparing the two regularisation schemes

- The asymptotic expansion of the discrepancies:

$$\sqrt{-h}\delta^{i_1i_2i_3i_4i_5i_6}_{j_1j_2j_3j_4j_5j_6}\mathcal{W}^{j_1j_2}_{i_1i_2}\mathcal{W}^{j_3j_4}_{i_3i_4}\mathcal{W}^{j_5j_6}_{i_5i_6} = \frac{\ell\sqrt{-g_{(0)}}}{z}\mathcal{W}^{j_1j_2}_{(0)i_1i_2}\mathcal{W}^{j_3j_4}_{(0)i_3i_4}\mathcal{W}^{j_5j_6}_{(0)i_5i_6}\,,$$

$$\sqrt{-h}\delta^{i_1i_2i_3i_4i_5}_{j_1j_2j_3j_4j_5}\mathcal{W}^{j_1j_2}_{i_1i_2}\mathcal{W}^{j_3j_4}_{i_3i_4}\mathcal{S}^{j_5}_{i_5} = \frac{\ell\sqrt{-g_{(0)}}}{z}\delta^{i_1i_2i_3i_4i_5}_{j_1j_2j_3j_4j_5}\mathcal{W}^{j_1j_2}_{(0)i_1i_2}\mathcal{W}^{j_3j_4}_{(0)i_3i_4}\mathcal{S}^{j_5}_{(0)i_5}\,,$$

$$\sqrt{-h}\delta^{i_1i_2i_3i_4}_{j_1j_2j_3j_4}\mathcal{W}^{j_1j_2}_{i_1i_2}\mathcal{S}^{j_2}_{i_2}\mathcal{S}^{j_3}_{i_3} = \frac{\ell\sqrt{-g_{(0)}}}{z}\delta^{i_1i_2i_3i_4}_{j_1j_2j_3j_4}\mathcal{W}^{j_1j_2}_{(0)i_1i_2}\mathcal{S}^{j_2}_{(0)i_2}\mathcal{S}^{j_3}_{(0)i_3}\,,$$

$$\sqrt{-h}\mathcal{S}^{i}_{j}\mathcal{B}^{j}_{i} = \frac{\ell\sqrt{-g_{(0)}}}{z}\mathcal{S}^{i}_{(0)j}\mathcal{B}^{j}_{(0)i}\,,$$

$$\begin{aligned}\sqrt{-h}\mathcal{W}^{ab}_{ij}\mathcal{W}^{ij}_{ab} = \frac{\ell^3\sqrt{-g_{(0)}}}{z^3}\Big[&\mathcal{W}^{ab}_{(0)ij}\mathcal{W}^{ij}_{(0)ab} + z^2\Big(4\mathcal{S}^{i}_{(0)j}\mathcal{S}^{a}_{(0)b}\mathcal{W}^{jb}_{(0)ia}\\ &-\frac{1}{8}\delta^{i_1i_2i_3i_4i_5}_{j_1j_2j_3j_4j_5}\mathcal{W}^{j_1j_2}_{(0)i_1i_2}\mathcal{W}^{j_3j_4}_{(0)i_3i_4}\mathcal{S}^{j_5}_{(0)i_5} + 2\mathcal{W}_{(0)abim}\nabla^{m}_{(0)}\mathcal{C}^{iab}_{(0)}\Big)\Big]\end{aligned}$$

- The mismatch between the two schemes vanishes for conformally flat conformal boundaries, like Schwarzschild-AdS black hole, where the topologically regularised action gives correct results. However, the topologically regularised action is not enough for conformal boundaries such as gravitational instantons and Taub-NUT-AdS, that are not conformally-flat.

Conformal gravity and the Q-curvature

- R. Gover and J. Petersen
  Commun. Math. Phys. 235, 339–378 (2003)

Conformally Invariant Powers of the Laplacian, Q-Curvature, and Tractor Calculus

Abstract: We describe an elementary algorithm for expressing, as explicit formulae in tractor calculus, the conformally invariant GJMS operators due to C.R. Graham et alia. These differential operators have leading part a power of the Laplacian. Conformal tractor calculus is the natural induced bundle calculus associated to the conformal Cartan connection. Applications discussed include standard formulae for these operators in terms of the Levi-Civita connection and its curvature and a direct definition and formula for T. Branson's so-called Q-curvature (which integrates to a global conformal invariant) as well as generalisations of the operators and the Q-curvature. Among examples, the operators of order 4, 6 and 8 and the related Q-curvatures are treated explicitly. The algorithm exploits the ambient metric construction of Fefferman and Graham and includes a procedure for converting the ambient curvature and its covariant derivatives into tractor calculus expressions. This is partly based on [12], where the relationship of the normal standard tractor bundle to the ambient construction is described.

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Conformally Invariant Powers of the Laplacian, *Q*-Curvature, and Tractor Calculus

$$
\begin{aligned}
&\frac{-12\left(8-4n+n^2\right)}{(-4+n)^2}\mathsf{P}_{ij|k}{}^{k}\mathsf{P}^{ij}{}_{|l}{}^{l} - 24\mathsf{P}_{ij|kl}\mathsf{P}^{ij}{}_{|}{}^{kl} - \frac{48(-18+7n)}{-6+n}\mathsf{P}_{ij}\mathsf{P}_{kl}\mathsf{P}^{ij}{}_{|}{}^{kl}\\
&-\frac{48(-2+n)}{-4+n}\mathsf{P}_{ij|k}\mathsf{P}^{ij}{}_{|}{}^{k}{}_{l}{}^{l} - \frac{12(-2+n)}{-6+n}\mathsf{P}_{ij}\mathsf{P}^{ij}{}_{|k}{}^{k}{}_{l}{}^{l} + \frac{192n}{-6+n}\mathsf{P}_{ij}\mathsf{P}_{kl}\mathsf{P}^{ik}{}_{|}{}^{jl}\\
&-\frac{48\left(-32+44n-12n^2+n^3\right)}{(-6+n)(-4+n)}\mathsf{P}_{ij}\mathsf{P}_{kl|}{}^{i}\mathsf{P}^{jk}{}_{|}{}^{l} + \frac{384(-16+3n)}{(-6+n)(-4+n)}\mathsf{P}_{ij}\mathsf{P}^{i}{}_{k|l}\mathsf{P}^{jk}{}_{|}{}^{l}\\
&+\frac{384\left(4-6n+n^2\right)}{(-6+n)(-4+n)^2}\mathsf{P}_{ij}\mathsf{P}^{i}{}_{k}\mathsf{P}^{jk}{}_{|l}{}^{l} + \frac{192\left(144-32n-4n^2+n^3\right)}{(-6+n)(-4+n)^2}\mathsf{P}_{ij}\mathsf{P}_{kl}\mathsf{P}^{ik}\mathsf{P}^{jl}\\
&+\frac{48\left(-32+20n-8n^2+n^3\right)}{(-6+n)(-4+n)}\mathsf{P}_{ij}\mathsf{P}^{i}{}_{k|l}\mathsf{P}^{jl}{}_{|}{}^{k}\\
&+\frac{6(-8+n)\left(256-32n-18n^2+3n^3\right)}{(-6+n)(-4+n)^2}\mathsf{P}_{ij}\mathsf{P}_{kl}\mathsf{P}^{ij}\mathsf{P}^{kl}\\
&+\frac{24\left(-176+80n-14n^2+n^3\right)}{(-6+n)(-4+n)}\mathsf{P}_{ij}\mathsf{P}_{kl|}{}^{i}\mathsf{P}^{kl}{}_{|}{}^{j} + \frac{4\left(48+80n-46n^2+5n^3\right)}{(-6+n)(-4+n)}\mathsf{P}_{ij|k}\mathsf{P}^{ij}{}_{|}{}^{k}\mathsf{J}\\
&+\frac{4\left(12-20n+5n^2\right)}{-6+n}\mathsf{P}_{ij}\mathsf{P}^{ij}{}_{|k}{}^{k}\mathsf{J} - \frac{384n}{-6+n}\mathsf{P}_{ij}\mathsf{P}^{i}{}_{k}\mathsf{P}^{jk}\mathsf{J}\\
&-\frac{96-80n-30n^2+5n^3}{-6+n}\mathsf{P}_{ij}\mathsf{P}^{ij}\mathsf{J}^2 + \frac{(-4+n)n(4+n)}{8}\mathsf{J}^4\\
&+\left(2+13n-2n^2\right)\mathsf{J}\mathsf{J}_{|i}\mathsf{J}_{|}{}^{i} - \frac{(-2+n)\left(200-38n+3n^2\right)}{-4+n}\mathsf{P}_{ij}\mathsf{J}_{|}{}^{i}\mathsf{J}_{|}{}^{j}\\
&+\frac{24(-2+n)\left(88-20n+n^2\right)}{(-6+n)(-4+n)}\mathsf{P}_{ij}\mathsf{P}^{i}{}_{k|}{}^{j}\mathsf{J}_{|}{}^{k} + \frac{16\left(-48+70n-17n^2+n^3\right)}{(-6+n)(-4+n)}\mathsf{P}_{ij}\mathsf{P}^{ij}{}_{|k}\mathsf{J}_{|}{}^{k}\\
&+\frac{16+4n-3n^2}{2}\mathsf{J}^2\mathsf{J}_{|i}{}^{i} + \frac{-8+3n}{2}\mathsf{J}_{|i}{}^{i}\mathsf{J}_{|j}{}^{j} + \frac{2\left(32-40n+5n^2\right)}{-4+n}\mathsf{P}_{ij}\mathsf{P}^{ij}\mathsf{J}_{|k}{}^{k}\\
&-\frac{16\left(4-8n+n^2\right)}{(-4+n)^2}\mathsf{P}_{ij|k}{}^{k}\mathsf{J}_{|}{}^{ij} + \frac{4n(-46+5n)}{-6+n}\mathsf{P}_{ij}\mathsf{J}\mathsf{J}_{|}{}^{ij}\\
&+\frac{2\left(-336+136n-20n^2+n^3\right)}{(-4+n)^2}\mathsf{J}_{|ij}\mathsf{J}_{|}{}^{ij} + \frac{24\left(1088-608n+144n^2-18n^3+n^4\right)}{(-6+n)(-4+n)^2}\mathsf{P}_{ij}\mathsf{P}^{i}{}_{k}\mathsf{J}_{|}{}^{jk}\\
&+(-38+5n)\mathsf{J}_{|i}\mathsf{J}_{|}{}^{i}{}_{j}{}^{j} - \frac{32(-7+n)}{-4+n}\mathsf{P}_{ij|k}\mathsf{J}_{|}{}^{ijk} + 2(-1+n)\mathsf{J}\mathsf{J}_{|i}{}^{i}{}_{j}{}^{j}\\
&-\frac{4(-54+7n)}{-6+n}\mathsf{P}_{ij}\mathsf{J}_{|}{}^{ij}{}_{k}{}^{k} - \mathsf{J}_{|i}{}^{i}{}_{j}{}^{j}{}_{k}{}^{k} - \frac{192\left(4-6n+n^2\right)}{(-6+n)(-4+n)^2}\mathsf{P}_{ij}\mathsf{P}_{kl|m}{}^{m}C^{ikjl}\\
&+\frac{96n}{-6+n}\mathsf{P}_{ij}\mathsf{P}_{kl}\mathsf{J}C^{ikjl} - \frac{24(-8+n)\left(-16-2n+n^2\right)}{(-6+n)(-4+n)^2}\mathsf{P}_{ij}\mathsf{J}_{|kl}C^{ikjl}\\
&-\frac{384(-5+n)}{(-6+n)(-4+n)}\mathsf{P}_{ij}\mathsf{P}_{kl|m}C^{ikjl}{}_{|}{}^{m} - \frac{192}{-4+n}\mathsf{P}_{ij|k}\mathsf{P}_{lm|}{}^{k}C^{iljm}\\
&+\frac{192\left(112-16n-6n^2+n^3\right)}{(-6+n)(-4+n)^2}\mathsf{P}_{ij}\mathsf{P}_{kl}\mathsf{P}^{i}{}_{m}C^{jklm} - \frac{96n}{-4+n}\mathsf{P}_{ij|k}\mathsf{P}^{i}{}_{l|m}C^{jlkm}\\
&+\frac{192}{-6+n}\mathsf{P}_{ij}\mathsf{P}_{kl}C^{i}{}_{m}{}^{k}{}_{p}C^{jmlp} - \frac{384}{-6+n}\mathsf{P}_{ij}\mathsf{P}_{kl}C^{i}{}_{m}{}^{k}{}_{p}C^{jplm}\\
&+\frac{192(-6+n)}{(-4+n)^2}\mathsf{P}_{ij}\mathsf{P}_{kl}C^{i}{}_{m}{}^{j}{}_{p}C^{kmlp}
\end{aligned}
$$

Fig. 5. The invariant $Q^{g}_{8,n}$

Conformal gravity and the Q-curvature

- The Euler-Lagrange variation of the 8D Q-curvature gives the *obstruction tensor*

$$\delta \int \mathrm{d}^{2m}x \, \mathcal{Q}_{2m} = - \int \mathrm{d}^{2m}x \sqrt{|g|}\, O^{(2m)}_{\mu\nu} \, \delta g^{\mu\nu}$$

1. *Naturality:* $O_{\mu\nu}$ is natural, i.e., it can be expressed as a universal polynomial in the metric, its inverse, the curvature, and covariant derivatives of the curvature.

2. *Linearisation:* $O_{\mu\nu}$ is a symmetric, conformally covariant tensor whose expression starts with

$$O_{\mu\nu} = \frac{1}{3-n} \triangle^{m-2} \left(\nabla^\alpha \nabla^\beta W_{\mu\alpha\nu\beta} \right) + \text{lower order terms}, \tag{6.1}$$

   where $\triangle = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ (or the D'Alembertian, in Lorentzian signature), and "lower order terms" involve terms with fewer derivatives of the curvature. The tensor $O_{\mu\nu}$ has conformal weight $2-n$, meaning that for $\widehat{g} = e^{2\omega} g$, $\widehat{O}_{\mu\nu} = e^{(2-n)\omega}\, O_{\mu\nu}$.

3. *Trace and divergence:* traceless and divergence-free: $g^{\mu\nu} O_{\mu\nu} = 0 = \nabla^\mu O_{\mu\nu}$.

4. *Vanishing on conformally Einstein metrics:* if g is conformal to an Einstein metric, then

$$O_{\mu\nu} = 0 \, . \tag{6.2}$$