Integrable auxiliary field deformations of dimensionally reduced gravity

Mattia Cesàro, Max Planck Institute for Gravitational Physics - AEI

Workshop on Quantum Gravity and Strings, Corfu

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Based on [MC, A.Kleinschmidt,D.Osten,arXiv:2409.04523], [MC, D.Osten, arXiv:2502.01750], [D.Bielli, MC, arXiv:2606.30717]

Plan of the talk

1. Kaluza-Klein reduction of $D = 4$ GR to $D = 2$: the BM model
2. $D = 2$ auxiliary field deformations
   - ν-frame
   - μ-frame
3. Uplifts to $D = 4$
4. Conclusions

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$D = 4$ Gravity with two space-like isometries

$D = 3$

$D = 2$

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$\big\downarrow S^1$

$D = 3$ $\mathbb{R}^+$-symmetry

dualisation of KK vector $\to$ SL(2)$_{(\mathrm{E})}$ symmetry

$D = 2$

Dimensionally reduced GR: the Breitenlohner-Maison model

KK reduction of $D = 4$ GR to $D = 2$

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$D = 2$ $\mathbb{R}^+ \times \mathsf{SL}(2)_{(\mathsf{E})}$ symmetry

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Dimensionally reduced GR: the Breitenlohner-Maison model

KK reduction of $D = 4$ GR to $D = 2$

$D = 4$ Gravity with two space-like isometries

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Geroch $\widehat{\mathsf{SL}(2)}$

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$$\mathcal{V} \in \mathsf{SL}(2)_{\mathsf{MM}}, \quad \mathcal{V}^{-1} \partial_\mu \mathcal{V} = P_\mu + Q_\mu \,, \qquad P_\mu \in \mathfrak{sl}_2 \ominus \mathfrak{so}_2 \,, \quad Q_\mu \in \mathfrak{so}_2 .$$

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Equations of motion:

$$\begin{cases} \Box \rho = 0\,, \\ D_\mu(\rho P^\mu) = 0\,, \end{cases}$$

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supplemented by Virasoro constraints (in lightcone coordinates),

$$\partial_\pm \rho \partial_\pm \sigma - \frac{1}{2}\partial_\pm\partial_\pm\rho - \frac{1}{2}\rho \mathsf{Tr}(P_\pm P_\pm) = 0\,.$$

- Linear system: [V. Belinskii, V. Sakharov, '78, '79] [P. Breitenlohner, D. Maison, '87].
- Extended linear system: [B. Julia, H. Nicolai, '96] (also [D. Bernard, N.Regnault, '01]).
- Hamiltonian integrability: [D. Korotkin, H. Samtleben, '98] .

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Plan of the talk

Auxiliary deformations: general framework

The ν-frame

Original work from [C. Ferko, L. Smith, '24] on the Principal Chiral Model, inspired from non linear deformations of Maxwell theory in [E. Ivanov, B. Zupnik, '02]. Later extended by [N.Baglioni, D.Bielli, MC, C. Ferko, M.Galli, A.Kleinschmidt, D.Osten, L.Smith, G.Tartaglino-Mazzucchelli,....].

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Start from an (integrable) seed theory,

$$\mathring{\mathcal{L}}(\phi).$$

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$$\tilde{\mathcal{L}} = -\mathring{\mathcal{L}}(\phi) + \mathcal{L}(\phi, v) + \mathcal{E}(\nu)\,,$$

with $\nu = \nu(v)$ a certain combination of auxiliary fields. The infinite family of theories preserves integrability!

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A set of deformations encompassing the "$T\bar{T}$" [A.B.Zamolodchikov, '04] and "root-$T\bar{T}$" [C.Ferko, A.Sfondrini, L.Smith, G.Tartaglino-Mazzucchelli, '22] , [H.Babaei-Aghbolagh, S.He, T.Morone, H.Ouyang, R.Tateo, '24] ones, but in principle more general.

Deform the Lagrangian through introduction of auxiliary fields [MC, D. Osten, '25],

$$\begin{aligned}\mathcal{L}^{(2,\nu)} =& - \partial_\mu \rho \partial^\mu \sigma + \frac{\rho}{2} \mathsf{Tr}(P_\mu P^\mu) \\ & - 2(\chi_{1\,\mu} \partial^\mu \rho + \chi_{2\,\mu} \partial^\mu \sigma + \chi_1^\mu \chi_{2\,\mu}) + \rho \left(\mathsf{Tr} \left[2 P_\mu v^\mu + v_\mu v^\mu \right] \right) + \mathcal{E}(\nu_2) \,,\end{aligned}$$

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with

$$\nu_2 = (\eta^{\alpha\beta}\eta^{\gamma\sigma} + \epsilon^{\alpha\beta}\epsilon^{\gamma\sigma}) \left(\chi_{1\,\alpha}\chi_{2\,\gamma} - \frac{\rho}{2}\mathsf{Tr}(v_\alpha v_\gamma) \right) \left(\chi_{1\,\beta}\chi_{2\,\sigma} - \frac{\rho}{2}\mathsf{Tr}(v_\beta v_\sigma) \right) \,.$$

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Deformed "currents"

$$\mathcal{P}_\mu = -(P_\mu + 2 v_\mu) \,, \quad \mathcal{R}_\mu = -(\partial_\mu \rho + 2\chi_{2\,\mu}) \,, \quad \mathcal{S}_\mu = -(\partial_\mu \sigma + 2 \chi_{1\,\mu}) \,,$$

are important ingredients.

Equations of motion: dynamical for scalars , dynamical for conformal factor and algebraic ones ,

$$\begin{cases} \partial_\mu \mathcal{R}^\mu = 0\,, \\ D_\mu(\rho \mathcal{P}^\mu) = 0\,, \end{cases}$$

$$\begin{cases} \partial_\mu \mathcal{S}^\mu = \mathsf{Tr}\left[\frac{1}{2}(P_\mu P^\mu) + 2P_\mu v^\mu + v_\mu v^\mu\right] + K^{\mu\nu}(\chi_{1,2}, v, \rho)\,\mathsf{Tr}(v_\mu v_\nu)\,, \end{cases}$$

$$\begin{cases} P_\mu = -v_\mu + K_\mu{}^\nu(\chi_{1,2}, v, \rho) v_\nu, \\ \partial_\mu \sigma = -\chi_{1\,\mu} + K_\mu{}^\nu(\chi_{1,2}, v, \rho)\chi_{1\,\nu}\,, \\ \partial_\mu \rho = -\chi_{2\,\mu} + K_\mu{}^\nu(\chi_{1,2}, v, \rho)\chi_{2\,\nu}\,, \end{cases}$$

where

$$K_\mu{}^\nu(\chi_{1,2}, v, \rho) = \mathcal{E}'(\nu_2)(\delta_\mu^\alpha \eta^{\gamma\nu} - \eta_{\mu\rho}\epsilon^{\rho\alpha}\epsilon^{\gamma\nu})\left(\chi_{1\,\alpha}\chi_{2\,\gamma} - \frac{\rho}{2}\mathsf{Tr}(v_\alpha v_\gamma)\right)\,.$$

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The model preserves (classical) integrability [MC, D. Osten, ‘25].

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Alternative duality-frame

the μ-frame

In [N.Baglioni, D.Bielli, M.Galli, G.Tartaglino-Mazzucchelli, '25] an alternative frame for such auxiliary field deformations was constructed.

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Start again from

$$\tilde{\mathcal{L}}^{(\nu)} = -\mathring{\mathcal{L}}(\phi) + \mathcal{L}(\phi, v) + \mathcal{E}(\nu)\,,$$

where we must now assume $\nu \to \sqrt{\nu_2}$.

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Operate a partial Legendre transform (Routhian),

$$-\nu\mathcal{E}_\nu + \mathcal{E} = \mathcal{H}(\mu)\,, \quad \text{with} \quad \mu = \mathcal{E}_\nu \equiv \frac{\partial\mathcal{E}(\nu)}{\partial\nu}, \quad \text{and} \quad \nu = -\mathcal{H}_\mu \equiv -\frac{\partial\mathcal{H}(\mu)}{\partial\mu}\,.$$

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We land in the so called μ-frame

$$\tilde{\mathcal{L}}^{(\mu)} = F(\mu)\mathring{\mathcal{L}}(\phi) + G(\mu)\mathcal{L}_{\text{non-poly}}(\phi) + \mathcal{H}(\mu).$$

Trade some complexity for simplicity and viceversa.

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	easy	convoluted
ν-frame	Lagrangian	Interaction function
μ-frame	Interaction function	Lagrangian

The sets of theories overlap only partially!

Alternative duality-frame

Deformed BM model in the ν-frame [MC, D. Osten, '25],

$$\mathcal{L}^{(2,\nu)} = -\frac{1}{2}\Big(-\partial_+\rho\partial_-\sigma - \partial_-\rho\partial_+\sigma - 2\chi_{1+}\partial_-\rho - 2\chi_{1-}\partial_+\rho - 2\chi_{2+}\partial_-\sigma - 2\chi_{2-}\partial_+\sigma$$
$$- 2\chi_{1+}\chi_{2-} - 2\chi_{1-}\chi_{2+} + \rho\mathsf{Tr}[P_+P_- + 2P_+v_- + 2P_-v_+ + 2v_+v_-]\Big) + \mathcal{E}(\nu),$$

where

$$\nu = \sqrt{\Big(\chi_{1+}\chi_{2+} - \frac{\rho}{2}\mathsf{tr}(v_+^2)\Big)\Big(\chi_{1-}\chi_{2-} - \frac{\rho}{2}\mathsf{tr}(v_-^2)\Big)}\,.$$

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Deformed BM model in the μ-frame [D.Bielli, MC, '26] ,

$$\mathcal{L}^{(2,\,\mu)} = +\frac{1}{2}\Big(\frac{\mu^2+4}{\mu^2-4}\Big)\Big(\partial_+\rho\partial_-\sigma + \partial_-\rho\partial_+\sigma - \rho\mathsf{Tr}(P_+P_-)\Big)$$
$$- \Big(\frac{4\mu}{\mu^2-4}\Big)\sqrt{\Big(\partial_+\rho\partial_+\sigma - \frac{\rho}{2}\mathsf{Tr}(P_+^2)\Big)\Big(\partial_-\rho\partial_-\sigma - \frac{\rho}{2}\mathsf{Tr}(P_-^2)\Big)} + \mathcal{H}(\mu)\,.$$

Plan of the talk

The $D = 4$ Einstein-Hilbert (EH) action can be written in terms of the anholonomicity coefficients,

$$\Omega_{AB}{}^{C} = E_{A}{}^{M} E_{B}{}^{N} (\partial_M E_N{}^{C} - \partial_N E_M{}^{C}) \,,$$

where $M = 0, .., 3$, $A = 0, ..., 3$, $E_M{}^{A}$ is the vierbein, as

$$S^{(4)}_{\mathsf{EH}}[E] = \int d^4x \mathcal{L}^{(4)}_{\mathsf{EH}} = -\frac{1}{8} \int d^4x E \left(\Omega^{ABC} \Omega_{ABC} + 2\Omega^{ABC} \Omega_{ACB} - 4\Omega_{AB}{}^{B} \Omega^{AC}{}_{C} \right) \,,$$

up to boundary contributions (“Teleparallel-Equivalent of GR” [Maluf , ‘13]).

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Starting point (inspired to [P.West, '01], [N.Boulanger, O.Hohm, '08])

$$\begin{aligned} S_{\mathsf{EH}}^{(4)}[E, Y] = +\frac{1}{8} \int d^4x E \Big(& + \Omega^{ABC}\Omega_{ABC} + 2\Omega^{ABC}\Omega_{ACB} - 4\Omega_{AB}{}^{B}\Omega^{AC}{}_{C} \\ & + 4(Y^{ABC}\Omega_{ABC} + 2Y^{ABC}\Omega_{ACB} - 4Y_{AB}{}^{B}\Omega^{AC}{}_{C}) \\ & + 2(Y^{ABC}Y_{ABC} + 2Y^{ABC}Y_{ACB} - 4Y_{AB}{}^{B}Y^{AC}{}_{C}) \Big). \end{aligned}$$

Integrating out $Y_{AB}{}^{C} = -\Omega_{AB}{}^{C}$ returns the original EH action.

Uplifts to D=4

Main challenge in uplifting the models:

- ν-frame: uplifting $\nu_2 = \Big(\chi_{1+}\chi_{2+} - \frac{\rho}{2}\mathsf{Tr}(v_+^2)\Big)\Big(\chi_{1-}\chi_{2-} - \frac{\rho}{2}\mathsf{Tr}(v_-^2)\Big)$.
- μ-frame: uplifting $\Big(\partial_+\rho\partial_+\sigma - \frac{\rho}{2}\mathsf{Tr}(P_+^2)\Big)\Big(\partial_-\rho\partial_-\sigma - \frac{\rho}{2}\mathsf{Tr}(P_-^2)\Big)$.

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Solution$\rightarrow$ consider as fundamental variables:

$$W_{CA}{}^{B} = E_C{}^{Q} E_A{}^{M} \partial_Q E_M{}^{B}, \qquad Z_{CA}{}^{B},$$

such that

$$\Omega_{AB}{}^{C} = 2W_{[AB]}{}^{C}, \qquad Y_{AB}{}^{C} = 2Z_{[AB]}{}^{C}.$$

Uplift of the $D=2$ action in the ν-frame, $S^{(2,\,\nu)} = \int d^2x \mathcal{L}^{(2,\,\nu)}$, is [D.Bielli, MC, '26] :

$$S^{(4,\,\nu)} = S^{(4)}_{\mathsf{EH}}[W(E), Z] + \int d^4x \; \mathcal{E}(\nu_4).$$

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$$S^{(4,\,\nu)} = S^{(4)}_{\mathsf{EH}}[W(E), Z] + \int d^4x\; \mathcal{E}(\nu_4).$$

Here,

$$\begin{aligned} S^{(4)}_{\mathsf{EH}}[W(E), Z] = +\frac{1}{8}\int d^4x E\Big(&+ \Omega^{ABC}\Omega_{ABC} + 2\Omega^{ABC}\Omega_{ACB} - 4\Omega_{AB}{}^{B}\Omega^{AC}{}_{C} \\ &+ 4(Y^{ABC}\Omega_{ABC} + 2Y^{ABC}\Omega_{ACB} - 4Y_{AB}{}^{B}\Omega^{AC}{}_{C}) \\ &+ 2(Y^{ABC}Y_{ABC} + 2Y^{ABC}Y_{ACB} - 4Y_{AB}{}^{B}Y^{AC}{}_{C})\Big), \end{aligned}$$

expressed in terms of W and Z.

Finally,

$$\nu_4 = (2\eta^{AB}\eta^{CD} - \eta^{AC}\eta^{BD})V_{AC}V_{BD}\,,$$

with V_{AB} symmetric in (A, B) ,

$$\begin{aligned} V_{AB} = \frac{E}{8}\Big[& -\Big(2Y_{ACD}Y_B{}^{CD} + 2Y_{ACD}Y_B{}^{DC} - 4Y_{AC}{}^{C}Y_{BD}{}^{D} + Y_{CDA}Y^{CD}{}_{B}\Big) \\ & + \frac{1}{2}\eta_{AB}\Big(Y_{CDE}Y^{CDE} + 2Y_{CDE}Y^{CED} - 4Y_{CD}{}^{D}Y^{CE}{}_{E}\Big) \\ & + 2\Big(Z_{CAD}Z^{C}{}_{B}{}^{D} - Z_{CDA}Z^{CD}{}_{B}\Big)\Big]. \end{aligned}$$

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$$\begin{aligned}V_{AB} = \frac{E}{8}\Bigg[&-\Big(2Y_{ACD}Y_B{}^{CD} + 2Y_{ACD}Y_B{}^{DC} - 4Y_{AC}{}^{C}Y_{BD}{}^{D} + Y_{CDA}Y^{CD}{}_B\Big)\\ &+\frac{1}{2}\eta_{AB}\Big(Y_{CDE}Y^{CDE} + 2Y_{CDE}Y^{CED} - 4Y_{CD}{}^{D}Y^{CE}{}_E\Big)\\ &+2\Big(Z_{CAD}Z^{C}{}_B{}^{D} - Z_{CDA}Z^{CD}{}_B\Big)\Bigg].\end{aligned}$$

Similarly, uplift of the $D=2$ action in the μ-frame, reads [D.Bielli, MC, '26] ,

$$S^{(4,\,\mu)} = \int d^4x \left[\frac{1}{2} \left(\frac{\mu^2+4}{\mu^2-4} \right) \mathcal{L}^{(4)}_{\mathsf{EH}}[E] - \left(\frac{4\mu}{\mu^2-4} \right) \sqrt{\mathcal{K}[E]} + \mathcal{H}(\mu) \right].$$

Uplifts to D=4: final result

The μ-frame

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Here,

$$\mathcal{L}^{(4)}_{\mathsf{EH}}[E] = -\frac{E}{8}\left(\Omega^{ABC}\Omega_{ABC} + 2\Omega^{ABC}\Omega_{ACB} - 4\Omega_{AB}{}^{B}\Omega^{AC}{}_{C}\right) ,$$

$$\mathcal{K}[E] = (2\eta^{AB}\eta^{CD} - \eta^{AC}\eta^{BD})t_{AC}t_{BD} ,$$

with t_{AB} symmetric in (A,B),

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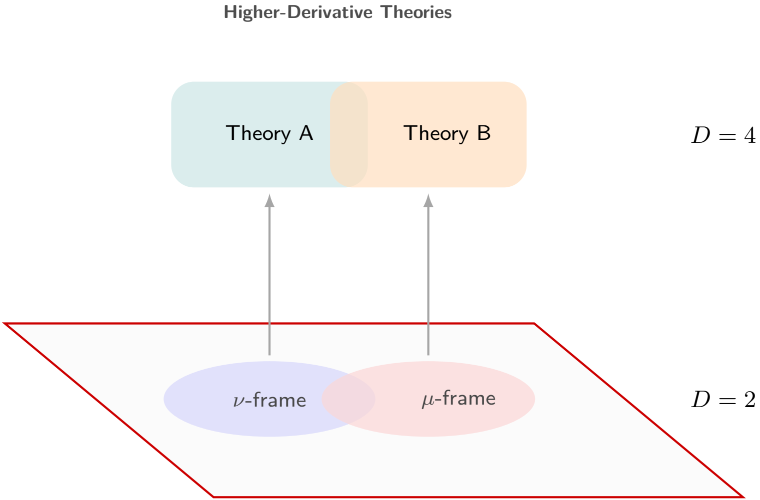
Auxiliary field deformations uplifted
A visual summary



Coda

What we have seen today:

- The integrable auxiliary field deformations of the BM model.

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What we have seen today:

- The integrable auxiliary field deformations of the BM model.
- The interplay between ν- and μ-frames.
- The uplift of the models back to $D = 4$.

Open problems and yet to be answered questions:

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- Dual graviton formulations?

Thank you

Parametrise

$$\mathcal{V}_{MM} = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \rho^{\frac{1}{2}}\Delta^{-\frac{1}{2}} & 0 \\ 0 & \rho^{-\frac{1}{2}}\Delta^{\frac{1}{2}} \end{pmatrix} .$$

The equations of motion allow for the definition of the dual potential B_E

$$\partial_+ \left(\Delta^2 \rho^{-1} \partial_- B\right) + \partial_- \left(\Delta^2 \rho^{-1} \partial_+ B\right) = 0 \quad \Rightarrow \quad \partial_\pm B_E \equiv \pm \Delta^2 \rho^{-1} \partial_\pm B \, ,$$

and therefore of

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$\mathfrak{sl}_{2\ MM}$ acts on $\mathcal{V}_{MM}$ by left multiplication via the matrices:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} .$$

Then one can prove:

$$\begin{aligned} [\delta_f, \delta_{f_E}] J_{\pm\ MM} &= \pm\big[\rho\mathcal{V}_{MM}^{-1}\, h\, \mathcal{V}_{MM}, J_{\pm\ MM} + J^T_{\pm\ MM}\big] \pm 2\partial_\pm\rho\, \mathcal{V}_{MM}^{-1}\, h\, \mathcal{V}_{MM} \,, \\ [\delta_f, [\delta_f, \delta_{f_E}]] J_{\pm\ MM} &= \mp 2\big[\rho\mathcal{V}_{MM}^{-1}\, f\, \mathcal{V}_{MM}, J_{\pm\ MM} + J^T_{\pm\ MM}\big] \mp 4\partial_\pm\rho\, \mathcal{V}_{MM}^{-1}\, f\, \mathcal{V}_{MM} \\ &\ \ \vdots \end{aligned}$$

Monodromy matrix

Define the generalised coset representative

$$\mathcal{V}(x;\gamma(x;z)) = \exp\left(\sum_{n=0}^{\infty} \phi_n T^n(\gamma)\right) ,$$

and its involutary automorphism,

$$\tau^{\infty}\mathcal{V}(\gamma) = (\mathcal{V}^T)^{-1}\left(\frac{1}{\gamma}\right) .$$

Then the monodromy matrix is

$$\mathcal{M} = \mathcal{V}\tau^{\infty}\mathcal{V}^{-1} = \mathcal{V}(x;\gamma)\mathcal{V}^T\left(x;\frac{1}{\gamma}\right) .$$

It holds that

$$\partial_\mu \mathcal{M} = \mathcal{V}\left(\mathcal{V}^{-1}\partial_\mu\mathcal{V} - \tau^{\infty}\left(\mathcal{V}^{-1}\partial_\mu\mathcal{V}\right)\right)\tau^{\infty}\mathcal{V}^{-1} = 0 \quad \rightarrow \quad \mathcal{M} = \mathcal{M}(z) .$$

- In the ν-frame:

$$\mathcal{E}(\mu;\lambda)_{T\bar{T}} = \lambda\nu_2$$

- In the μ-frame:

$$H(\mu;\lambda)_{T\bar{T}} = \frac{1}{4\lambda}\left(1 + \frac{\mu^2+1}{\mu^2-1}\right)$$

Uplifts to D=4

Consider

- Generic $D = 2$ seed theory $\mathring{\mathcal{L}}$.
- Easiest possible deformation in the ν-frame $\mathcal{E}(\nu_2) = \nu_2$.

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At linear order in a perturbative expansion,

$$\nu_{2\ \mathsf{lin}} = (2\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma})T_{\mu\nu}T_{\rho\sigma},$$

where $T_{\mu\nu}$ canonically defined (for a flat background metric) in $D = 2$ as

$$T_{\mu\nu} = \frac{\delta\mathring{\mathcal{L}}}{\delta\eta^{\mu\nu}} - \frac{1}{2}\eta_{\mu\nu}\mathring{\mathcal{L}}.$$

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Naive expectation

Check at linear order in a perturbative expansion, if the $D = 4$ ansatz

$$\nu_{4\,\text{lin}} = (2\eta^{AC}\eta^{BD} - \eta^{AB}\eta^{CD})T_{AB}T_{CD},$$

with

$$T_{AB} = \frac{\delta\mathcal{L}^{(4)}_{\text{EH}}}{\delta\eta^{AB}} - \frac{1}{2}\eta_{AB}\mathcal{L}^{(4)}_{\text{EH}},$$

reduces to $\nu_{2\,\text{lin}}$ in $D = 2$.

Uplifts to D=4

Consider

- Generic $D = 2$ seed theory $\mathring{\mathcal{L}}$.
- Easiest possible deformation in the ν-frame $\mathcal{E}(\nu_2) = \nu_2$.

At linear order in a perturbative expansion,

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$$\nu_{4\ \mathrm{lin}} = (2\eta^{AC}\eta^{BD} - \eta^{AB}\eta^{CD})T_{AB}T_{CD},$$

with

$$T_{AB} = \frac{\delta\mathcal{L}^{(4)}_{\mathsf{EH}}}{\delta\eta^{AB}} - \frac{1}{2}\eta_{AB}\mathcal{L}^{(4)}_{\mathsf{EH}},$$

reduces to $\nu_{2\ \mathrm{lin}}$ in $D = 2$. Incorrect!