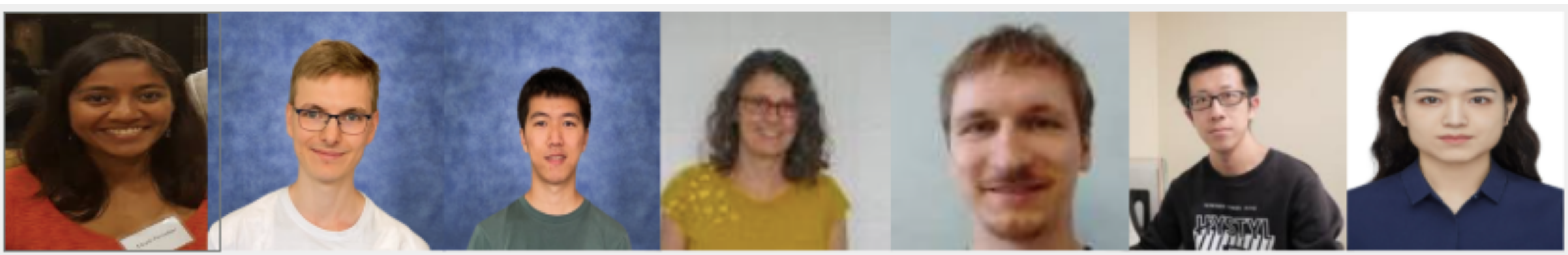
Cluster Bootstrap & Quadrangular Polylogarithms for dS Cosmology

BROWN

Workshop on Quantum Gravity and Strings, Corfu 2026

Based on 2603.08670, 2605.06542



Paranjape, Skowronek, Weng, Ferro, Lukowski, Ren, Zhang

MS, A. Volovich

September 7, 2026

(My) Motivation

In $\mathcal{N}=4$ super Yang-Mills theory the forefront of perturbative calculations is enabled by a bootstrap program [1] that rests on two basic pillars:

(1) the hypothesis that all $n=6$ and 7-particle amplitudes (at fixed loop order) are multiple polylogarithm functions, ie they have symbols and can be studied via symbology [2], and

(2) adjacent symbol letters are compatible [3] cluster variables [4] of the $Gr(4,n)$ cluster algebra (closely connected to extended Steinmann relations [5])

[1] Caron-Huot, Dixon, Drummond, Dulat, Foster, Gurdogan, von Hippel, McLeod, Papathanasiou 2005.06735 (review, more refs below)

[2] Goncharov, MS, Vergu, Volovich 1006.5703

[3] Drummond, Foster, Gurdogan 1710.10953

[4] Golden, Goncharov, MS, Vergu, Volovich 1305.1617

[5] Caron-Huot, Dixon, Dulat, von Hippel, McLeod, Papathanasiou 1906.07116

Lightning Review (highly abbreviated for this talk!)

(1) $$\mathrm{Li}_{m_1,\ldots,m_d}(z_1,\ldots,z_d) = \sum_{0<n_1<\cdots<n_d} \frac{z_1^{n_1}}{n_1^{m_1}} \cdots \frac{z_d^{n_d}}{n_d^{m_d}}$$

multiple polylogarithm function of depth d and weight $\sum_{i=1}^{d} m_i = w$

(2) $\mathcal{S}(\mathrm{Li}_{m_1,\ldots,m_d}(z_1,\ldots,z_d))$ = the symbol of $\mathrm{Li}_{m_1,\ldots,m_d}$

= a linear combination of elements of a formal w-fold tensor product

$$\cdots + a_1 \otimes a_2 \otimes \cdots \otimes a_w + \cdots$$

where the a's — called symbol letters — are simple functions of the z's

(3) cluster variables [Fomin, Zelevinsky] are certain sets of certain variables related to each other in a certain way

$$\{a_1, a_3, a_4\}, \quad \{a_2, a_3, a_5\}, \quad \{a_1, a_4, a_5\}$$

(4) Two variables are compatible if they appear together in a cluster.

a_1 is compatible with a_3, a_4, a_5 but not with a_2

Lightning Review (highly abbreviated for this talk!)

(5) A symbol satisfies cluster adjacency if, in every term

$$\cdots + a_1 \otimes a_2 \otimes \cdots \otimes a_w + \cdots$$

a_i & a_j may appear next to each other only if they are compatible.

(My) Motivation

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Together with other physical input (from Regge & near-collinear limits, etc.) — as well as physically mysterious input (notably antipodal duality [6]) this program has enabled the computation of six- and seven-particle amplitudes through eight and five loops, respectively [7].

[6] Dixon, Gurdogan, McLeod, Wilhelm 2112.06243

[7] Dixon, Liu 2308.08199, Dixon, Li 2606.22380
He, Jiang, Li, Liu 2511.09669

An Aside (Apparantely)

A couple of years ago mathematicians Matveiakin and Rudenko introduced and studied quadrangular polylogarithms, which Rudenko used to prove Goncharov's depth conjecture. [2012.05599, 2208.01564]

The defining feature of these functions is that they satisfy what we call total compatibility : in every term in the symbol

$$\cdots + a_1 \otimes a_2 \otimes \cdots \otimes a_w + \cdots$$

all of the a's that appear are mutually compatible with each other.

In 2306.04630 Ren, MS, Vergu, Volovich found an application of (a modified version called alternating polylogarithms by Rudenko), but they seemed to have no role to play in $\mathcal{N}=4$ SYM — amplitudes there do not satisfy total compatibility.

$\square$ square peg $\bigcirc$ round hole

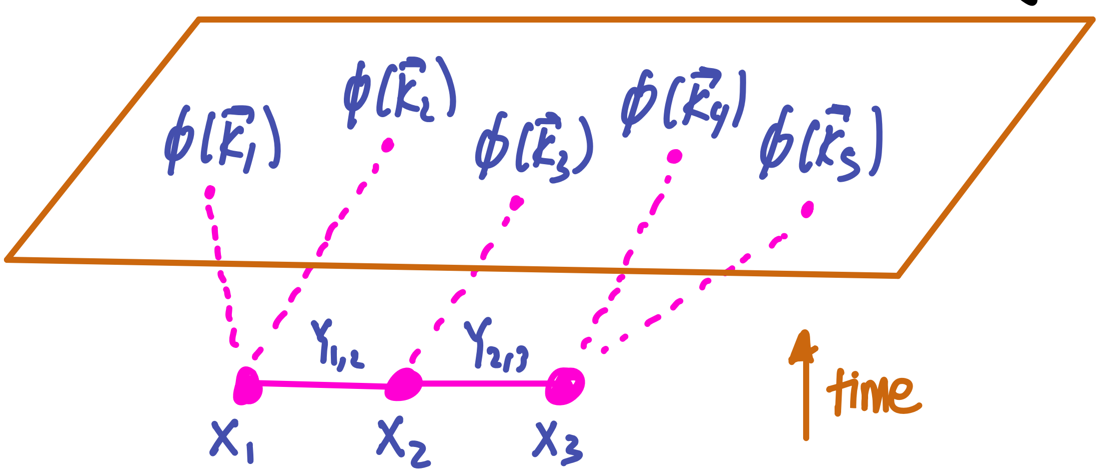
We left it on the shelf as a tantalizing curiosity.

Cosmological Wavefunction Coefficients in de Sitter

The CWCs of a conformally coupled scalar field in de Sitter space are multiple polylogarithm functions whose symbols can be computed recursively by simple differential equations.

1709.02813 Arkani-Hamed, Benincasa, Postnikov
1912.09450 Hillman
[8] 2312.05303 Arkani-Hamed, Baumann, Hillman, Joyce, Lee, Pimentel
2407.17715 He, Jiang, Liu, Yang, Zhang
many others...

We consider the contribution from a single graph, eg for ϕ^3 theory



vertex variables

$X_1 = |\vec{k}_1| + |\vec{k}_2| \quad X_2 = |\vec{k}_3| \quad X_3 = |\vec{k}_4| + |\vec{k}_5|$

edge variables

$Y_{1,2} = |\vec{k}_1 + \vec{k}_2| \quad Y_{2,3} = |\vec{k}_4 + \vec{k}_5|$

The symbol of this quantity has 104 terms and symbol alphabet

$\{X_1 \pm Y_{1,2},\ X_2 \pm Y_{1,2} \pm Y_{2,3},\ X_3 \pm Y_{2,3},\ X_1+X_2 \pm Y_{2,3},\ X_2+X_3 \pm Y_{1,2},\ X_1+X_2+X_3\}$

and Hillman 1912.09450 found the explicit formula

$$
\begin{aligned}
\Psi_3 = \sum_{\{\sigma_1,\sigma_2\}\in\{-,+\}^2} \sigma_1\sigma_2 \Bigg[& \mathrm{Li}_3\left(-\frac{u_{\sigma_1}v_{\sigma_2}}{1-u_{\sigma_1}-v_{\sigma_2}}\right) + \mathrm{Li}_3\left(\frac{u_{\sigma_1}}{1-v_{\sigma_2}}\right) + \mathrm{Li}_3\left(\frac{v_{\sigma_2}}{1-u_{\sigma_1}}\right) \\
& + \mathrm{Li}_3\left(\frac{1-u_{\sigma_1}-v_{\sigma_2}}{1-v_{\sigma_2}}\right) + \mathrm{Li}_3\left(\frac{1-u_{\sigma_1}-v_{\sigma_2}}{1-u_{\sigma_1}}\right) \\
& -\frac{1}{2}\mathrm{Li}_1\left(\frac{1-u_{\sigma_1}-v_{\sigma_2}}{1-v_{\sigma_2}}\right)\mathrm{Li}_1^2\left(\frac{u_{\sigma_1}}{1-v_{\sigma_2}}\right) - \frac{1}{2}\mathrm{Li}_1\left(\frac{1-u_{\sigma_1}-v_{\sigma_2}}{1-u_{\sigma_1}}\right)\mathrm{Li}_1^2\left(\frac{v_{\sigma_2}}{1-u_{\sigma_1}}\right) \\
& +\frac{1}{6}\mathrm{Li}_1^3\left(\frac{v_{\sigma_2}}{1-u_{\sigma_1}}\right) + \frac{1}{6}\mathrm{Li}_1^3\left(\frac{u_{\sigma_1}}{1-v_{\sigma_2}}\right) - \frac{1}{6}\mathrm{Li}_1^3\left(u_{\sigma_1}+v_{\sigma_2}\right)\Bigg] \\
& +\mathrm{Li}_1(1-u_+)\left[\mathrm{Li}_2\left(\frac{1-v_+-u_+}{1-v_--u_+}\right) - \mathrm{Li}_2\left(\frac{v_-(1-v_+-u_+)}{v_+(1-v_--u_+)}\right)\right. \\
& \left. -\mathrm{Li}_2\left(\frac{1-v_+-u_-}{1-v_--u_-}\right) + \mathrm{Li}_2\left(\frac{v_-(1-v_+-u_-)}{v_+(1-v_--u_-)}\right)\right] \\
& +\mathrm{Li}_1(1-v_+)\left[\mathrm{Li}_2\left(\frac{1-v_+-u_+}{1-v_+-u_-}\right) - \mathrm{Li}_2\left(\frac{u_-(1-v_+-u_+)}{u_+(1-v_+-u_-)}\right)\right. \\
& \left. -\mathrm{Li}_2\left(\frac{1-v_--u_+}{1-v_--u_-}\right) + \mathrm{Li}_2\left(\frac{u_-(1-v_--u_+)}{u_+(1-v_--u_-)}\right)\right] \\
& +\mathrm{Li}_1(1-u_+)\,\mathrm{Li}_1(1-v_+)\,\mathrm{Li}_1\left(\frac{(v_--v_+)(u_--u_+)}{(1-u_+-v_-)(1-u_--v_+)}\right)
\end{aligned}
$$

Where we have used the variables

$$u_\pm = \frac{X_1 \pm Y_1}{X_1+X_2+X_3} \qquad v_\pm = \frac{X_3 \pm Y_2}{X_1+X_2+X_3}$$

This remained the state of the art, as far as explicit formulas, until 2026.

Cosmology Meets Cluster Algebra

In 2512.14854 Mazloumi, Xu and
2512.14859 Capuano, Ferro, Lukowski, Palazio observed that the symbol letters of the n-chain graph function ψ_n could be realized as cluster variables of the A_{2n-2} cluster algebra!

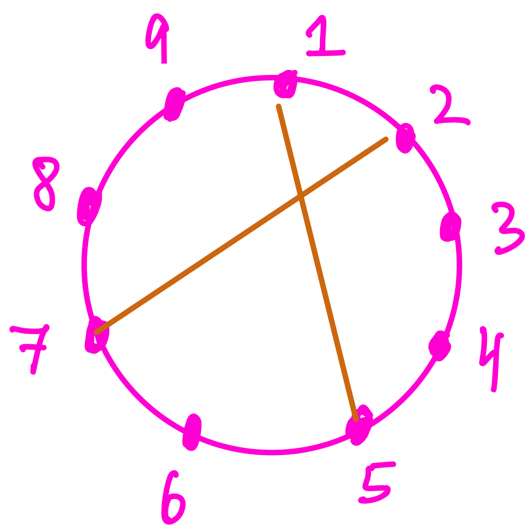
In a nutshell consider the $2 \times (2n+1)$ matrix

$$C_n = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 0 & X_1 + Y_{1,2} & X_1 - Y_{1,2} & X_1 + X_2 + Y_{1,2} & X_1 + X_2 - Y_{1,2} & \cdots & X_1 + \cdots + X_n & 1 \end{pmatrix}$$

all symbol letters are 2×2 minors of this matrix, ie

$$\Delta_{ij} = \det(C_i \;\; C_j)$$

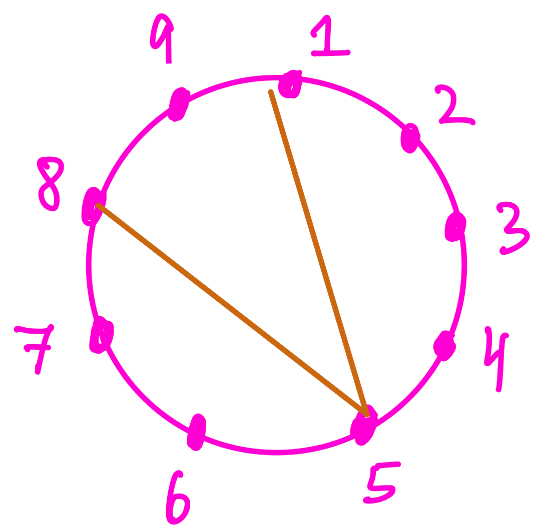
Two cluster variables Δ_{ij}, $\Delta_{k\ell}$ are compatible in the A_{2n-2} cluster algebra if and only if the chords (ij) & $(k\ell)$ don't cross.



Δ_{15} and Δ_{27}

are <u>not</u> compatible

in A_6



Δ_{15} and Δ_{58}

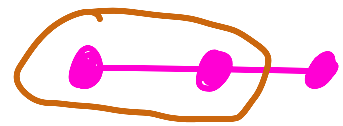
are compatible

in A_6

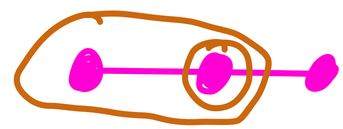
Cluster Bootstrap for Cosmological Correlators

In 2603.08670 Paranjape, Skowronek, MS, Volovich, Weng pointed out that the tubing rules [8] for writing down the recursive differential equation for the n-chain graph CWC Ψ_n make it manifest that it satisfies cluster adjacency!

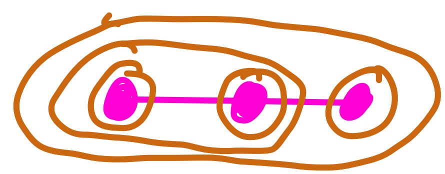
cluster variables $\longleftrightarrow$ tubes



compatible cluster variables $\longleftrightarrow$ non-overlapping tubes



clusters $\longleftrightarrow$ maximal sets of non-overlapping tubes



We could then play the bootstrap game, using input like

first-entry condition: "±" only "+"   soft limits: Ψ_n vanishes as any $Y \to 0$

$\boxed{\text{However, we missed the forest for the trees!}}$

Generalized Cluster Adjacency for Cosmology

In 2603.09965 Capuano, Ferro, Lukowski, Palazio, Zhang pointed out that the same argument makes it clear that Ψ_n satisfies total compatibility!

So now it is finally clear what Rudenko's quadrangular polylogs are "good for" in physics!

Quadrangular Polylogarithm (QLi) Functions

There is beautiful mathematical structure underlying the quadrangular polylogarithm functions that I don't have time to explain. They:

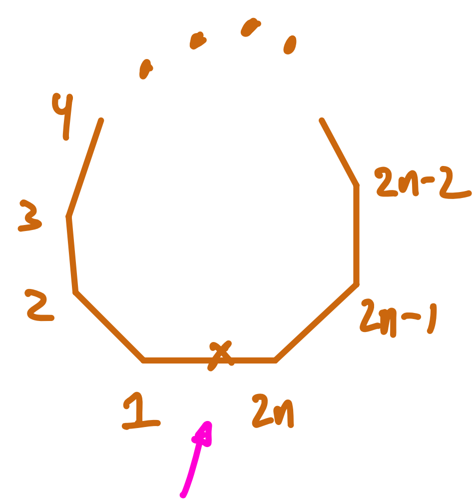
(1) come in two varieties that we call even (+) and odd (-)

(2) are defined on any index set $\underbrace{\{a,b,c,d,\ldots\}}_{2n}$ of increasing integers of even length ≥ 4

(3) are defined for any weight $k \geq n$, but fortunately we only needed the cases $k=n$, $k=n+1$.

In all that follows, keep a picture of an even rooted $2n$-gon in your mind, with

$0 < a < b < c < d < \cdots < 2n+1$



marked edge $2n$ to 1

QLi Functions – Some Illustrative Examples

(subscript on QLi is for us always the weight)

$$\mathrm{QLi}_k^+(a,b,c,d) = (-1)^k\,\mathrm{Li}_k(q_{abcd})\,,$$
$$\mathrm{QLi}_k^-(a,b,c,d) = (-1)^{k+1}\,\mathrm{Li}_k(1/q_{abcd})\,.$$

weight $k \geq 1$

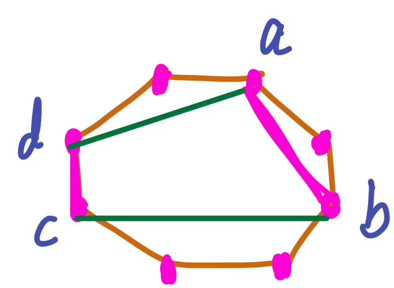
$$\begin{aligned}
\mathrm{QLi}_2^+(a,b,c,d,e,f) = {} & \mathrm{Li}_{1,1}(q_{abcf}, q_{cdef}) - \mathrm{Li}_{1,1}(q_{abef}, 1/q_{bcde}) \\
& + \mathrm{Li}_{1,1}(q_{adef}, q_{abcd})\,, \\
\mathrm{QLi}_2^-(a,b,c,d,e,f) = {} & \mathrm{Li}_2(1/(q_{abcd}\,q_{adef})) + \mathrm{Li}_{1,1}(1/q_{abcf}, 1/q_{cdef}) \\
& - \mathrm{Li}_{1,1}(1/q_{abef}, q_{bcde}) + \mathrm{Li}_{1,1}(1/q_{adef}, 1/q_{abcd})
\end{aligned}$$

weight 2

$$\begin{aligned}
\mathrm{QLi}_3^+(a,b,c,d,e,f) = {} & -\mathrm{Li}_{2,1}(q_{abcf}, q_{cdef}) - \mathrm{Li}_{1,2}(q_{abcf}, q_{cdef}) \\
& + \mathrm{Li}_{2,1}(q_{abef}, 1/q_{bcde}) + \mathrm{Li}_{1,2}(q_{abef}, 1/q_{bcde}) \\
& - \mathrm{Li}_{2,1}(q_{adef}, q_{abcd}) - \mathrm{Li}_{1,2}(q_{adef}, q_{abcd})\,, \\
\mathrm{QLi}_3^-(a,b,c,d,e,f) = {} & -2\,\mathrm{Li}_3(1/(q_{abcd}\,q_{adef})) \\
& - \mathrm{Li}_{2,1}(1/q_{abcf}, 1/q_{cdef}) - \mathrm{Li}_{1,2}(1/q_{abcf}, 1/q_{cdef}) \\
& + \mathrm{Li}_{2,1}(1/q_{abef}, q_{bcde}) + \mathrm{Li}_{1,2}(1/q_{abef}, q_{bcde}) \\
& - \mathrm{Li}_{2,1}(1/q_{adef}, 1/q_{abcd}) - \mathrm{Li}_{1,2}(1/q_{adef}, 1/q_{abcd})\,.
\end{aligned}$$

weight 3

q_{abcd} = "quadrangular cross-ratio" = $\dfrac{\Delta_{ab}\,\Delta_{cd}}{\Delta_{bc}\,\Delta_{da}}$



Chain Graph CWC from Quadrangular Polylogarithms: Step 1

We (Ferro, Lukowski, Ren, MS, Volovich, Weng, Zhang; 2605.06542) first worked at the level of the Lie coalgebra — this means modulo products of functions of lower weight.

By checking a few cases at small n we formulated the guess that

$$\boxed{\psi_n = F_n(2n{+}1, 1, \ldots, 2n{-}1) - F_n(2n, 1, \ldots, 2n{-}1) + F_n(2n, 2n{+}1, 2, \ldots, 2n{-}1)\,.}$$

where

$$\boxed{F_n(0,1,\ldots,2n-1) = (-1)^n \, \mathrm{QLi}_n(0,1,\ldots,2n-1)}$$

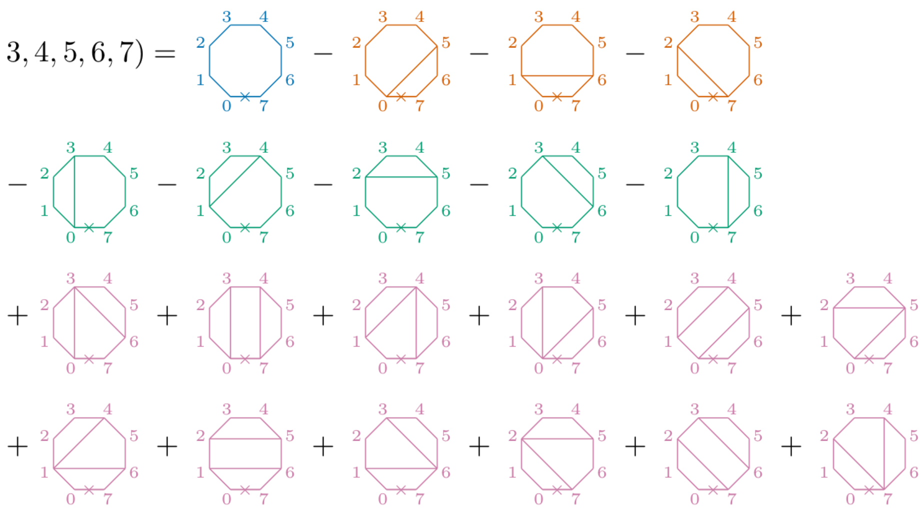
+ products of QLi functions associated to smaller polygons

Total compatibility places strong constraints on the missing terms — you can only multiply QLi functions of non-overlapping polygons!

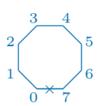
Chain Graph CWC from Quadrangular Polylogarithms: Step 2

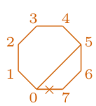
After some playing around, my collaborators found the structure of all product terms. For example for $n=4$

$F_4(0,1,2,3,4,5,6,7) =$

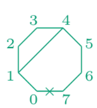


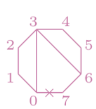
where the four different types of terms are represented by

 $= \mathrm{QLi}_4^+(0,1,2,3,4,5,6,7) + \mathrm{QLi}_3^+(0,1,2,3,4,5,6,7)\log(q_{01234567})\,,$

 $= \big(\mathrm{QLi}_2^+(0,5,6,7) + \mathrm{QLi}_1^+(0,5,6,7)\log(q_{0567})\big)\,\mathrm{QLi}_2^+(0,1,2,3,4,5)\,,$

(5.11)

 $= \big(\mathrm{QLi}_3^+(0,1,4,5,6,7) + \mathrm{QLi}_2^+(0,1,4,5,6,7)\log(q_{014567})\big)\,\mathrm{QLi}_1^-(1,2,3,4)\,,$

 $= \big(\mathrm{QLi}_2^+(0,3,6,7) + \mathrm{QLi}_1^+(0,3,6,7)\log(q_{0367})\big)\,\mathrm{QLi}_1^+(0,1,2,3)\,\mathrm{QLi}_1^-(3,4,5,6)\,.$

Chain Graph CWC from Quadrangular Polylogarithms: Step 3

Our guess for the general n function was then

$$\psi_n = F_n(2n{+}1, 1, \ldots, 2n{-}1) - F_n(2n, 1, \ldots, 2n{-}1) + F_n(2n, 2n{+}1, 2, \ldots, 2n{-}1)\,.$$

where

$$F_n(0, \ldots, 2n{-}1) = \sum_{k=0}^{n-2} \sum_{\substack{S_0 \sqcup S_1 \sqcup \cdots \sqcup S_k \\ = \{0, \ldots, 2n-1\}}} (-1)^{k+n} \left(\mathrm{QLi}^{+}_{\frac{|S_0|}{2}}(S_0) + \mathrm{QLi}^{+}_{\frac{|S_0|}{2}-1}(S_0) \log(q_{S_0}) \right) \prod_{i=1}^{k} \mathrm{QLi}^{(-)^{S_i(1)}}_{\frac{|S_i|}{2}-1}(S_i)\,,$$

FLRSVWZ

and the sum runs over all dissections of the initial polygon into even sub-polygons.

We proved that this formula satisfies the differential equation of He, Jiang, Liu, Yang, Zhang 2407.17715 which in our streamlined notation is simply

just some logarithm

$$\boxed{d\psi_n = \sum_{i=1}^{2n-1} \psi_{n-1}(\hat{i}, \widehat{i{+}1})\; d\,\mathrm{QLi}_1^{(-1)^{i-1}}(i{-}1, i, i{+}1, i{+}2)}$$

Chain Graph CWC from Quadrangular Polylogarithms: Step 4

This proved our guess — at function level (not just symbol level) — up to a possible additive constant, but it is simple to check that the expression given above vanishes in any soft limit $Y_{i,i+1} \to 0$ so no such constant is missing.

$\Rightarrow$ We have an explicit analytic formula for the n-site chain graph cosmological wavefunction coefficient.

Outlook (Capuano, Ferro, Lukowski, Palazio, Paranjape, Ren, MS, Volovich, Weng, Zhang)

(1) It is interesting to explore the expansion away from dS space to general power-law FRW cosmologies.

(2) "Correlators are simpler than wavefunctions" — 2512.23795 Arkani-Hamed, Glew, Vazao — is $\mathbb{Q}L_i$ useful there?

(3) In 2603.08670 we also showed that the symbol of the n-gon graph CWC satisfies cluster adjacency (indeed, total compatibility) with respect to the B_{2n-1} cluster algebra.

(4) Indeed it is manifest from the tubing picture that the symbol of any graph satisfies total compatibility with respect to the variables associated to its tubes — it's just that for general graphs, the collections of such variables do not correspond to cluster variables of some cluster algebra.

Conclusion

The last comment indicates that CWCs suggest some natural way of generalizing cluster algebras and totally compatible polylogarithm functions associated to them.

Just like the amplitudes of $\mathcal{N}=4$ SYM suggest structures that generalize cluster variables.