Higher-Order Newton-Cartan Gravity

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Higher-Order Gravity Theories

Goal

Obtain the Non-Relativistic limit of the family of higher-order theories below at the level of action and equations of motion.

Higher-Order Theories:the Action

We consider the following set of theories in D dimensions, parametrized by the couplings α, β and γ

$$\mathcal{S} = \int d^D x \sqrt{-g}\left(\mathrm{R} + \alpha\,\mathrm{R}^2 + \beta\,\mathrm{R}_{\mu\nu}\mathrm{R}^{\mu\nu} + \gamma\,\mathrm{R}_{\mu\nu\rho\sigma}\mathrm{R}^{\mu\nu\rho\sigma}\right),$$

We study them non-perturbatively (α, β and γ are not necessarily small).

Some Relevant Features

- The equations of motion contain terms with more than 2 derivatives.
- The pure quadratic theory defined by $\beta = -4\alpha\,, \gamma = \alpha$ is called Gauss-Bonnet gravity.
  - It has second-order equations of motion.
  - It is a topological term in 4D.

Taking the Non-Relativistic Limit

① Lorentz Breaking

The flat index $\hat{A}$ is decomposed in a time and space part $\hat{A} = \{0, a\}$

$$\hat{A} = \begin{cases} 0 & \textbf{Time} \\ a = 1, ..., D-1 & \textbf{Space} \end{cases}$$

particle limit.

② Redefinition Ansatz

The relativistic fields (and parameters) are redefined (in an invertible way) in terms of would be non-relativistic fields (and parameters) with a dimensionless parameter c. We consider the vielbein $E_\mu{}^{\hat{A}}$ $(g_{\mu\nu} = E_\mu{}^{\hat{A}} E_\nu{}^{\hat{B}} \eta_{\hat{A}\hat{B}})$,

$$E_\mu{}^0 = c\tau_\mu\,, \qquad E_\mu{}^a = e_\mu{}^a\,, \,.$$

and the Lorentz symmetry parameter

$$\underbrace{\Lambda_{0a} = \frac{1}{c}\lambda_a\,,}_{\text{Boost}} \qquad \underbrace{\Lambda_{ab} = \lambda_{ab}\,.}_{\text{Spatial Rotations}}$$

③ Substitution

The ansatz is substituted in the expressions we want to take the limit of. For example the Lorentz transformation rules

$$\delta E_\mu{}^0 = \Lambda^0{}_b E_\mu{}^b\,,$$
$$\delta E_\mu{}^a = \Lambda^a{}_b E_\mu{}^b + \Lambda^a{}_0 E_\mu{}^0\,,$$

become

$$\delta\tau_\mu = -\frac{1}{c^2}\lambda_b e_\mu{}^b\,,$$
$$\delta e_\mu{}^a = \lambda^a{}_b e_\mu{}^b - \lambda^a \tau_\mu\,.$$

④ The Limit

The parameter c is sent to ∞. The limit is consistent if it does not develop **divergent** terms. In the example

$$\delta\tau_\mu = 0\,,$$
$$\delta e_\mu{}^a = \lambda^a{}_b e_\mu{}^b - \lambda^a \tau_\mu\,.$$

Limit of the Action

Einstein-Hilbert Action

$$\mathcal{S}_{EH} = \int d^D x \sqrt{-g}\,\mathrm{R}$$

Index	Definition & Values
$\mu, \nu, \rho, \ldots$	D Curved ,
$\hat{A}, \hat{B}, \hat{C}, \ldots$	D Flat $(\hat{A} = \{0, a\})$,
$a, b, c \ldots$	Spatial Flat $a = 1, \ldots, D-1$.

Ansatz

$$E_\mu{}^0 = c\tau_\mu\,,$$
$$E^\mu{}_0 = \frac{1}{c}\tau^\mu$$
$$E_\mu{}^a = e_\mu{}^a\,,$$
$$E^\mu{}_a = e^\mu{}_a\,.$$

Orthogonality Relations

$$\tau^\mu \tau_\mu = 1\,,$$
$$e^\mu{}_a e_\mu{}^b = \delta^b_a\,,$$
$$e^\mu{}_a \tau_\mu = e_\mu{}^a \tau^\mu = 0\,,$$
$$\tau_\mu \tau^\nu + e_\mu{}^a e^\nu{}_a = \delta^\nu_\mu\,.$$

Ricci Scalar Expansion

$$\mathrm{R} = \frac{c^2}{4} t_{ab} t^{ab} + \mathcal{O}(c^0)\,,$$

with $t_{ab} = e^\mu{}_a e^\nu{}_b t_{\mu\nu}$ and $t_{\mu\nu} = 2\partial_{[\mu}\tau_{\nu]}$ **(intrinsic torsion).**

Divergence

The leading order term is divergent, we need a way to cancel it and access the finite order.

Cancellation Mechanism

Einstein-Hilbert+Maxwell

$$\mathcal{S} = \int d^{10}x\,\sqrt{-g}\left(\mathrm{R} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)$$

with $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$.

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Expansion

$$\mathrm{R} = \frac{c^2}{4}t_{ab}t^{ab} + \mathcal{O}(c^0)\,,$$

$$F_{\mu\nu}F^{\mu\nu} = c^2 t_{ab}t^{ab} + \mathcal{O}(c^0)\,,$$

with $t_{ab} = e^\mu{}_a e^\nu{}_b t_{\mu\nu}$ and $t_{\mu\nu} = 2\partial_{[\mu}\tau_{\nu]}$ **(intrinsic torsion).**

Ansatz

$$E_\mu{}^0 = c\tau_\mu\,,$$

$$E^\mu{}_0 = \frac{1}{c}\tau^\mu$$

$$E_\mu{}^a = e_\mu{}^a\,,$$

$$E^\mu{}_a = e^\mu{}_a\,.$$

$$A_\mu = c\tau_\mu + \frac{1}{c}a_\mu\,.$$

Orthogonality Relations

$$\tau^\mu\tau_\mu = 1\,,$$

$$e^\mu{}_a e_\mu{}^b = \delta^b_a\,,$$

$$e^\mu{}_a\tau_\mu = e_\mu{}^a\tau^\mu = 0\,,$$

$$\tau_\mu\tau^\nu + e_\mu{}^a e^\nu{}_a = \delta^\nu_\mu\,.$$

Non-relativistic Limit

The leading order term coming from the the Einstein-Hilbert lagrangian cancels against an analogous term coming from the Maxwell lagrangian. The result of the limit is:

$$\mathcal{S} = \int d^Dx\, e\left(e^{\mu a}e^\nu_a\tilde{\mathrm{R}}_{\mu\nu} + 2\tau^\mu e^{\nu a}e^\rho_a\tilde{\nabla}_\nu t_{\mu\rho} + \right.$$
$$\left. - \frac{3}{2}t_{0a}t_0{}^a - t^{ab}f_{ab}\right)\,,$$

[Bergshoeff et al. 2023; Hansen et al. 2020; Hartong et al. 2023]

The Strategy

Mimicking the 2-derivative case we can use a gauge field to cancel the divergences of the following actions

$$\mathcal{S} = \int d^D x \sqrt{-g} \left(\mathrm{R} + \alpha\, \mathrm{R}^2 + \beta\, \mathrm{R}_{\mu\nu}\mathrm{R}^{\mu\nu} + \gamma\, \mathrm{R}_{\mu\nu\rho\sigma}\mathrm{R}^{\mu\nu\rho\sigma} \right),$$

A Taste of the Divergences

$$\begin{aligned}\mathrm{R}_{\mu\nu}\mathrm{R}^{\mu\nu} = & \frac{1}{16} c^4 \left(t_{ab}t^{ab}t_{cd}t^{cd} + 4 t_a{}^b t_b{}^c t_c{}^d t_d{}^a \right) + , \\ & + \frac{1}{4} c^2 \Big[4 t^{ab} t_a{}^c (\tilde{\mathrm{R}}_{bc} + \tilde{\nabla}(t)_{b0c} - 3 t_{0b} t_{0c}) + t^{ab} t_{ab} (-2 \tilde{\nabla}(t)_{c0}{}^c + \\ & + 2 t_{0c} t_0{}^c + t^{cd} f_{cd}) + 2 (\tilde{\nabla}(t)_a{}^{ab} - 4 t^{ab} t_{0a}) \tilde{\nabla}(t)_{cb}{}^b \Big] + \mathcal{O}(c^0),\end{aligned}$$

The Strategy

Mimicking the 2-derivative case we can use a gauge field to cancel the divergences of the following actions

$$\mathcal{S} = \int d^D x \sqrt{-g} \left(\mathrm{R} + \alpha\, \mathrm{R}^2 + \beta\, \mathrm{R}_{\mu\nu} \mathrm{R}^{\mu\nu} + \gamma\, \mathrm{R}_{\mu\nu\rho\sigma} \mathrm{R}^{\mu\nu\rho\sigma} \right),$$

The Issues

There are a few critical points:

- Adding just the Maxwell terms is not enough, we need higher-order terms ,
- Coupling between the gravity sector and Maxwell field could be necessary ,
- The higher-order terms will produce extra divergences (at orders c^4 and c^2). These should all be cancelled otherwise the contribution of the EH term is removed from the action.

Limit of the Action: Setting Up

The Tools

We add the following terms to the action, each with an arbitrary coupling and we fix the couplings requiring full cancellation of the divergences,

$$\mathcal{B} = \Big\{ (F_{\mu\nu}F^{\mu\nu})^2 \,, F^{\mu}{}_{\nu}F^{\nu}{}_{\rho}F^{\rho}{}_{\sigma}F^{\sigma}{}_{\mu} \,, F^{\mu\nu}F^{\rho\sigma}\mathrm{R}_{\mu\nu\rho\sigma} \,, F^{\mu\rho}F^{\nu}{}_{\rho}\mathrm{R}_{\mu\nu} \,, \\ F_{\mu\nu}F^{\mu\nu}\mathrm{R} \,, \nabla_{\mu}F_{\nu\rho}\nabla^{\mu}F^{\nu\rho} \,, \nabla_{\mu}F^{\mu\rho}\nabla_{\nu}F^{\nu}{}_{\rho} \Big\} \,.$$

Voluntarily overparametrized (avoid integration by parts).

Remarks

- In the Einstein-Hilbert-Maxwell case the coefficient of the Maxwell term can be modified by redefinition.
- In the higher-order case not all the coeffcients in $\mathcal{B}$ can be arbitrarily fine-tuned via non-perturbative redefinitions.
- Cancellation of divergences would be highly non-trivial!

Limit of the Action: Selected Theories

The Result

- It is possible to cancel all the divergences coming from the higher order terms.
- For each of the quadratic terms there is only one! combination in $\mathcal{B}$ cancelling the divergent part (only certains theories admit the non-relativistic limit).
- The coeffcients do not depend on the dimension.

Relativistic Theories Admitting a Non-Relativistic Limit

For each of the quadratic terms in the action the following combinations guarantee a finite limit of the action

$$\mathcal{L}_\alpha = \left(\mathrm{R} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)^2 ,$$

$$\mathcal{L}_\beta = \mathrm{R}_{\mu\nu}\mathrm{R}^{\mu\nu} + \frac{1}{16}F^4 + \frac{1}{4}F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\sigma F^\sigma{}_\mu - \frac{1}{2}\nabla_\mu F^{\mu\nu}\nabla_\rho F_\nu{}^\rho - F^{\mu\rho}F^\nu{}_\rho \mathrm{R}_{\mu\nu} ,$$

$$\mathcal{L}_\gamma = \mathrm{R}_{\mu\nu\rho\sigma}\mathrm{R}^{\mu\nu\rho\sigma} + \frac{3}{8}F^4 + \frac{5}{8}F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\sigma F^\sigma{}_\mu + \nabla_\mu F_{\nu\rho}\nabla^\mu F^{\nu\rho} - \frac{3}{2}F^{\mu\nu}F^{\rho\sigma}\mathrm{R}_{\mu\nu\rho\sigma} .$$

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Why only those theories?

- The limit knows about some physical features of the possible higher-order theories?
- Can the existence of the non-relativistic limit be used to gain informations on the higher-order gravity theories?
- Can we characterize more precisely the theories admitting the limit?

Pure Gravitational Theory in D+1 Dimensions

The Lagrangian admitting a finite non-relativistic limit $\mathcal{L}_\alpha, \mathcal{L}_\beta$ and $\mathcal{L}_\gamma$ can be obtained considering their pure gravitational part in one dimension higher and performing a compactification followed by a truncation of the scalar field.

Higher-Dimensional Origin of the Theories with Finite Limit

$$\mathbf{(D+1)\text{-dim.}} \quad \longrightarrow \quad \mathbf{D\text{-dim.}}$$

$$\mathrm{R}^2 \quad \longrightarrow \quad \left(\mathrm{R} - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}\right)^2$$

$$\mathrm{R}_{MN}\mathrm{R}^{MN} \quad \longrightarrow \quad \mathrm{R}_{\mu\nu}\mathrm{R}^{\mu\nu} + \tfrac{1}{16} F^4 + \tfrac{1}{4} F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\sigma F^\sigma{}_\mu - \tfrac{1}{2} \nabla_\mu F^{\mu\nu} \nabla_\rho F_\nu{}^\rho + \\ - F^{\mu\rho} F^\nu{}_\rho \mathrm{R}_{\mu\nu}$$

$$\mathrm{R}_{MNPQ}\mathrm{R}^{MNPQ} \quad \longrightarrow \quad \mathrm{R}_{\mu\nu\rho\sigma}\mathrm{R}^{\mu\nu\rho\sigma} + \tfrac{3}{8} F^4 + \tfrac{5}{8} F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\sigma F^\sigma{}_\mu + \nabla_\mu F_{\nu\rho} \nabla^\mu F^{\nu\rho} + \\ - \tfrac{3}{2} F^{\mu\nu} F^{\rho\sigma} \mathrm{R}_{\mu\nu\rho\sigma}$$

Taking the Limit of the Equations of Motion

The limit $c \to \infty$ just select the leading order of the expansion of the equations of motion,

$$[X] = c^n \overset{(n)}{[X]} + c^{n-2} \overset{(n-2)}{[X]} + \mathcal{O}(c^{n-4}) = 0\,,$$

giving

$$\overset{(n)}{[X]} = 0\,.$$

Equations of Motion: Einstein-Hilbert-Maxwell Case

The equations of motion coming from the action we have considered are:

$$[G]^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} F^{\mu}{}_{\rho} F^{\nu\rho} = 0\,, \qquad [A]^{\mu} = \nabla_{\nu} F^{\nu\mu} = 0\,,$$

and with flat indices:

$$[G]_{\hat{A}\hat{B}} = [G]^{\mu\nu} E_{\mu\hat{A}} E_{\nu\hat{B}} = 0\,, \qquad [A]^{\hat{A}} = [A]^{\mu} E_{\mu}{}^{\hat{A}} = 0\,.$$

The Poisson Equation in Newtonian Gravity

The Poisson equation is characteristic of Newtonian gravity (and Newton-Cartan gravity), $\Delta\Psi = 4\pi G\rho$, with ρ mass density distribution and Ψ Newton potential. We could expect an analogous equation from our limit.

Where is the Poisson Equation?

We can consider a general scalar combination

$$[P_+] := \mathtt{A}\, E^{\mu}{}_{a} E^{\nu a} [G]_{\mu\nu} + \mathtt{C} E^{\mu}{}_{0} E^{\nu}{}_{0}\, [G]_{\mu\nu} + \mathtt{B}\, E_{\mu}{}^{0} [A]^{\mu}$$

where $\mathtt{A}, \mathtt{B}$ and $\mathtt{C}$ are three constant parameters. The expansion of $[P_+]$

$$[P_+] = c^2 [\overset{(2)}{P}_+] + [\overset{(0)}{P}_+] + c^{-2} [\overset{(-2)}{P}_+] + \mathcal{O}(c^{-4})\,,$$

has a term that potentially could be interpreted as a Poisson equation at order c^{-2}. [Andringa et al. 2012; Bergshoeff et al. 2021].

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The Issue!

The order c^{-2} is sub-sub-leading: it cannot be obtained from the limit of the corresponding equations unless...

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The Solution

...the leading (c^2) and sub-leading (c^0) orders cancels by choosing $\mathtt{A}$, $\mathtt{B}$ and $\mathtt{C}$ properly .

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Poisson Expansion

$$[\overset{(2)}{P_+}] = \frac{1}{4}(\mathtt{C} - 2\mathtt{B}) t_{ab} t^{ab}\,,$$

$$[\overset{(0)}{P_+}] = \frac{1}{2}(3\mathtt{A} - D\mathtt{A} + \mathtt{C}) \tilde{\mathrm{R}}_{a}{}^{a} + (2\mathtt{A} - D\mathtt{A} + \mathtt{B}) \tilde{\nabla}(t)_{a0}{}^{a} + \\ + \frac{1}{4}(-5\mathtt{A} - \mathtt{C} + 3D\mathtt{A}) t_{0a} t_{0}{}^{a} + \frac{1}{2}(-3\mathtt{A} - 2\mathtt{B} + D\mathtt{A}) t^{ab} f_{ab}\,,$$

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...the leading (c^2) and sub-leading (c^0) **cannot** be canceled by choosing the proper combination...

The Solution

We can add, by hand, another equation (a constraint)

$$F_{\mu\nu} = 0 \,.$$

[Bergshoeff et al. 2015; Bergshoeff et al. 2025]

Einstein-Hilbert-Maxwell and the Poisson Equation

Proper Combinations

$$[G]_{\{ab\}}\,, \qquad [G]_{0a}\,,$$
$$[P_\pm] := \mathtt{A}\, E^{\mu}{}_{a} E^{\nu a} [G]_{\mu\nu} \pm \mathtt{A}(D-3) E^{\mu}{}_{0} E^{\nu}{}_{0}\, [G]_{\mu\nu}\,.$$

+

Constraint

$$F_{\mu\nu} = 0$$

The Effect of the Constraint

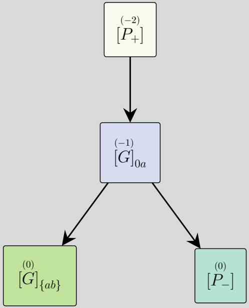
$$F_{\mu\nu} = c\, t_{\mu\nu} - \frac{1}{c} f_{\mu\nu} = 0 \longrightarrow t_{\mu\nu} = \frac{1}{c^2} f_{\mu\nu}\,.$$

The constraints allows to shift some c power to a lowest order. Its limit implies **zero torsion** $t_{\mu\nu} = 0$.

The Poisson Equation

$$\overset{(-2)}{[P]} = (D-2)\mathtt{A}\tilde{\mathrm{R}}_{00} = (D-2)\mathtt{A}\mathrm{R}(\mathrm{G})_{0a}{}^{a}\,,$$

Boost Irrep (Reducible-Indecomposable)



Towards Higher Derivative Non-Relativistic Equations of Motion

Remarks

- We should just add the higher-derivative contributions to the equations of motion.
- The combination for the Poisson equations is fully fixed by the 2-derivative terms.

What could go wrong?

The constraint could be not enough to eliminate the terms at the leading and subleading orders produced by the higher-order terms of the action.

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Does it really happen?

Yes

Action & EOMs

$$\mathcal{S}_{GB} = \int d^D x \sqrt{-g}\Big[\mathcal{L}_{EHM} + \alpha(\mathcal{L}_\alpha - 4\mathcal{L}_\beta + \mathcal{L}_\gamma)\Big]\,,$$

$$\mathcal{L}_\alpha = \left(\mathrm{R} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)^2\,,$$

$$\mathcal{L}_\beta = \mathrm{R}_{\mu\nu}\mathrm{R}^{\mu\nu} + \frac{1}{16}F^4 + \frac{1}{4}F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\sigma F^\sigma{}_\mu - \frac{1}{2}\nabla_\mu F^{\mu\nu}\nabla_\rho F_\nu{}^\rho - F^{\mu\rho}F^\nu{}_\rho \mathrm{R}_{\mu\nu}\,,$$

$$\mathcal{L}_\gamma = \mathrm{R}_{\mu\nu\rho\sigma}\mathrm{R}^{\mu\nu\rho\sigma} + \frac{3}{8}F^4 + \frac{5}{8}F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\sigma F^\sigma{}_\mu + \nabla_\mu F_{\nu\rho}\nabla^\mu F^{\nu\rho} - \frac{3}{2}F^{\mu\nu}F^{\rho\sigma}\mathrm{R}_{\mu\nu\rho\sigma}\,.$$

ACTION & EOMS

$$
\begin{aligned}
[A]^\nu :=& \nabla_\mu \left[F^{\mu\nu} - \frac{3}{2}\alpha F^2 F^{\mu\nu} + 3\alpha F^{\mu\alpha} F_{\alpha\beta} F^{\nu\beta} \right] + \\
& + 8\alpha \mathrm{R}^{\rho[\mu} \nabla_\mu F^{\nu]}{}_\rho + \alpha \mathrm{R}^{\mu\nu\rho\sigma} \nabla_\mu F_{\rho\sigma} + 2\alpha \nabla_\mu F^{\mu\nu} \mathrm{R} = 0\,, \\
[G]_{\alpha\beta} :=& \mathrm{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \mathrm{R} - \frac{1}{2} F_{\alpha\mu} F_\beta{}^\mu + \frac{1}{8} g_{\alpha\beta} F^2 + \\
& + \alpha \Big[- \frac{3}{2} F_\alpha{}^\mu F_\mu{}^\nu F_\nu{}^\rho F_{\rho\beta} + \frac{3}{4} F_\alpha{}^\mu F_{\beta\mu} F^2 + \frac{3}{16} g_{\alpha\beta} F_\sigma{}^\mu F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\sigma - \frac{3}{32} g_{\alpha\beta} F^4 + \\
& + \frac{3}{4} g_{\alpha\beta} F^{\mu\nu} F^{\rho\sigma} \mathrm{R}_{\mu\nu\rho\sigma} + 2 F_\mu{}^\nu F^{\mu\rho} \mathrm{R}_{\alpha\rho\beta\nu} - \frac{3}{2} F^{\nu\rho} (F_\beta{}^\mu \mathrm{R}_{\alpha\mu\nu\rho} + F_\alpha{}^\mu \mathrm{R}_{\beta\mu\nu\rho}) + \\
& - \frac{1}{2} \mathrm{R}_{\alpha\beta} F^2 - 2 F_\beta{}^\mu F_\mu{}^\nu \mathrm{R}_{\alpha\nu} - 2 F_\alpha{}^\mu F_\mu{}^\nu \mathrm{R}_{\beta\nu} + 3 F_\alpha{}^\mu F_\beta{}^\nu \mathrm{R}_{\mu\nu} - 2 g_{\alpha\beta} F_\mu{}^\nu F^{\mu\rho} \mathrm{R}_{\nu\rho} + \\
& - F_{\alpha\mu} F_\beta{}^\mu \mathrm{R} + \frac{1}{4} g_{\alpha\beta} F^2 \mathrm{R} + \frac{1}{2} \nabla_\alpha F_{\mu\nu} \nabla_\beta F^{\mu\nu} - \nabla_\mu F_\alpha{}^\mu \nabla_\nu F_\beta{}^\nu + \nabla_\alpha F_{\beta\mu} \nabla_\nu F^{\mu\nu} + \\
& + \nabla_\beta F_{\alpha\mu} \nabla_\nu F^{\mu\nu} + \nabla_\mu F_{\alpha\nu} \nabla^\mu F_\beta{}^\nu - \frac{1}{2} g_{\alpha\beta} \nabla_\mu F_{\nu\rho} \nabla^\mu F^{\nu\rho} - g_{\alpha\beta} \nabla_\mu F^{\mu\nu} \nabla^\rho F_{\nu\rho} + \\
& - 4 \mathrm{R}_\alpha{}^\mu \mathrm{R}_{\beta\mu} + 2 g_{\alpha\beta} \mathrm{R}_{\mu\nu} \mathrm{R}^{\mu\nu} + 2 \mathrm{R}_{\alpha\beta} \mathrm{R} - \frac{1}{2} g_{\alpha\beta} \mathrm{R}^2 + \\
& - 4 \mathrm{R}^{\mu\nu} \mathrm{R}_{\alpha\mu\beta\nu} + 2 \mathrm{R}_\alpha{}^{\mu\nu\rho} \mathrm{R}_{\beta\mu\nu\rho} - \frac{1}{2} g_{\alpha\beta} \mathrm{R}_{\mu\nu\rho\sigma} \mathrm{R}^{\mu\nu\rho\sigma} \Big] = 0\,,
\end{aligned}
$$

Poisson

$$[P_+] = c^2 \overset{(2)}{[P_+]} + \overset{(0)}{[P_+]} + c^{-2} \overset{(-2)}{[P_+]} + \mathcal{O}(c^{-2}) \,,$$

with

$$\begin{aligned}\overset{(2)}{[P_+]} = (\mathtt{C} - 2\mathtt{B})\alpha \Bigg[& \frac{1}{4} t_{ab} t^{ab} \left(\alpha^{-1} + 2\tilde{\mathrm{R}}_c{}^c + \frac{1}{3} t_0{}^c t_{0c} - 2 t^{cd} f_{cd} \right) + \\ & + t^{ab} \left(- t_{0a} \tilde{\nabla}(t)_{cb}{}^c + \frac{2}{3} t_0{}^c \tilde{\nabla}(t)_{(ac)b} - t_a{}^c \tilde{\mathrm{R}}_{bc} - \frac{1}{6} t_a{}^c t_{0b} t_{0c} + t_a{}^c t_b{}^d f_{cd} \right) + \\ & + \tilde{\nabla}(t)_a{}^{ab} \tilde{\nabla}(t)_{cb}{}^c + \frac{2}{3} \tilde{\nabla}(t)^{abc} \tilde{\nabla}(t)_{(ab)c} \Bigg] \,,\end{aligned}$$

$$\overset{(0)}{[P_+]} = \frac{1}{2}(3\mathtt{A} + \mathtt{C} - D\mathtt{A}) \tilde{\mathrm{R}}_a{}^a + \frac{1}{2}(5\mathtt{A} + \mathtt{C} - D\mathtt{A})\alpha \left[(\tilde{\mathrm{R}}_a{}^a)^2 - 4 \tilde{\mathrm{R}}^{ab} \tilde{\mathrm{R}}_{ab} + \tilde{\mathrm{R}}^{abcd} \tilde{\mathrm{R}}_{abcd} \right] + \ldots \,,$$

Poisson

Supplementing the system of relativistic equations (and $F_{\mu\nu} = 0$ constraint coming from Einstein-Hilbert-Maxwell case) with the constraint

$$\langle C \rangle := \mathrm{R}^2 - 4\mathrm{R}_{\mu\nu}\mathrm{R}^{\mu\nu} + \mathrm{R}_{\mu\nu\rho\sigma}\mathrm{R}^{\mu\nu\rho\sigma} = 0$$

and considering the combination

$$[P_+] := \mathtt{A}\, E^{\mu}{}_{a} E^{\nu a} [G]_{\mu\nu} + (D-3)\mathtt{A} E^{\mu}{}_{0} E^{\nu}{}_{0}\, [G]_{\mu\nu}\,,$$

we are able to get the Poisson equation.

Limit of the Constraints

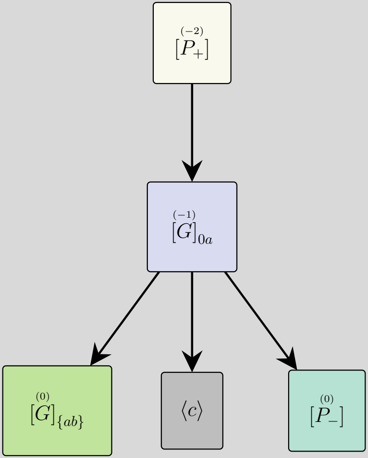
$$t_{\mu\nu} = \mathrm{R}(\mathrm{H})_{\mu\nu} = 0\,, \qquad\qquad \langle c \rangle := \mathrm{Ric}(\mathrm{J}) = 0\,,$$

Equations of Motion

$$\begin{aligned}
\overset{(-2)}{[P_+]} :=\,& \mathrm{R}(\mathrm{G})_{0a}{}^{a} + 4\mathrm{R}(\mathrm{G})_{ab}{}^{a}\mathrm{R}(\mathrm{G})^{bc}{}_{c} + \frac{1}{2}\mathrm{R}(\mathrm{G})^{abc}\mathrm{R}(\mathrm{G})_{abc} + \\
& + 4\mathrm{R}(\mathrm{G})_{0}{}^{ab}\mathrm{Ric}(\mathrm{J})_{ab} + \frac{1}{2}\mathrm{R}(\mathrm{J})^{0abc}(\mathrm{R}(\mathrm{J})_{0abc} - 2\mathrm{R}(\mathrm{G})_{bca}) = 0\,,\\
\overset{(0)}{[P_-]} :=\,& \mathrm{R}(\mathrm{J})^{abcd}\mathrm{R}(\mathrm{J})_{abcd} - 4\mathrm{Ric}(\mathrm{J})^{ab}\mathrm{Ric}(\mathrm{J})_{ab} = 0\,,\\
\overset{(-1)}{[G]}_{0a} :=\,& \mathrm{R}(\mathrm{G})_{ab}{}^{b} - 2\mathrm{R}(\mathrm{J})^{bc}{}_{a}{}^{d}\mathrm{R}(\mathrm{G})_{bcd} + 4\mathrm{Ric}(\mathrm{J})_{ab}\mathrm{R}(\mathrm{G})^{bc}{}_{c} + \\
& + 4\mathrm{Ric}(\mathrm{J})_{ab}\mathrm{R}(\mathrm{G})^{bc}{}_{c} + 4\mathrm{Ric}(\mathrm{J})^{bc}\mathrm{R}(\mathrm{G})_{abc} = 0\,,\\
\overset{(0)}{[G]}_{\{ab\}} :=\,& -\mathrm{Ric}(\mathrm{J})_{ab} + 2\mathrm{R}(\mathrm{J})_{a}{}^{cde}\mathrm{R}(\mathrm{J})_{bcde} - 4\mathrm{Ric}(\mathrm{J})_{a}{}^{c}\mathrm{Ric}(\mathrm{J})_{bc} + \\
& - 4\mathrm{Ric}(\mathrm{J})^{cd}\mathrm{R}(\mathrm{J})_{acbd} = 0\,,
\end{aligned}$$

Boost Representation (Reducible-Indecomposable)

$$
\begin{aligned}
\delta_G \mathrm{R}(\mathrm{H})_{\mu\nu} &= 0\,,\\
\delta_G \langle c\rangle &= 0\,,\\
\delta_G \overset{(-2)}{[P_+]} &= -\,2\lambda^a \overset{(-1)}{[G]}_{0a}\,,\\
\delta_G \overset{(0)}{[P_-]} &= 0\,,\\
\delta_G \overset{(-1)}{[G]}_{0a} &= -\,\lambda^b \overset{(0)}{[G]}_{\{ab\}} - 2\lambda_a \overset{(0)}{[P_-]} +\\
&\quad + \lambda_a \langle c\rangle (1 - 2[\phi])\,,\\
\delta_G \overset{(0)}{[G]}_{\{ab\}} &= 0\,,
\end{aligned}
$$



Conclusion & Outlooks

- Taking the non-relativistic limit of higher-deriviative quadratic theory at the level of the action requires coupiling the gravity sector to a gauge field,
- At the level of the EOMs requires introducing further constraints on the system,

- Finding the proper constraints.
- Change the foliation.
- Study solutions.
- Include further cases (cubic gravity and ...).
- Can the non-relativistic limit be used to acquire info on higher-order gravity theories?

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Thank You!

The Poisson of Ricci Scalar Squared

$$\overset{(2)}{[P]} = \frac{1}{4}(\mathtt{C} - 2\mathtt{B})t_{ab}t^{ab}\left[1 + 4\tilde{\nabla}(t)_{c0}{}^{c} + 2\tilde{\mathrm{R}}_{c}{}^{c} - 3t_{0c}t_{0}{}^{c} - 2f^{cd}t_{cd}\right],$$

$$\overset{(0)}{[P]} = \frac{1}{2}(3\mathtt{A} - D\mathtt{A} + \mathtt{C})\tilde{\mathrm{R}}_{a}{}^{a} + \frac{1}{2}(5\mathtt{A} - D\mathtt{A} + \mathtt{C})\alpha(\tilde{\mathrm{R}}_{a}{}^{a})^{2} + \\ + 2(D\mathtt{A} - 2\mathtt{A} - \mathtt{C})h^{\mu\nu}\tilde{\nabla}_{\mu}\tilde{\nabla}_{\nu}\tilde{\mathrm{R}}_{a}{}^{a} + \ldots,$$

The Issue

The constraint $F_{\mu\nu}$ is not enough to remove all the terms at order c^{0}.

Constraints

$$F_{\mu\nu} = 0,$$

$$\langle C\rangle := \mathrm{R}^{2} + 2\nabla_{\mu}\nabla^{\mu}\mathrm{R} = 0.$$

Limit of the Constraints

The two constraints after the limit become:

$$t_{\mu\nu} = \mathrm{R(H)}_{\mu\nu} = 0\,,$$
$$\overset{(0)}{\langle C \rangle} := \mathrm{Ric(J)}^2 - 2h^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu \mathrm{Ric(J)} = 0\,,$$

Full Set of EOMs

The full set of non-relativistic equations is:

$$\overset{(-2)}{[P_+]} := \mathrm{R(G)}_{0a}{}^{a}(1 - 2\mathrm{Ric(J)}) + 2\tau^\mu\tau^\nu\tilde{\nabla}_\mu\partial_\nu \mathrm{Ric(J)} = 0\,,$$
$$\overset{(0)}{[P_-]} := \mathrm{Ric(J)}(1 - 3\mathrm{Ric(J)}) = 0\,,$$
$$\overset{(-1)}{[G]}_{0a} := -\,\mathrm{Ric(J)}_{0a} + 2\mathrm{Ric(J)}_{0a}\mathrm{Ric(J)} + 2\tau^{(\mu}e^{\nu)}{}_a\tilde{\nabla}_\mu\partial_\nu \mathrm{Ric(J)} = 0\,,$$
$$\overset{(0)}{[G]}_{\{ab\}} := -\,\mathrm{Ric(J)}_{\{ab\}}(1 - 2\mathrm{Ric(J)}) + 2e^\mu{}_{\{a}e^\nu{}_{b\}}\tilde{\nabla}_\mu\partial_\nu \mathrm{Ric(J)} = 0\,,$$

Equations of Motion & Constraints Structure

Boost Representation (Reducible-Indecomposable)

$$
\begin{aligned}
\delta_G \mathrm{R}(\mathrm{H})_{\mu\nu} &= 0\,,\\
\delta_G \overset{(0)}{\langle C\rangle} &= 0\,,\\
\delta_G \overset{(-2)}{[P_+]} &= -\,2\lambda^a \overset{(-1)}{[G]}_{0a}\,,\\
\delta_G \overset{(0)}{[P_-]} &= 0\,,\\
\delta_G \overset{(-1)}{[G]}_{0a} &= -\,\lambda^b \overset{(0)}{[G]}_{\{ab\}} + \frac{\lambda_a}{D-1}\overset{(0)}{[P_-]}\,,\\
\delta_G \overset{(0)}{[G]}_{\{ab\}} &= 0\,.
\end{aligned}
$$