Conformally colored gravity in 3D

Based on: WIP with O. Fuentealba, D. Tempo, and R. Troncoso;

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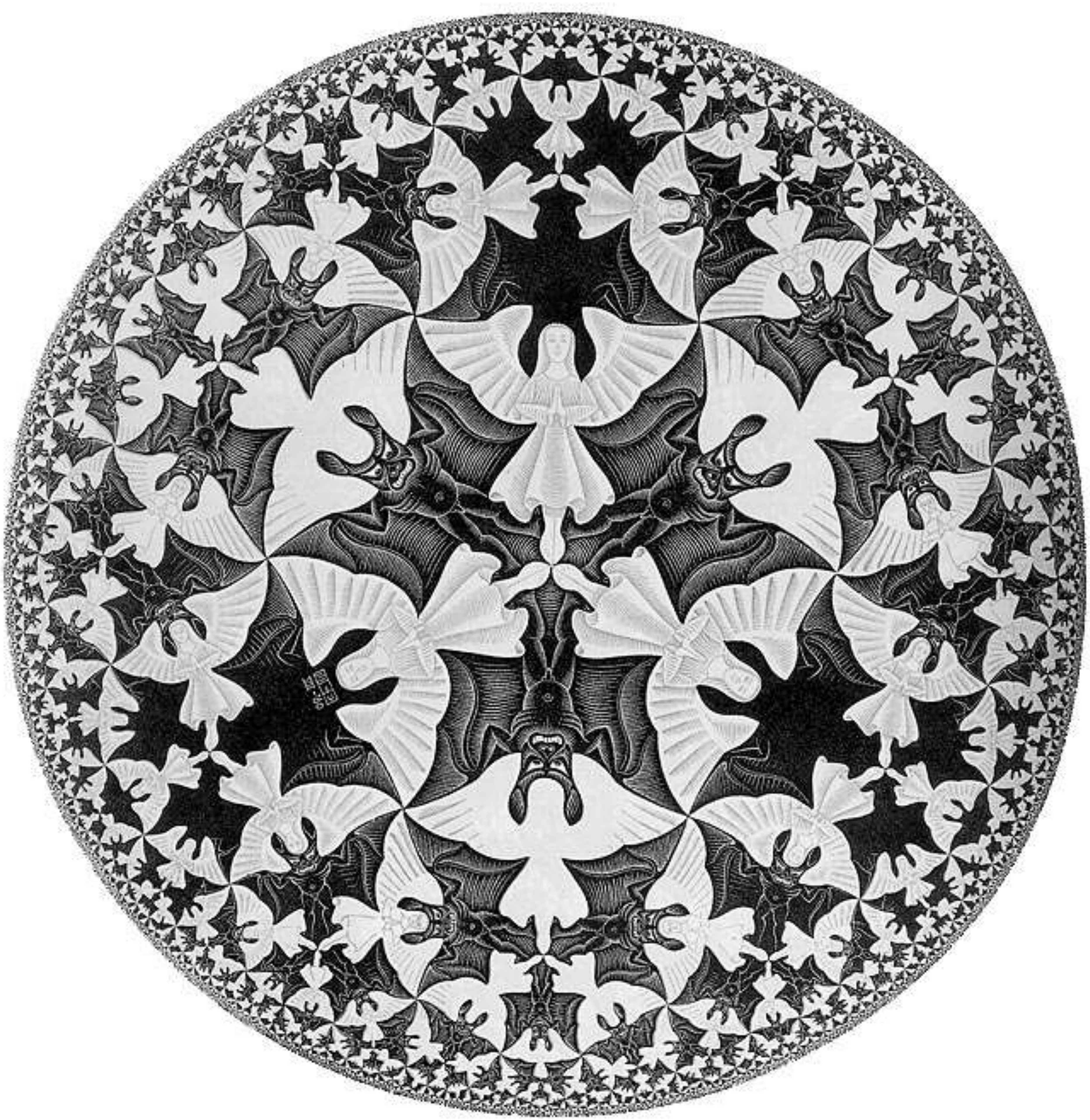


Outline

- Introduction and motivation
- Metric formulation
- Symmetries
- Chern-Simons approach
- Black hole
- Summary

Introduction and motivation

Introduction and motivation

- **Colored gravity?**
- **Photon**
- Yang-Mills: “coloured photon” self-interacting, non-linear gauge symmetry
- **Graviton**
- No go results:

Cutler and Wald;
Boulanger, Henneaux, and Damour
(including ways to circumvent them)

Introduction and motivation

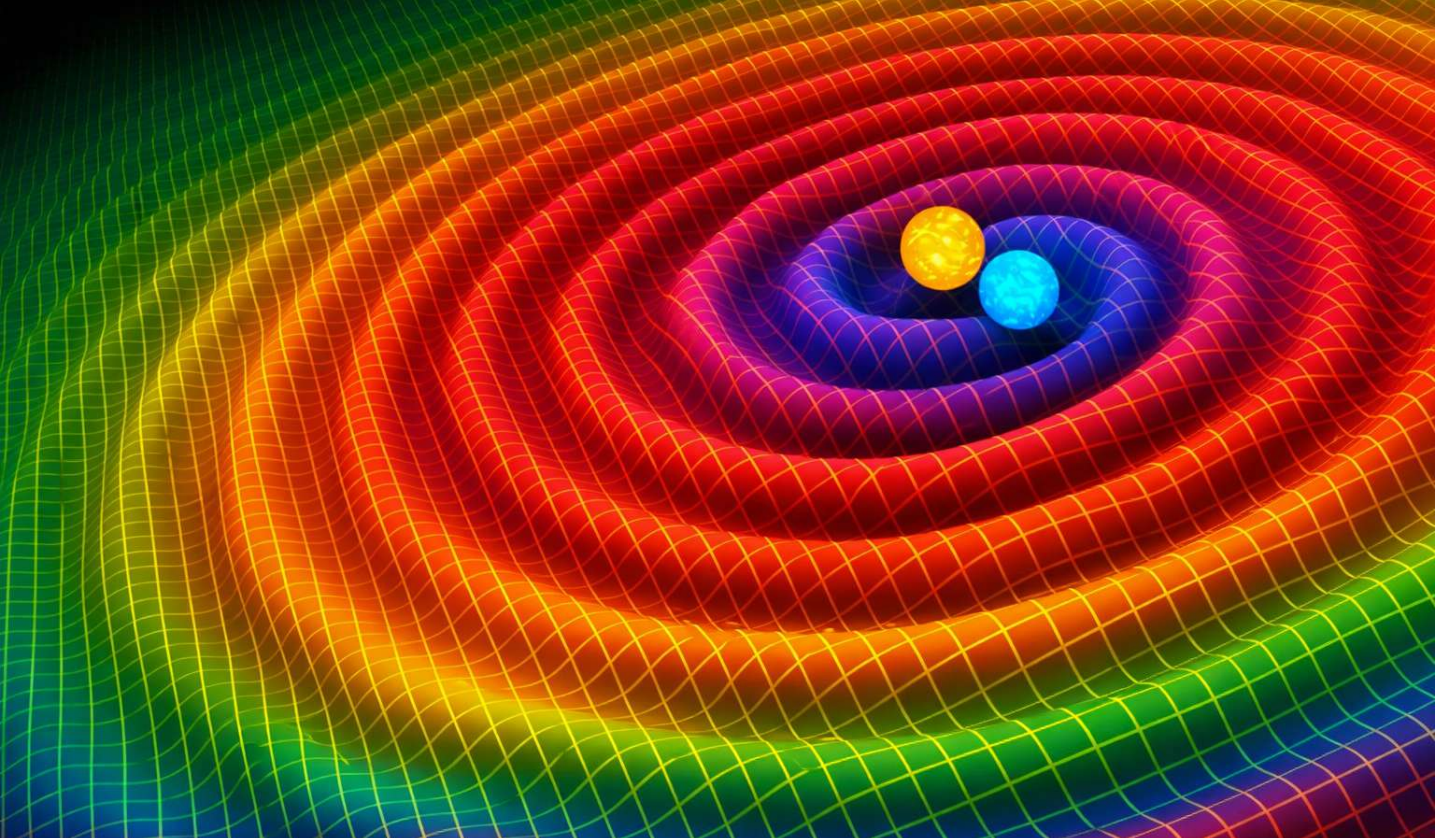
- agrees with Coleman-Mandula, Weinberg th,..
- 3D gravity: no-propagating d.o.f. (hence, no need for S-matrix)
- for that reason 3D color gravity exists
- **Proposals for coloured gravity in 3D**
- Boulanger, Gualtieri; Gwak, Joung, Mkrtchyan, Rey; Gomis, Jong, Kleinschmidt, Mkrtchyan

Introduction and motivation

- Main results:
- requiring local conformal invariance yields extremely simple theory
- There is a **new kind of gauge invariance mixing spacetime and internal symmetries**
- Ingredients $g_{\mu\nu}$ Riemannian spacetime metric

  $a_\mu{}^I{}_J$ so(N) gauge field

  $s^I_{\mu\nu}$ N-colored spin-2 fields (symmetric and traceless)



Metric formulation

Metric formulation

$$I = \int d^3x \epsilon^{\mu\nu\rho} \left[\Gamma^\sigma{}_{\mu\lambda} \partial_\nu \Gamma^\lambda{}_{\rho\sigma} + \frac{2}{3} \Gamma^\sigma{}_{\mu\lambda} \Gamma^\lambda{}_{\nu\tau} \Gamma^\tau{}_{\rho\sigma} - s^{I}_{\mu}{}^{\sigma} \mathfrak{D}_\nu s_{I\rho\sigma} + 4a^{IJ} \left(da_{IJ} + \frac{2}{3} a^K_I a_{KJ} \right) \right]$$

Metric formulation

Conformal gravity in 3d

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Metric formulation

Conformal gravity in 3d

Color fields:
${a_\mu}^I{}_J$ is so(N) gauge field

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Metric formulation

Conformal gravity in 3d

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N spin-2 fields of color I, minimally coupled to gravity,
and so(N) gauge field

Metric formulation

Conformal gravity in 3d

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N spin-2 fields of color I, minimally coupled to gravity, and so(N) gauge field

Full covariant derivative

$$\mathfrak{D}_\mu V_\nu^{aI} = \partial_\mu V_\nu^{aI} - \Gamma^\rho_{\mu\nu} V_\rho^{Ia} + \epsilon^a{}_{bc}\omega^b V_\nu^{cI} + \mathfrak{a}^I{}_J V_\nu^{aJ}$$

Metric formulation

Conformal gravity in 3d

Color fields:
$a_\mu{}^I{}_J$ is so(N) gauge field

$$I = \int d^3x \epsilon^{\mu\nu\rho} \left[\boxed{\Gamma^\sigma{}_{\mu\lambda}\partial_\nu \Gamma^\lambda{}_{\rho\sigma} + \frac{2}{3}\Gamma^\sigma{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\tau}\Gamma^\tau{}_{\rho\sigma}} - \boxed{s^{I}_{\mu}{}^{\sigma}\mathfrak{D}_\nu s_{I\rho\sigma}} + \boxed{4a^{IJ}\left(da_{IJ} + \frac{2}{3}a^K_I a_{KJ}\right)} \right]$$

N spin-2 fields of color I, minimally coupled to gravity,
and so(N) gauge field

Full covariant derivative

$$\mathfrak{D}_\mu V^{aI}_\nu = \partial_\mu V^{aI}_\nu - \Gamma^\rho_{\mu\nu} V^{Ia}_\rho + \epsilon^a{}_{bc}\omega^b V^{cI}_\nu + \mathfrak{a}^I{}_J V^{aJ}_\nu$$

$$\nabla_\mu V_\nu = \mathfrak{D}_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\rho_{\mu\nu} V_\rho$$

$$D_\mu V^a = \mathfrak{D}_\mu V^a = \partial_\mu V^a + \epsilon^a{}_{bc}\omega^b{}_\mu V^c$$

Metric formulation

- By varying the action, we obtain the equations of motion (eom) for the fields

- For the metric, eom is the Cotton tensor and tensor made from $s^I_{\mu\nu}$

$$C^{\mu\nu} = T^{\mu\nu}$$

- Cotton tensor has a standard definition

$$\boxed{C^{\mu\nu} \equiv \epsilon^{\mu\rho\sigma}\nabla_\rho (R_\sigma{}^\nu - \frac{1}{4}R\delta_\sigma{}^\nu)}$$

Metric formulation

- Stress-energy tensor, is given in terms of field $s^I_{\mu\nu}$, and it is symmetric, divergenless and traceless on shell

$$\boxed{T^{\mu\nu} = 2\left[\epsilon^{\alpha\beta\sigma} s^{I(\mu}{}_{\alpha}\mathfrak{D}_{\beta} s^{\nu)}_{I\sigma} + \epsilon^{\alpha\beta(\mu|}\nabla_{\sigma}\left(s^{\sigma}_{I\ \alpha} s^{I|\nu)}{}_{\beta}\right)\right]}$$

- Eom for the field $s^I_{\mu\nu}$ is

$$\boxed{\epsilon^{\mu\nu(\alpha}\nabla_{\mu} s_{I\ \nu}{}^{\beta)} = 0}$$

Metric formulation

- By setting N=1 and using the eom for $s^I_{\mu\nu}$, $C^{\mu\nu} = T^{\mu\nu}$ will equal the equation in Pope et al, and Grigoriev et al

$$\boxed{C^{\mu\nu}} + \boxed{\epsilon^{\rho\sigma(\mu}\nabla_\rho s^{I\nu)\lambda} s_{I\lambda\sigma} + \epsilon^{\rho\sigma(\mu} s^{I\nu)}{}_\rho \nabla_\lambda s^\lambda_{I\,\sigma}} = 0$$

- We have an additional eom for the color gauge bosons, where the contribution beside from the Chern-Simons part, comes from the covariant derivative

Metric formulation

- 3rd eom:

$$F^{IJ} = \mathcal{F}^{IJ}{}_{\mu\nu} + \partial_\mu a_{\nu ij} - \partial_\nu a_{\mu ij} + \tau^{kl} a_{\mu li} a_{\nu kj} - \tau^{kl} a_{\nu li} a_{\mu kj}$$

$$\boxed{\mathcal{F}^{IJ}{}_{\mu\nu} = 4 s^{I\rho}{}_{[\mu} s^{J}{}_{\nu]\rho}}$$

field strength

sourced by the dual of the conserved current (on-shell)

$$J^\lambda_{IJ} = 4\epsilon^{\lambda\mu\nu} s^\rho_{I\ \mu} s_{J\nu\rho}$$

$$\mathfrak{D}_\lambda J^\lambda_{IJ} = 0$$

Symmetries

Symmetries

- No propagating d.o.f.s because of the gauge invariances
- **Gauge invariances:**
- diffeomorphisms $\delta\Phi = \mathcal{L}_\xi \Phi$
- $\boxed{\text{local conformal transformations}}$

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu} \qquad \delta_\gamma g_{\mu\nu} = 2\gamma g_{\mu\nu} \ ,$$

$$s^I_{\mu\nu} \to \Omega s^I_{\mu\nu} \qquad \delta_\gamma s^I_{\mu\nu} = \gamma s^I_{\mu\nu} \ ,$$

$$a^I{}_J \to a^I{}_J \qquad \delta_\gamma a^{IJ}{}_\mu = 0$$

$$\Omega = e^\gamma$$

Symmetries

- so(N) infinitesimal gauge transformations

$$\delta g_{\mu\nu} = 0 \,, \qquad \delta s^I{}_{\mu\nu} = -\gamma^I{}_J s^J_{\mu\nu}$$

$$\boxed{\delta a^I{}_{J\mu} = \mathfrak{D}_\mu \gamma^I{}_J = \partial_\mu \gamma^I{}_J + a^I{}_{L\mu}\gamma^L_J - a^L{}_{J\mu}\gamma^I{}_L}$$

New gauge transformation:

It acts on the metric as: $\boxed{\delta_{\eta^I} g_{\mu\nu} = -2\eta_I s^I_{\mu\nu}}$

It acts on the fields $s^I_{\mu\nu}$ and $a_\mu{}^I{}_J$ in the very non-trivial way:

$$\delta_{\eta^I} s^J_{\mu\nu} = \frac{1}{2}\left[\mathfrak{D}_{(\mu}\mathfrak{D}_{\nu)} - \frac{1}{3}g_{\mu\nu}\mathfrak{D}_\rho\mathfrak{D}^\rho + R_{\mu\nu} - \frac{1}{3}g_{\mu\nu}R\right]\eta^I + \eta^{[I}s^{J]}{}_{\lambda(\mu}s^\lambda_{\nu)(J)} - \frac{1}{3}g_{\mu\nu}\eta^{[I}s^{J]}_{\lambda\rho}s^{\lambda\rho}_J$$

$$\delta_{\eta^I} a^{JK}{}_\mu = \left(\mathfrak{D}^\rho s^{[I}_{\rho\mu}\right)\eta^{J]} - 2s^{\rho[I}_\mu \mathfrak{D}_\rho \eta^{J]}$$

New gauge transformation:

It acts on the metric as: $$\delta_{\eta^I} g_{\mu\nu} = -2\eta_I s^I_{\mu\nu}$$

It acts on the fields $s^I_{\mu\nu}$ and $a_\mu{}^I{}_J$ in the very non-trivial way:

reminiscent of gauge symmetry of partially massless gravitons

$$\delta_{\eta^I} s^J_{\mu\nu} = \frac{1}{2}\left[\mathfrak{D}_{(\mu}\mathfrak{D}_{\nu)} - \frac{1}{3}g_{\mu\nu}\mathfrak{D}_\rho\mathfrak{D}^\rho + R_{\mu\nu} - \frac{1}{3}g_{\mu\nu}R\right]\eta^I$$
$$+\,\eta^{[I}s^{J]}{}_{\lambda(\mu}s^{\lambda}{}_{\nu)(J)} - \frac{1}{3}g_{\mu\nu}\eta^{[I}s^{J]}_{\lambda\rho}s^{\lambda\rho}_J$$

$$\delta_{\eta^I} a^{JK}{}_\mu = \left(\mathfrak{D}^\rho s^{[I}_{\rho\mu}\right)\eta^{J]} - 2s^{\rho[I}_\mu \mathfrak{D}_\rho \eta^{J]}$$

Symmetries

- The algebra mixes internal gauge transformations with spacetime symmetries (diffeomorphisms and conformal transformations) in a non-trivial way
- T: diffeomorphisms + conformal transformations + so(N)gauge transformation
- G: new gauge transformations

$$[T,T] \sim T$$
$$[T,G] \sim G$$
$$[G,G] \sim T$$

Symmetries

$$[\delta_{Z^\mu}, \delta_{\xi_K}]\, g_{\mu\nu} = \delta_{\boxed{\eta_K}} g_{\mu\nu} \ ,$$

$$[\delta_{\gamma_c}, \delta_{\xi_K}]\, g_{\mu\nu} = \delta_{\boxed{\tilde{\eta}_K}} g_{\mu\nu} \ ,$$

$$[\delta_{\eta_I}, \delta_{\xi_J}]\, g_{\mu\nu} = \mathcal{L}_{X^\mu} g_{\mu\nu} + 2\gamma g_{\mu\nu} \ ,$$

$$= \delta_{\boxed{X^\mu}} g_{\mu\nu} + \delta_{\boxed{\gamma}} g_{\mu\nu},$$

$$[\delta_{\gamma_{KL}}, \delta_{\xi_J}]\, g_{\mu\nu} = \delta_{\boxed{\hat{\eta}_K}} g_{\mu\nu} \ ,$$

$$\boxed{\eta_K} = -Z^\rho \partial_\rho \xi_K \ ,$$

$$\boxed{\tilde{\eta}_K} = -\gamma_c \xi_K \ ,$$

$$\boxed{\gamma} = -\frac{1}{3} \nabla_\mu X^\mu \ ,$$

$$\boxed{X_\mu} = -\frac{1}{2} \left(\xi^K \mathfrak{D}_\mu \eta_K - \eta^K \mathfrak{D}_\mu \xi_K \right)$$

$$\boxed{\hat{\eta}_K} = -\gamma_{KL} \xi^L \ ,$$

Symmetries

$$
\begin{aligned}
[\delta_{Z^\mu}, \delta_{\xi_K}]\, g_{\mu\nu} &= \delta_{\boxed{\eta_K}} g_{\mu\nu} \ , \\
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\end{aligned}
$$

- The form (possibly its generalised version) is kept for the fields $a_\mu{}^I{}_J$ and $s^I_{\mu\nu}$ with the parameters defined as

$$
\begin{aligned}
\boxed{\eta_K} &= -Z^\rho \partial_\rho \xi_K \ , \\
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Symmetries

$$[\delta_{Z^\mu}, \delta_{\xi_K}]\, g_{\mu\nu} = \delta_{\boxed{\eta_K}} g_{\mu\nu} \;,$$
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$$= \delta_{\boxed{X^\mu}} g_{\mu\nu} + \delta_{\gamma} g_{\mu\nu},$$
$$[\delta_{\gamma_{KL}}, \delta_{\xi_J}]\, g_{\mu\nu} = \delta_{\boxed{\hat\eta_K}} g_{\mu\nu} \;,$$

- The form (possibly its generalised version) is kept for the fields $a_\mu{}^I{}_J$ and $s^I_{\mu\nu}$ with the parameters defined as

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$$\boxed{X_\mu} = -\frac{1}{2} \left(\xi^K \mathfrak{D}_\mu \eta_K - \eta^K \mathfrak{D}_\mu \xi_K \right)$$
$$\boxed{\hat\eta_K} = -\gamma_{KL} \xi^L \;,$$

- Possibly algebra closes on shell, and requires auxiliary fields (checking)

Chern-Simons formulation

Chern-Simons formulation

- In the Chern-Simons formulation we recover eom and gauge transformations
- This explains that our action has no propagating d.o.f.s, since it can result from the topological action

$$I\left[A\right]=\frac{k}{4\pi}\int\left\langle AdA+\frac{2}{3}A^3\right\rangle ,$$

- Requirements for this are integrating out the fields, and suitable gauge fixing

Chern-Simons formulation

- Additional fields:
- some of them are **auxiliary** (algebraic) and have eom and others are **dynamical** and can be consistently gauged away
- CS for so(3,2) → so(N+3,2)

  [Horne-Witten]

Chern-Simons formulation

- so(N+3,2) in the suitable basis so that the CS connection reads
$$A = \omega^a J_a + e_i^a P_a^i - \frac{1}{2}\mathfrak{a}_{ij}T^{ij}$$

Chern-Simons formulation

- so(N+3,2) in the suitable basis so that the CS connection reads

$$A = \omega^a \boxed{J_a} + e_i^a \boxed{P_a^i} - \frac{1}{2}\mathfrak{a}_{ij} \boxed{T^{ij}}$$

so(2,1)

so(N+1,1)

colored translations
(non-Abelian)

Chern-Simons formulation

- We can single out Poincare translations + special conformal transformations

i=+,-,*I*
I=1,…,*N*

$$D = T_{+-} \qquad U_I = T_{I+}$$

$$V_I = T_{I-} \qquad P_a = P_a^+$$

$$K_a = P_a^-$$

$$A = \omega^a J_a + e^a P_a + f^a K_a + s_I^a P_a^I - bD - u_I T^{I+} - v_I T^{I-} - \frac{1}{2}\mathfrak{a}_{IJ} T^{IJ}$$

auxiliary

dynamical, gauged away

auxiliary

Chern-Simons formulation

- The gauge transformations $\delta A = d\eta + [A, \eta]$

gauge parameter, so(N+3,2)-valued zero form

$$\eta = t^b J_b + \rho^b P_b^+ + \sigma^b P_b^- + \lambda_I^b P_b^I + \gamma D + \xi_I T^{I+} + \zeta_I T^{I-} - \frac{1}{2}\gamma_{IJ} T^{IJ} \ ,$$

Lorentz

conformal

new gauge transformation

so(N)

- Remaining parameters are fixed by the independent ones

Chern-Simons formulation

- The field equations are $F = dA + A^2 = 0$

- We recover eom and gauge transformations from the metric formulation

Colored black hole

Colored black hole

- Example:
- Eom for the metric allows for the solution, which is asymptotically (A)dS or locally flat

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\phi^2 \ ,$$

$$f(r) = \left[c\cos\left(\frac{2\sigma}{r} + \varphi_0\right) + \mathcal{C}\right] r^2$$

$$|\mathcal{C}| \leq c$$

- We have four independent constants

Colored black hole

- The field s which allows for this solution is symmetric and traceless, and it has one non-vanishing component

$$s_{\mu\nu} dx^\mu dx^\nu = \frac{\sigma}{\sqrt{f(r)} r^2} \left(dr^2 - r^2 f(r) d\phi^2 \right)$$

$$s^\mu{}_\nu = \sigma \frac{\sqrt{f(r)}}{r^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

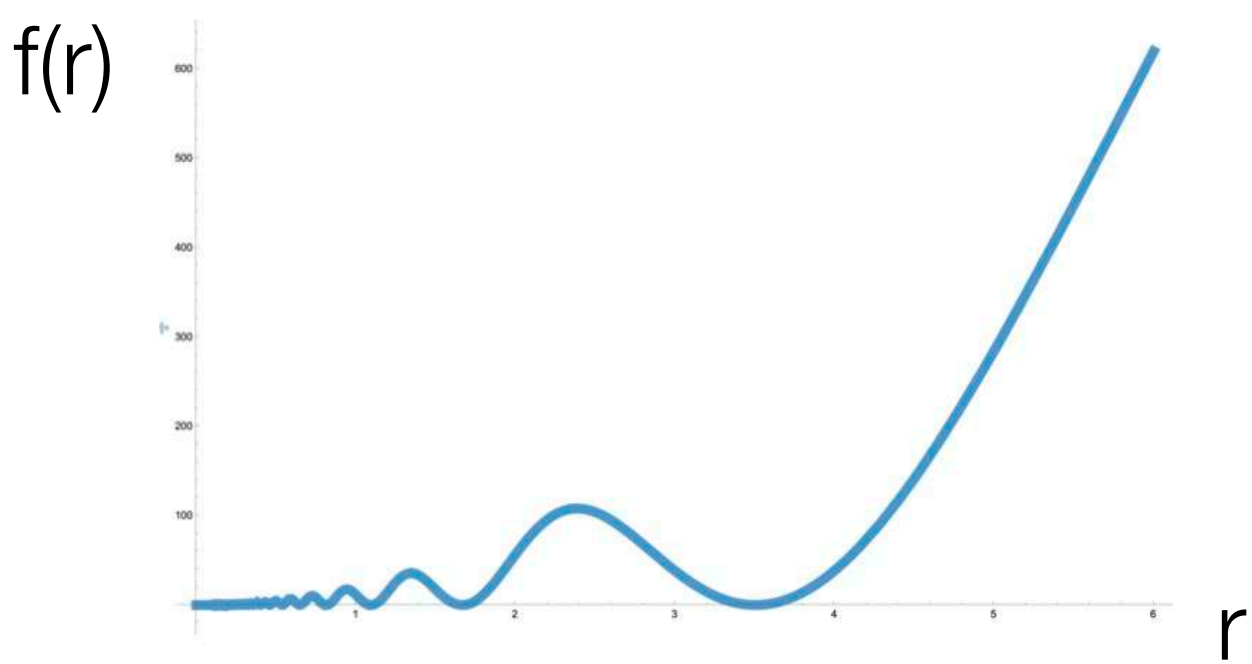
- traceless

Colored black hole

- There is an outer horizon, and due to periodicity of cos we can have **infinite** number of inner horizons

$$r_h(n) = \frac{2\sigma}{\pm \arccos\left(-\mathcal{C}/c\right) - \varphi_0 + 2\pi n} > 0$$

- complicated causal structure



Colored black hole

- Hawking temperature is the same for all horizons

$$T = \frac{f(r)'}{4\pi}\Big|_{r_h}$$

$$T = 2\sigma\sqrt{c^2 - \mathcal{C}^2}$$

The bound saturates at the extremal case: T=0

$\Rightarrow$ infinite number of extremal horizons

Colored black hole

- Ricci scalar is singular at the origin, r=0 (null singularity)

$$R = -6\mathcal{C} - \frac{8c\sigma}{r}\sin\left(\frac{2\sigma}{r} + \varphi_0\right) + \left(\frac{4c\sigma^2}{r^2} - 6c\right)\cos\left(\frac{2\sigma}{r} + \varphi_0\right)$$

- Open issue: full thermodynamics (TD)
- The global charges and TD can be obtained from the asymptotic behaviour already studied in our earlier paper 2501.00439 *Enhanced conformal* BMS_3 *symmetries*

Summary

Summary

- In the **metric formalism** we have spin-2 fields minimally coupled to metric and so(N) fields
- We have a **new type of gauge symmetry** mixing of spacetime and internal symmetries (not a supersymmetry and not a higher spin symmetry)
- Since it admits a **CS formulation** the algebra closes off shell
- It has an intriguing **BH solution** with infinite number of inner horizons, all of the same temperature. Solution is not conformally flat, it is asymptotically (A)dS or locally flat

Thank you for your attention!

Chern-Simons formulation

$$
\begin{aligned}
\delta\omega^a &= Dt^a - 2\epsilon_{abc}e^a\sigma^b - 2\epsilon_{abc}f^a\rho^b - 2\epsilon_{abc}s^a_I\lambda^{bI} \ , \\
\delta e^a &= D\rho^a + b\rho^a + e^a\gamma - s^a_I\xi^I - u^I\lambda^a_I + \epsilon^{abc}e_b t_c \ , \\
\delta f^a &= D\sigma^a - b\sigma^a - f^a\gamma - v^I\lambda^a_I - s^a_I\zeta^I + \epsilon^{abc}f_b t_c \ , \\
\delta s^a_I &= \mathfrak{D}\lambda^a_I + f^a\xi_I + e^a\zeta_I + s^a_J\gamma^J{}_I + v_I\rho^a + u_I\sigma^a + \epsilon^{abc}s_{bI}t_c \ , \\
\delta b &= -d\gamma + u^I\zeta_I - v^I\xi_I - 2e^a\sigma_a + 2f^a\rho_a \ , \\
\delta u_I &= -\mathfrak{D}\xi_I - b\xi_I + u_I\gamma + u^K\gamma_{KI} - 2s^a_I\rho_a + 2e^a\lambda_{aI} \ , \\
\delta v_I &= -\mathfrak{D}\zeta_I + b\zeta_I - v_I\gamma + v^K\gamma_{KI} - 2s^a_I\sigma_a + 2f^a\lambda_{aI} \ , \\
\delta\mathfrak{a}_{IJ} &= \mathfrak{D}\gamma_{IJ} + v_I\xi_J - v_J\xi_I + u_I\zeta_J - u_J\zeta_I - 2s^a_I\lambda_{aJ} + 2s^a_J\lambda_{aI} \ ,
\end{aligned}
$$

Chern-Simons formulation

$$
\begin{aligned}
\delta\omega^a &= Dt^a - 2\epsilon_{abc}e^a\sigma^b - 2\epsilon_{abc}f^a\rho^b - 2\epsilon_{abc}s^a_I\lambda^{bI} \ , \\
\delta e^a &= D\rho^a + b\rho^a + e^a\gamma - s^a_I\xi^I - u^I\lambda^a_I + \epsilon^{abc}e_b t_c \ , \\
\delta f^a &= D\sigma^a - b\sigma^a - f^a\gamma - v^I\lambda^a_I - s^a_I\zeta^I + \epsilon^{abc}f_b t_c \ , \\
\delta s^a_I &= \mathfrak{D}\lambda^a_I + f^a\xi_I + e^a\zeta_I + s^a_J\gamma^J{}_I + v_I\rho^a + u_I\sigma^a + \epsilon^{abc}s_{bI}t_c \ , \\
\delta b &= -d\gamma + u^I\zeta_I - v^I\xi_I - 2e^a\sigma_a + 2f^a\rho_a \ , \\
\delta u_I &= -\mathfrak{D}\xi_I - b\xi_I + u_I\gamma + u^K\gamma_{KI} - 2s^a_I\rho_a + 2e^a\lambda_{aI} \ , \\
\delta v_I &= -\mathfrak{D}\zeta_I + b\zeta_I - v_I\gamma + v^K\gamma_{KI} - 2s^a_I\sigma_a + 2f^a\lambda_{aI} \ , \\
\delta\mathfrak{a}_{IJ} &= \mathfrak{D}\gamma_{IJ} + v_I\xi_J - v_J\xi_I + u_I\zeta_J - u_J\zeta_I - 2s^a_I\lambda_{aJ} + 2s^a_J\lambda_{aI} \ ,
\end{aligned}
$$

Fields b, u_I and $s^{I\mu}{}_\mu$ can be consistently gauged away

Chern-Simons formulation

$$
\begin{aligned}
\delta\omega^a &= Dt^a - 2\epsilon_{abc}e^a\sigma^b - 2\epsilon_{abc}f^a\rho^b - 2\epsilon_{abc}s^a_I\lambda^{bI} \ , \\
\delta e^a &= D\rho^a + e^a\gamma - s^a_I\xi^I + \epsilon^{abc}e_b t_c \ , \\
\delta f^a &= D\sigma^a - f^a\gamma - v^I\lambda^a_I - s^a_I\zeta^I + \epsilon^{abc}f_b t_c \ , \\
\delta s^a_I &= \mathfrak{D}\lambda^a_I + f^a\xi_I + e^a\zeta_I + s^a_J\gamma^J_{\ I} + v_I\rho^a + \epsilon^{abc}s_{bI}t_c \ , \\
0 &= -d\gamma - v^I\xi_I - 2e^a\sigma_a + 2f^a\rho_a \ , \\
0 &= -\mathfrak{D}\xi_I - 2s^a_I\rho_a + 2e^a\lambda_{aI} \ , \\
\delta v_I &= -\mathfrak{D}\zeta_I - v_I\gamma + v^K\gamma_{KI} - 2s^a_I\sigma_a + 2f^a\lambda_{aI} \ , \\
\delta\mathfrak{a}_{IJ} &= \mathfrak{D}\gamma_{IJ} + v_I\xi_J - v_J\xi_I - 2s^a_I\lambda_{aJ} + 2s^a_J\lambda_{aI} \ ,
\end{aligned}
$$

Chern-Simons formulation

$$\delta\omega^a = Dt^a - 2\epsilon_{abc}e^a\sigma^b - 2\epsilon_{abc}f^a\rho^b - 2\epsilon_{abc}s^a_I\lambda^{bI} \ ,$$
$$\delta e^a = D\rho^a + e^a\gamma - s^a_I\xi^I + \epsilon^{abc}e_b t_c \ ,$$
$$\delta f^a = D\sigma^a - f^a\gamma - v^I\lambda^a_I - s^a_I\zeta^I + \epsilon^{abc}f_b t_c \ ,$$
$$\delta s^a_I = \mathfrak{D}\lambda^a_I + f^a\xi_I + e^a\zeta_I + s^a_J\gamma^J_{\ I} + v_I\rho^a + \epsilon^{abc}s_{bI}t_c \ , \longrightarrow \zeta_I$$
$$0 = -d\gamma - v^I\xi_I - 2e^a\sigma_a + 2f^a\rho_a \ , \longrightarrow \sigma_a$$
$$0 = -\mathfrak{D}\xi_I - 2s^a_I\rho_a + 2e^a\lambda_{aI} \ , \longrightarrow \lambda_{aI}$$
$$\delta v_I = -\mathfrak{D}\zeta_I - v_I\gamma + v^K\gamma_{KI} - 2s^a_I\sigma_a + 2f^a\lambda_{aI} \ ,$$
$$\delta\mathfrak{a}_{IJ} = \mathfrak{D}\gamma_{IJ} + v_I\xi_J - v_J\xi_I - 2s^a_I\lambda_{aJ} + 2s^a_J\lambda_{aI} \ ,$$

Chern-Simons formulation

$$\delta\omega^a = Dt^a - 2\epsilon_{abc}e^a\sigma^b - 2\epsilon_{abc}f^a\rho^b - 2\epsilon_{abc}s_I^a\lambda^{bI} \ ,$$

$$\delta e^a = D\rho^a + e^a\gamma - s_I^a\xi^I + \epsilon^{abc}e_b t_c \ , \longrightarrow \boxed{\delta g_{\mu\nu} = 2\gamma g_{\mu\nu} - 2s_{(\mu\nu)}^{(I)}\xi_i}$$

$$\delta f^a = D\sigma^a - f^a\gamma - v^I\lambda_I^a - s_I^a\zeta^I + \epsilon^{abc}f_b t_c \ ,$$

$$\delta s_I^a = \mathfrak{D}\lambda_I^a + f^a\xi_I + e^a\zeta_I + s_J^a\gamma^J{}_I + v_I\rho^a + \epsilon^{abc}s_{bI}t_c \ , \longrightarrow \zeta_I \ , \boxed{\delta s_{\mu\nu}^I}$$

$$0 = -d\gamma - v^I\xi_I - 2e^a\sigma_a + 2f^a\rho_a \ , \longrightarrow \sigma_a$$

$$0 = -\mathfrak{D}\xi_I - 2s_I^a\rho_a + 2e^a\lambda_{aI} \ , \longrightarrow \lambda_{aI}$$

$$\delta v_I = -\mathfrak{D}\zeta_I - v_I\gamma + v^K\gamma_{KI} - 2s_I^a\sigma_a + 2f^a\lambda_{aI} \ ,$$

$$\delta\mathfrak{a}_{IJ} = \mathfrak{D}\gamma_{IJ} + v_I\xi_J - v_J\xi_I - 2s_I^a\lambda_{aJ} + 2s_J^a\lambda_{aI} \ ,$$

$$\boxed{\delta\mathfrak{a}_{ij\mu} = \mathcal{D}_\mu\gamma_{ij} - \frac{1}{2}\left(D_\alpha s_{(i)}^{\alpha}{}_{\mu}\xi_j - D_\alpha s_{(j)}^{\alpha}{}_{\mu}\xi_i\right) + s_{(i)\mu}^{\alpha}D_\alpha\xi_j - s_{(j)\mu}^{\alpha}D_\alpha\xi_i \ ,}$$

Chern-Simons formulation

$$\delta\omega^a = Dt^a - 2\epsilon_{abc}e^a\sigma^b - 2\epsilon_{abc}f^a\rho^b - 2\epsilon_{abc}s_I^a\lambda^{bI} \ , \longrightarrow \text{consistent}$$

$$\delta e^a = D\rho^a + e^a\gamma - s_I^a\xi^I + \epsilon^{abc}e_b t_c \ , \longrightarrow \boxed{\delta g_{\mu\nu} = 2\gamma g_{\mu\nu} - 2s^{(I)}_{(\mu\nu)}\xi_I}$$

$$\delta f^a = D\sigma^a - f^a\gamma - v^I\lambda_I^a - s_I^a\zeta^I + \epsilon^{abc}f_b t_c \ , \longrightarrow \text{consistent}$$

$$\delta s_I^a = \mathfrak{D}\lambda_I^a + f^a\xi_I + e^a\zeta_I + s_J^a\gamma^J{}_I + v_I\rho^a + \epsilon^{abc}s_{bI}t_c \ , \longrightarrow \zeta_I \ , \boxed{\delta s^I_{\mu\nu}}$$

$$0 = -d\gamma - v^I\xi_I - 2e^a\sigma_a + 2f^a\rho_a \ , \longrightarrow \sigma_a$$

$$0 = -\mathfrak{D}\xi_I - 2s_I^a\rho_a + 2e^a\lambda_{aI} \ , \longrightarrow \lambda_{aI}$$

$$\delta v_I = -\mathfrak{D}\zeta_I - v_I\gamma + v^K\gamma_{KI} - 2s_I^a\sigma_a + 2f^a\lambda_{aI} \ , \longrightarrow \text{consistent}$$

$$\delta\mathfrak{a}_{IJ} = \mathfrak{D}\gamma_{IJ} + v_I\xi_J - v_J\xi_I - 2s_I^a\lambda_{aJ} + 2s_J^a\lambda_{aI} \ ,$$

$$\boxed{\delta\mathfrak{a}_{ij\mu} = \mathcal{D}_\mu\gamma_{ij} - \frac{1}{2}\left(D_\alpha s^\alpha_{(i)\ \mu}\xi_j - D_\alpha s^\alpha_{(j)\ \mu}\xi_i\right) + s^\alpha_{(i)\mu}D_\alpha\xi_j - s^\alpha_{(j)\mu}D_\alpha\xi_i \ ,}$$

Chern-Simons formulation

- The field equations are $F = dA + A^2 = 0$

$$
\begin{aligned}
F^a &= R^a - 2\epsilon^{abc} e_b f_c - \epsilon^{abc} s_{bI} s_c^I \ , \\
T_+^a &= T^a \ , \\
T_-^a &= Df^a - v^I s_I^a \ , \\
T_I^a &= \mathfrak{D} s_I^a + v_I e^a \\
F_{+-} &= -2e^a f_a \ , \\
F_{I+} &= -2s_I^a e_a \ , \\
F_{I-} &= \mathfrak{D} v_I - 2s_I^a f_a \ , \\
F_{IJ} &= \mathcal{F}_{IJ} - 2s_I^a s_{aJ} \ ,
\end{aligned}
$$

Chern-Simons formulation

- The field equations are $F = dA + A^2 = 0$

$$F^a = R^a - 2\epsilon^{abc} e_b f_c - \epsilon^{abc} s_{bI} s_c^I \ ,$$

$$T_+^a = T^a \ , \longrightarrow \text{torsion is zero}$$

$$T_-^a = Df^a - v^I s_I^a \ ,$$

$$T_I^a = \mathfrak{D} s_I^a + v_I e^a$$

$$F_{+-} = -2e^a f_a \ , \longrightarrow f_{[\mu\nu]} = 0$$

$$F_{I+} = -2s_I^a e_a \ , \longrightarrow s^I_{[\mu\nu]} = 0$$

$$F_{I-} = \mathfrak{D} v_I - 2s_I^a f_a \ ,$$

$$F_{IJ} = \mathcal{F}_{IJ} - 2s_I^a s_{aJ} \ ,$$

Chern-Simons formulation

- The field equations are $F = dA + A^2 = 0$

$$f_{\mu\nu} = \frac{1}{2}\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right) + s^{(I)}_{\mu\alpha}s^{\alpha}_{(I)\nu} - \frac{1}{4}g_{\mu\nu}s^{(I)}_{\alpha\beta}s_{(I)\alpha\beta}$$

$$F^a = R^a - 2\epsilon^{abc}e_b f_c - \epsilon^{abc}s_{bI}s^I_c\ ,$$

$$T^a_+ = T^a\ , \longrightarrow \text{torsion is zero}$$

$$T^a_- = Df^a - v^I s^a_I\ ,$$

$$T^a_I = \mathfrak{D}s^a_I + v_I e^a$$

$$F_{+-} = -2e^a f_a\ , \longrightarrow f_{[\mu\nu]} = 0$$

$$F_{I+} = -2s^a_I e_a\ , \longrightarrow s^I_{[\mu\nu]} = 0$$

$$F_{I-} = \mathfrak{D}v_I - 2s^a_I f_a\ ,$$

$$F_{IJ} = \mathcal{F}_{IJ} - 2s^a_I s_{aJ}\ ,$$

Chern-Simons formulation

- The field equations are $F = dA + A^2 = 0$

$$f_{\mu\nu} = \frac{1}{2}\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right) + s^{(I)}_{\mu\alpha}s^{\alpha}_{(I)\nu} - \frac{1}{4}g_{\mu\nu}s^{(I)}_{\alpha\beta}s_{(I)\alpha\beta}$$

$$F^a = R^a - 2\epsilon^{abc}e_b f_c - \epsilon^{abc}s_{bI}s^I_c \; , \longrightarrow$$

$$T^a_+ = T^a \; , \longrightarrow \text{torsion is zero}$$

$$T^a_- = Df^a - v^I s^a_I \; , \longrightarrow \text{eom for } g_{\mu\nu}$$

$$T^a_I = \mathfrak{D}s^a_I + v_I e^a \longrightarrow \text{eom for } s^I_{\mu\nu} \text{ and } v_{I\mu} = \frac{1}{2}\mathfrak{D}_\alpha s^\alpha_{(I)\ \mu}$$

$$F_{+-} = -2e^a f_a \; , \longrightarrow f_{[\mu\nu]} = 0$$

$$F_{I+} = -2s^a_I e_a \; , \longrightarrow s^I_{[\mu\nu]} = 0$$

$$F_{I-} = \mathfrak{D}v_I - 2s^a_I f_a \; ,$$

$$F_{IJ} = \mathcal{F}_{IJ} - 2s^a_I s_{aJ} \; , \longrightarrow \text{eom for } a^I{}_J$$

Chern-Simons formulation

- The field equations are $F = dA + A^2 = 0$

$$\boxed{\begin{aligned} f_{\mu\nu} &= \frac{1}{2}\left(R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R\right) \\ &\quad + s^{(I)}_{\mu\alpha} s^{\alpha}_{(I)\nu} - \frac{1}{4} g_{\mu\nu} s^{(I)}_{\alpha\beta} s_{(I)\alpha\beta} \end{aligned}}$$

$$F^a = R^a - 2\epsilon^{abc} e_b f_c - \epsilon^{abc} s_{bI} s^I_c \,, \longrightarrow$$

$$T^a_+ = T^a \,, \longrightarrow \text{ torsion is zero}$$

$$T^a_- = D f^a - v^I s^a_I \,, \longrightarrow \text{ eom for } g_{\mu\nu}$$

$$T^a_I = \mathfrak{D} s^a_I + v_I e^a \longrightarrow \text{ eom for } s^I_{\mu\nu} \text{ and } v_{I\mu} = \frac{1}{2} \mathfrak{D}_\alpha s^{\alpha}_{(I)\ \mu}$$

$$F_{+-} = -2e^a f_a \,, \longrightarrow f_{[\mu\nu]} = 0$$

$$F_{I+} = -2s^a_I e_a \,, \longrightarrow s^I_{[\mu\nu]} = 0$$

$$F_{I-} = \mathfrak{D} v_I - 2s^a_I f_a \,, \longrightarrow \text{ consistency condition for the eom for } s^I_{\mu\nu}$$

$$F_{IJ} = \mathcal{F}_{IJ} - 2s^a_I s_{aJ} \,, \longrightarrow \text{ eom for } a^I{}_J$$