

The group geometric approach to supergravity

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Group geometric construction of supergravity theories, Torino group 80's

originates from Ne'eman and Regge (1978), then D'Adda, D' Auria, Fré, LC, van Nieuwenhuizen, Townsend, Pilch, Ceresole, ...

Old Reviews: D' Adda, D' Auria, Fré, Regge 1980,
Fré, van Nieuwenhuizen, LC 1981
Fré, Giani, Pilch, van Nieuwenhuizen, LC 1983
van Nieuwenhuizen, 1983
D' Auria, Fré, Townsend, van Nieuwenhuizen 1983
LC 1991

Book: D' Auria, Fré, LC 1991

Recent Reviews: LC 2018, D' Auria 2019
D' Auria, Fré, LC 2026

Related approaches: Chamseddine, West 1977

The group-geometric approach provides a **systematic and algorithmic procedure** to construct locally supersymmetric actions



Tullio Regge (1931-2014)

1. Group geometric approach
2. Geometric action
3. Variational principle
4. Building rules for the Lagrangian $L^{(d)}$
5. Symmetries
6. Symmetries of d=4 N=1 Supergravity
7. Equations of motion of d=4, N=1 Supergravity
8. Free differential algebras (FDA)
9. Selfdual forms
10. Group manifold Lagrangians for d=6 and d=10 SG

1. Group geometric approach

- Supergravity theories can be naturally formulated in a (super) group-geometric framework.
- Tangent vectors t_A on a G group manifold, defined by the infinitesimal action of G on itself, satisfy the Lie commutation relations

$$[t_A, t_B] = C^C_{AB} t_C$$

- The dual cotangent basis of one-forms σ^A (with $\sigma^A(t_B) = \delta^A_B$) satisfies the Cartan-Maurer eq.s.

$$d\sigma^A + \frac{1}{2} C^A_{BC} \sigma^B \wedge \sigma^C = 0$$

$$d^2 = 0 \iff C^A_{B[C} C^B_{DE]} = 0$$

- The basic fields correspond to the σ^A , the Vielbeine of G

- To introduce **dynamics**, we allow σ^A to fluctuate, $\longrightarrow \tilde{\sigma}^A$ describing a **deformation** of the group manifold

$$G \longrightarrow \tilde{G}$$

- **Curvature** of $\tilde{G}$

$$R^A \equiv d\tilde{\sigma}^A + \frac{1}{2}C^A_{BC} \tilde{\sigma}^B \wedge \tilde{\sigma}^C \neq 0$$

measures the deformation

- **Bianchi identities**

$$dR^A - C^A_{BC} R^B \tilde{\sigma}^C = 0$$

- The fields $\tilde{\sigma}^A$ depend on **all the $\tilde{G}$ coordinates**
- Thus **diff.s, supersymmetry, gauge transformations** are all **diffeomorphisms** on the “soft” (super)group manifold $\tilde{G}$
- They are **invariances** of a geometric action on $\tilde{G}$

NB *no tilde in the following*

Example: super Poincaré group manifold

- Lie algebra (superPoincaré)

$$[P_a, P_b] = 0$$

$$[M_{ab}, M^{cd}] = 4\delta_{[a}^{[c} M_{b]}^{d]}$$

$$\{Q_\alpha, Q_\beta\} = i(C\Gamma^a)_{\alpha\beta} P_c$$

$$[M_{ab}, P^c] = 2\delta_{[a}^c P_{b]}$$

$$[Q_\alpha, P_b] = 0$$

$$[M_{ab}, Q_\alpha] = \frac{1}{4}(\Gamma_{ab})_\alpha{}^\beta Q_\beta$$

- Dynamical fields

G - coordinates

$$\sigma^A \left\{ \begin{array}{ll} V^a \longleftrightarrow P_a & \text{vierbein} \\ \omega^{ab} \longleftrightarrow M_{ab} & \text{spin connection} \\ \psi^\alpha \longleftrightarrow Q_\alpha & \text{gravitino} \end{array} \right. \quad \begin{array}{l} x \\ y^{ab} \\ \theta^\alpha \end{array}$$

$$A = (a, ab, \alpha)$$

- Curvatures

$$R^a = dV^a - \omega^a_b V^b - \frac{i}{2} \bar{\psi} \Gamma^a \psi$$

Torsion

$$R^{ab} = d\omega^{ab} - \omega^a_c \omega^{cb}$$

Lorentz curvature

$$\rho = d\psi - \frac{1}{4} \omega^{ab} \Gamma_{ab} \psi$$

gravitino curvature

- Bianchi identities

$$dR^a - \omega^a_b R^b + R^a_b V^b - i \bar{\psi} \Gamma^a \rho = 0$$

$$dR^{ab} + 2\omega^{[a}_c R^{b]c} = 0$$

$$d\rho - \frac{1}{4} \omega^{ab} \Gamma_{ab} \psi + \frac{1}{4} \Gamma_{ab} \psi R^{ab} = 0$$

- Curvatures

$$R^a = dV^a - \omega^a_b V^b - \frac{i}{2} \bar{\psi} \Gamma^a \psi$$

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gravitino curvature

- Bianchi identities

$$dR^a - \omega^a_b R^b + R^a_b V^b - i \bar{\psi} \Gamma^a \rho = 0$$

$$dR^{ab} + 2\omega^{[a}_c R^{b]c} = 0$$

$$d\rho - \frac{1}{4} \omega^{ab} \Gamma_{ab} \psi + \frac{1}{4} \Gamma_{ab} \psi R^{ab} = 0$$

scaling invariance

$$V^a, R^a \longrightarrow \lambda V^a, \lambda R^a$$

$$\psi, \rho \longrightarrow \sqrt{\lambda} \psi, \sqrt{\lambda} \rho$$

$$\omega^{ab}, R^{ab} \longrightarrow \omega^{ab}, R^{ab}$$

2. Geometric action

- Task: construct an action containing the G vielbeins and curvatures, invariant under diffeomorphisms on the G -group manifold

→ integral on G of a $\dim G$ - form

- Then the symmetries of the theory are given by the G -diffeomorphisms, generated by the Lie derivative along tangent vectors $\varepsilon = \varepsilon^A t_A$:

$$\begin{aligned}\ell_\varepsilon \sigma^A &= (di_\varepsilon + i_\varepsilon d) \sigma^A \\ &= d\varepsilon^A + C^A_{BC} \sigma^B \varepsilon^C + i_\varepsilon R^A \\ &= \boxed{(\nabla_G \varepsilon)^A + i_\varepsilon R^A}\end{aligned}$$

- When the curvature term $i_\varepsilon R^A$ vanishes, the diff. takes the form of a **gauge transformation**, and the curvature is said to be **horizontal** along the G -directions corresponding to $\varepsilon = \varepsilon^A t_A$
- Diff.s on G are **reinterpreted** as
 - **ordinary diff.s** in the spacetime directions of G : in these directions the curvatures are not horizontal
 - **gauge** variations
 - **susy** transformations (diff.s in the fermionic directions)

- So far all fields depend on **all the G coordinates**
- How can we obtain a **d -dim field theory** where the fields depend only on **d** space-time coordinates (for ex. **$d = 4$**) ?



construct a Lagrangian **d** - form and integrate it on a **d** - dim **submanifold S** of **G**

$$\int_S L^{(d)}$$

- For ex. $G = \text{SuperPoincaré}$, and we want a 4 dim theory. Then construct a **Lagrangian 4-form**, and integrate it on a **4-surface S** embedded into G , parametrized by spacetime coordinates x

3. Variational principle

- must vary **fields** (1-form vielbeins) in L , and also **submanifold** S
- a continuous variation of S can be compensated by a diffeomorphism on the fields $\implies$ **vary only the fields**:

$$\delta \int_S L(\phi, d\phi) = \int_S \delta\phi \frac{\partial L}{\partial\phi} + d\delta\phi \frac{\partial L}{\partial(d\phi)}$$

- field equations:

$$\frac{\partial L}{\partial\phi} + d \frac{\partial L}{\partial(d\phi)} = 0$$

$(d-1)$ -form equations in G

- NB: since variation of S can be compensated by variation (diff) on the fields, **on shell the action is invariant under small changes in S .** Then

$$dL = 0 \text{ on shell}$$

because of Stokes theorem

4. Building rules for the Lagrangian $L^{(d)}$

- Choose a Lie (super)algebra G , containing d translations P_a that can be associated to d spacetime dimensions, and a Lorentz-like subalgebra H .
For ex: superPoincaré in d dim, $OSp(N|2^{[d/2]})$, ...
- Construct the most general d - form on G , by multiplying (exterior product) 1 - form vielbeins σ^A and 2 - form curvatures R^A , without bare Lorentz connection ω and contracting indices with H - invariant tensors.
- Require vacuum solution ($R = 0 \rightarrow$ rigid G), correct scaling and parity.

Example: N=1 supergravity in d=4

- Most general 4 - form made out of super Poincaré curvatures $R^A = (R^a, R^{ab}, \rho^\alpha)$ and vielbeins $\sigma^A = (V^a, \omega^{ab}, \psi^\alpha)$

$$L = \Lambda^{(4)} + R^A \nu_A^{(2)} + R^A R^B \nu_{AB}^{(0)}$$

with

$$\Lambda^{(4)} = C_{ABCD} \sigma^A \sigma^B \sigma^C \sigma^D$$

$$\nu_A^{(2)} = C_{ABC} \sigma^B \sigma^C$$

$$\nu_{AB}^{(0)} = C_{AB}$$

- All terms **bilinear** in curvatures are **topological terms** (Euler and Pontryagin) or reduce to terms **linear** in curvatures after integration by parts and use of Bianchi identities

- Applying the building rules yields the group manifold L :

$$L = R^{ab}V^cV^d\varepsilon_{abcd} + 4\bar{\psi}\gamma_5\gamma_a\rho V^a$$

- All fields are forms in G and depend on all G coordinates x , y^{ab} , θ^α
- The **spacetime restriction** of L coincides with the usual $d = 4$ SG Lagrangian, found in 1976.

Ferrara, Friedman, van Nieuwenhuizen 1976

Deser, Zumino 1976

In the group manifold construction:

- spacetime restriction gives **correct L**
- spacetime diff. invariance and Lorentz invariance are automatic from the building rules
- supersymmetry variations can be found as diff.s in the fermionic directions, via **Lie derivative** on the vielbeins
- but is the group manifold action invariant under fermionic diff.s ? **(supersymmetry)**

5. Symmetries

$\int_S L^{(d)}$ is not, in general, invariant under G diff.s

- Indeed the diff. variation of L is

$$\ell_\varepsilon L^{(d)} = \iota_\varepsilon dL + d\iota_\varepsilon L$$

When L = top form, this variation is a total derivative since $dL = 0$.
But here L is a d -form, with $d < \dim G$, so that $\iota_\varepsilon dL$ survives

- Therefore $\int_S L^{(d)}$ is invariant under G diff.s along ε if and only if

$$\iota_\varepsilon dL = 0$$

up to a total derivative. This condition entails constraints on curvature components $\iota_\varepsilon R^A$

Reminder: curvature components and contractions

Expansion of the 2-form R^A on the vielbein basis:

$$R^A = R^A_{BC} \sigma^B \sigma^C$$

Contractions along tangent vectors t_A :

$$\iota_A R^B = 2R^B_{AC} \sigma^C$$

For ex. super Poincaré curvatures:

$$R^a = \overset{\text{inner}}{\boxed{R^a_{bc}}} V^b V^c + \overset{\text{outer}}{\boxed{R^a_{b\alpha}}} V^b \psi^\alpha + \overset{\text{outer}}{\boxed{R^a_{\alpha\beta}}} \psi^\alpha \psi^\beta + \text{terms containing } \omega^{ab}$$

$$R^{ab} = R^{ab}_{cd} V^c V^d + R^{ab}_{c\alpha} V^c \psi^\alpha + R^{ab}_{\alpha\beta} \psi^\alpha \psi^\beta + \text{terms containing } \omega^{ab}$$

$$\rho^\alpha = \rho^\alpha_{cd} V^c V^d + \rho^\alpha_{c\beta} V^c \psi^\beta + \rho^\alpha_{\beta\gamma} \psi^\beta \psi^\gamma + \text{terms containing } \omega^{ab}$$

- Thus when $\iota_\varepsilon dL = 0$ the G diff variation of L is a **total derivative**

$$\ell_\varepsilon L^{(d)} = d\iota_\varepsilon L$$

- The G diff variation of L can be computed also using the **diff. variation of the fields** σ^A in L :

$$\ell_\varepsilon \sigma^A = (\nabla_G \varepsilon)^A + i_\varepsilon R^A$$

and using the **Leibniz rule** for ℓ_ε

- Inserting the values for $\iota_\varepsilon R^A$ determined by $\iota_\varepsilon dL = 0$ **we find the infinitesimal transformations on the fields** $\delta_\varepsilon \sigma^A$ under which **L varies into a total derivative.**

- All equations being form equations...

The restriction of the d -form L on S , i.e. **the spacetime Lagrangian** (i.e. the spacetime restriction of the d -form L , depending only on spacetime coordinates x), is then **invariant under the spacetime restriction of the transformation rules $\delta_\varepsilon \sigma^A$**

- This protocol provides a systematic method to find **geometric** (diff. invariant) actions with local **Lorentz** and **supersymmetry** invariances

Note:

$$\iota_\varepsilon dL = 0$$

usually relates outer components of the curvatures to inner components (“**rheonomic constraints**”)

Inner components are usually **left free** and only the **eq. of motion** will constrain them to satisfy differential conditions.

6. Symmetries of d=4 N=1 Supergravity

- Using curvature definitions and Bianchi identities:

$$dL = 2R^{ab}R^cV^d\varepsilon_{abcd} + 4\bar{\rho}\gamma_5\gamma_a\rho V^a - 4\bar{\psi}\gamma_5\gamma_a\rho R^a$$

- requesting invariance of L **under all G diff.s**, i.e. $\iota_\varepsilon dL = 0$ for all tangent vectors ε , implies the constraints on the outer components of the curvatures:

$$\begin{aligned} R^a &= R^a_{bc}V^bV^c \\ R^{ab} &= R^{ab}_{cd}V^cV^d + \boxed{\bar{\theta}_c^{ab}}\psi V^c \\ \rho^\alpha &= \rho^\alpha_{cd}V^cV^d \end{aligned}$$

 **outer**

where $\bar{\theta}_c^{ab} = \varepsilon^{abef}\bar{\rho}_{ef}\gamma_5\gamma_c + \varepsilon^{efg[a}\bar{\rho}_{ef}\gamma_5\gamma_g\delta_c^{b]}$

the inner components remaining free

- thus L is invariant under the transformations

$$\delta_\varepsilon \sigma^A = \ell_\varepsilon \sigma^A = (\nabla_G \varepsilon)^A + i_\varepsilon R^A$$

specializing the G -diff. parameter ε yields

Lorentz transformations (generated by $\ell_{\varepsilon^{cd} t_{cd}}$):

$$\delta V^a = \varepsilon^a_b V^b$$

$$\delta \omega^{ab} = d\varepsilon^{ab} + 2\omega^{[a}_c \varepsilon^{b]c} = \mathcal{D}\varepsilon^{ab}$$

$$\delta \psi = \frac{1}{4} \varepsilon^{ab} \gamma_{ab} \psi$$

diff.s in d dimensions (generated by $\ell_{\varepsilon^c t_c}$):

$$\delta V^a = d\varepsilon^a - \omega^a_b \varepsilon^b + 2R^a_{bc} \varepsilon^b V^c$$

$$\delta \omega^{ab} = 2R^{ab}_{cd} \varepsilon^c V^d - \bar{\theta}^{ab}_c \psi \varepsilon^c$$

$$\delta \psi = 2\rho_{ab} \varepsilon^a V^b$$

supersymmetry transformations (generated by $\ell_{\epsilon^\alpha t_\alpha}$):

$$\delta V^a = i\bar{\epsilon} \gamma^a \psi$$

$$\delta \omega^{ab} = \bar{\theta}^{ab}_c \epsilon V^c$$

$$\delta \psi = \mathcal{D}\epsilon \equiv d\epsilon - \frac{1}{4} \omega^{ab} \gamma_{ab} \epsilon$$

under these transformations the group manifold
lagrangian

$$L = R^{ab}V^cV^d\varepsilon_{abcd} + 4\bar{\psi}\gamma_5\gamma_a\rho V^a$$

transforms into a total derivative.

Considering now all fields and variation parameters
as only x-dependent, **the spacetime restriction of L
is invariant under the spacetime restriction of the
variations.**

The spacetime restriction of L is the
d=4 N=1 SG Lagrangian found in 1976

N=1 supergravity in d=4

Ferrara, Friedman, van Nieuwenhuizen 1976
Deser, Zumino 1976

Action

$$I_{SG} = \int_{M^4} (R\sqrt{-g} + \bar{\psi}_\mu \gamma_5 \gamma_a \mathcal{D}_\nu \psi_\rho V_\sigma^a \epsilon^{\mu\nu\rho\sigma}) d^4x$$

in form language:

$$I_{SG} = \int_{M^4} R^{ab} \wedge V^c \wedge V^d \epsilon_{abcd} + 4 \bar{\psi} \wedge \gamma_5 \gamma_d \rho \wedge V^d$$

with $R^{ab} = d\omega^{ab} - \omega^a_c \wedge \omega^{cb}$
 $\rho = d\psi - \frac{1}{4}\omega^{ab}\gamma_{ab}\psi$

vierbein

spin connection

gravitino

$$\textcircled{V^a} = V_\mu^a dx^\mu, \textcircled{\omega^{ab}} = \omega_\mu^{ab} dx^\mu, \textcircled{\psi} = \psi_\mu dx^\mu$$
$$R^{ab} = R^{ab}_{\mu\nu} dx^\mu \wedge dx^\nu = (\partial_{[\mu} \omega_{\nu]}^{ab} - \omega_{[\mu}^{ac} \omega_{\nu]}^b{}_c) dx^\mu \wedge dx^\nu$$

Invariances

- diffeomorphisms
- local Lorentz rotations
- local supersymmetry

$$\delta_\epsilon V^a = i \bar{\epsilon} \gamma^a \psi$$
$$\delta_\epsilon \psi = d\epsilon - \frac{1}{4} \omega^{ab} \gamma_{ab} \epsilon$$

2026: 50 years of supergravity

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Progress toward a theory of supergravity*

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As a new approach to supergravity, an action containing only vierbein and Rarita-Schwinger fields ($V_{\mu\nu}$ and ψ_μ) is presented together with supersymmetry transformations for these fields. The action is explicitly shown to be invariant except for a ψ^5 term in its variation. This term may also vanish, depending on a complicated calculation. (Added note: This term has now been shown to vanish by a computer calculation, so that the action presented here does possess full local supersymmetry.)

Even early in the development of the Fermi-Bose supersymmetry concept, it was thought that the new fermionic symmetry transformation might be important for the theory of gravitation.¹ Two similar but apparently inequivalent approaches to this theory of "supergravity" have been formulated by Arnowitt, Nath, and Zumino² and by Zumino.³ These approaches exploit the geometry of "superspace,"⁴ a manifold parametrized by four anticommuting spinor coordinates θ_α in addition to the normal Riemannian coordinates x^μ . The theories are formulated in terms of superfields which contain a very large number of ordinary fields—i.e., vectors, tensors, spinors, etc. Although it is expected that some component fields are merely generalized gauge excitations and not true physical fields, the physical content of the Arnowitt-Nath-Zumino theories has never been spelled out, but there are indications² that, as is perhaps desirable, the approaches necessarily bring in gauge vector and spin- $\frac{1}{2}$ particles in addition to tensor and spin- $\frac{3}{2}$ particles.

In this note we report on progress in a very different approach to supergravity in which we commit ourselves from the start to a formulation without superspace in which the only fields in the gravitational supermultiplet are the metric tensor $g_{\mu\nu}(x)$ [or, equivalently, the vierbein field $V_{\mu\nu}(x)$] and a Rarita-Schwinger field $\psi_\mu(x)$. If fully successful, we would then expect to adjoin matter supermultiplets of lower-spin fields in much the same way as matter fields are treated in conventional gravitation.

There is a theorem⁵ in the usual theory of global supersymmetry which demonstrates the existence of irreducible representations of the graded Lie algebra of supersymmetry charges and Poincaré group generators. Some of these representations act in the Hilbert space of helicity states of two

massless particles, one neutral boson and one Majorana fermion of adjacent spins J and $J + \frac{1}{2}$ (for any $J=0, \frac{1}{2}, 1, \dots$). It is therefore known that a representation exists containing states of massless spin- $\frac{3}{2}$ and spin-2 particles, and it was suggested⁶ earlier that these particles form the gravitational supermultiplet. The theorem does not guarantee that there exists a corresponding interacting quantum field theory, but it is reasonable to hope that it exists, and this is the basic mathematical motivation for our approach. Many questions can be asked about the physical motivation and consistency of both this treatment and the entire concept of supergravity. We shall discuss some of them at the end of this note, and we proceed now to the formulation.

The starting point of our approach is the generally covariant action⁷

$$I = \int d^4x (\mathcal{L}_2 + \mathcal{L}_{3/2}) \\ = \int d^4x \left[\frac{1}{4} \kappa^{-2} \sqrt{-g} R - \frac{1}{2} \epsilon^{\lambda\mu\nu} \bar{\psi}_\lambda(x) \gamma_5 \gamma_\mu D_\nu \psi_\rho(x) \right] \quad (1)$$

describing the interaction of vierbein fields and Rarita-Schwinger⁸ fields subject to the Majorana constraint $\psi_\rho(x) = C \bar{\psi}_\rho(x)^T$. The covariant derivative⁹

$$D_\nu \psi_\rho(x) = \partial_\nu \psi_\rho(x) - \Gamma_{\nu\rho}^\sigma \psi_\sigma + \frac{1}{2} \omega_{\nu ab} \sigma^{ab} \psi_\rho \quad (2)$$

involves the standard Christoffel symbol (although it cancels in $\mathcal{L}_{3/2}$ because of the tensor density $\epsilon^{\lambda\mu\nu}$) and the vierbein connection

$$\omega_{\nu ab} = \frac{1}{2} [V_a^\mu (\partial_\nu V_{b\mu} - \partial_\mu V_{b\nu}) + V_a^\rho V_b^\sigma (\partial_\sigma V_{\rho\mu}) V_{\nu}^\mu] \\ - (a \leftrightarrow b), \quad (3)$$

while

$$\sigma_{ab} = \frac{1}{4} [\gamma_a, \gamma_b].$$

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CONSISTENT SUPERGRAVITY

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A combined spin 2 – spin 3/2 extension of general relativity is given which is both free of the usual higher spin inconsistencies and invariant under local supersymmetry transformations.

The unification of the gravitational field with a spin 3/2 system is a natural goal within the framework of supersymmetry [1]. In constructing such a theory one faces obvious problems due to the highly non-linear nature of general relativity [2] and to the well-known difficulties encountered in coupling higher spin fields in a consistent way. We shall show here that the simplest candidate, namely the sum of the Einstein action and that for a massless, minimally coupled, Rarita-Schwinger [3] Majorana field fulfills the consistency criteria. As we shall also see, this fact is related to the invariance of the theory under local supersymmetry transformations^{†1}.

The key to our results lies in the use of the first order formalism for gravitation, in which vierbeins and connection coefficients are treated independently (a convenient description of the first order formalism can be found in a paper by Kibble [5]). Minimal coupling in this sense implies the existence of torsion, or of non-minimal contact interactions in second order language. The first order formulation with torsion is closely related to the description of supergravity in superspace [1, 6].

The combined Lagrangian has the form^{†2}

$$L = -\frac{1}{2} eR - \frac{1}{2} \epsilon^{\lambda\mu\nu\rho} \bar{\psi}_\lambda \gamma_5 \gamma_\mu D_\nu \psi_\rho \quad (1)$$

where

$$e = \det e_{\mu a}, \quad R = e_a^\mu e_b^\nu R_{\mu\nu}{}^{ab}. \quad (2)$$

The covariant derivative on ψ_μ is defined according to its spin 1/2 content only

$$D_\mu = \partial_\mu - \frac{1}{2} \omega_{\mu ab} \Sigma^{ab}, \quad \Sigma^{ab} = \frac{1}{4} [\gamma^a, \gamma^b] \quad (3)$$

and satisfies

$$[D_\mu, D_\nu] = -\frac{1}{2} R_{\mu\nu ab} \Sigma^{ab}. \quad (4)$$

This form for D_μ is strongly suggested by the Maxwell-like gauge invariance of the flat space action under $\delta\psi_\mu = 2\partial_\mu \alpha$ and preserves the simplest definition of the curl. This results in a particularly simple form of the torsion. The vierbeins $e_{\mu a}$, the connection coefficients $\omega_{\mu ab}$ and the Majorana vector-spinor ψ_μ are to be varied independently. Note that we have not introduced any auxiliary fields. They will be useful, however, when coupling to matter is included. The equations of motion are

$$R^\lambda{}_\mu \equiv \epsilon^{\lambda\mu\nu\rho} (\gamma_\mu D_\nu \psi_\rho - \frac{1}{4} \gamma_\tau C_{\mu\nu}{}^\tau \psi_\rho) = 0 \quad (5)$$

$$C_{\mu\nu}{}^\tau = \frac{1}{2} \bar{\psi}_\mu \gamma^\tau \psi_\nu \quad (6)$$

and

$$G^{\tau\mu} = \frac{1}{2} \epsilon^{\lambda\mu\nu\rho} \bar{\psi}_\lambda \gamma_5 \gamma^\tau D_\nu \psi_\rho. \quad (7)$$

Here

$$G^\tau{}_a = R^\tau{}_a - \frac{1}{2} e_a{}^\tau R, \quad R_{\tau a} = R_{\tau\lambda a}{}^\lambda \quad (8)$$

is the (non-symmetric) Einstein tensor and

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^{†1} Local supersymmetry transformations, with parameters depending arbitrarily upon the space-time co-ordinates were first discussed in ref. [4], where their existence was stated for two space-time dimensions and conjectured for four dimensions. The commutator of two local supersymmetry transformations contains a general co-ordinate transformation.

^{†2} We choose units in which $\kappa = 1$. Our gamma matrices satisfy $(\gamma_0)^2 = (\gamma_5)^2 = -1$. If we use the Majorana representation the matrices γ_μ and γ_5 are real and the field ψ_μ Hermitean, with $\bar{\psi}_\mu = \psi_\mu \gamma^0$. We also take $\epsilon^{0123} = -\epsilon_{0123} = 1$.

7. Equations of motion of d=4 N=1 Supergravity

- Expanding the field equations $\frac{\partial L}{\partial \phi} + d \frac{\partial L}{\partial (d\phi)} = 0$
(3-form equations) on a complete G basis $\sigma^A \sigma^B \sigma^C$
with $\sigma^A = (V^a, \omega^{ab}, \psi^\alpha)$

1) determines the **outer components** of the curvatures

$$\begin{aligned} R^a &= R^a_{bc} V^b V^c \\ R^{ab} &= R^{ab}_{cd} V^c V^d + \boxed{\bar{\theta}_c^{ab}} \psi V^c \\ \rho^\alpha &= \rho^\alpha_{cd} V^c V^d \end{aligned}$$

 **outer**

where $\bar{\theta}_c^{ab} = \varepsilon^{abef} \bar{\rho}_{ef} \gamma_5 \gamma_c + \varepsilon^{efg[a} \bar{\rho}_{ef} \gamma_5 \gamma_g \delta_c^{b]}$

2) yields the **propagation equations**

$$\begin{aligned} R^a_{bc} &= 0 && \text{(torsion eq.)} \\ R^{ac}_{bc} - \frac{1}{2} \delta_b^a R^{cd}_{cd} &= 0 && \text{(Einstein eq.)} \\ \gamma^a \rho_{ab} &= 0 && \text{(gravitino eq.)} \end{aligned}$$

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same as from

$$\iota_\varepsilon dL = 0$$

to be expected

since dL=0 on shell

where $\bar{\theta}_c^{ab} = \varepsilon^{abef} \bar{\rho}_{ef} \gamma_5 \gamma_c + \varepsilon^{efg[a} \bar{\rho}_{ef} \gamma_5 \gamma_g \delta_c^{b]}$

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$$R^a_{bc} = 0 \quad \text{(torsion eq.)}$$

$$R^{ac}_{bc} - \frac{1}{2} \delta_b^a R^{cd}_{cd} = 0 \quad \text{(Einstein eq.)}$$

$$\gamma^a \rho_{ab} = 0 \quad \text{(gravitino eq.)}$$

8. Free differential algebras (FDA)

Sullivan, 1977
D'Auria, Fré 1982

- convenient algebraic setting for field theories with antisymmetric tensors (***p*-forms**)
- generalize Cartan-Maurer eq.s of group manifold G
1-form vielbeins σ^A , by including ***p*-forms** B^i
- **example** : ordinary Cartan-Maurer 1-forms σ^A supplemented by a **single *p*-form** B^i in a representation D^i_j of G

$$\begin{aligned}
 d\sigma^A + \frac{1}{2}C^A_{BC} \sigma^B \sigma^C &= 0 \\
 \underbrace{dB^i + C^i_{Aj} \sigma^A B^j}_{\nabla B^i} + \frac{1}{(p+1)!} \underbrace{C^i_{A_1 \dots A_{p+1}} \sigma^{A_1} \dots \sigma^{A_{p+1}}}_{C^i} &= 0
 \end{aligned}$$

suggests a multibracket

- dual formulation: **L_∞ algebras**

- In the multibracket formulation, L_∞ algebras were first introduced in Stasheff 1992, Lada and Stasheff 1993 and later recognized to be dual to FDA's Fiorenza, Sati and Schreiber 2014; D' Auria, LC 2026

9. Selfdual forms

- There are instances where $\iota_\varepsilon dL = 0$ implies constraints also on inner (VV) components of the curvatures. This happens notably when selfdual p-forms are present, as in $d=6$ or $d=10$ SG (there constraints are selfduality conditions)
- In this case L varies into a total derivative only modulo the selfduality condition. In particular L is not supersymmetric, only the e.o.m are.
- The e.o.m derived by varying the group manifold lagrangian include the selfduality condition on the p-form curvatures (in their outer projections)

- The e.o.m derived by varying the spacetime lagrangian **do not include the selfduality condition**, and the spacetime lagrangian is supersymmetric only modulo the selfduality condition. In this case the e.o.m. are not supersymmetric, since they do not include selfduality.
- Strategies to find supersymmetric spacetime Lagrangians containing self-dual forms: **Siegel 1984, Pasti Sorokin Tonin 1995, Sen 2015**

Table 1. Fields of 2b supergravity in the SU(1,1)/U(1) formulation.

Field	SU(1,1) repres.	U(1) charge	Deg. of freed. (on-shell)
V_μ^a	1	0	35
ψ_μ	1	$\frac{1}{2}$	2×56
$A_{\mu\nu}^\alpha$	2	0	2×28
$B_{\mu\nu\rho\sigma}$	1	0	35
λ	1	$\frac{3}{2}$	2×8
$V_\pm^\alpha$	2	± 1	2

D = 10 supergravity (2b)

FDA

LC 1987

Table 2. The free differential algebra of 2b supergravity.

$$\begin{aligned}
 R^a &= \mathcal{D}V^a - i\bar{\psi}\Gamma^a\psi = 0 \\
 R^{ab} &= d\omega^{ab} - \omega^{ac}\omega^{cb} = 0 \\
 \rho &= \mathcal{D}\psi \equiv d\psi - \frac{1}{4}\omega^{ab}\Gamma^{ab}\psi - \frac{1}{2}iQ\psi = 0 \\
 R(A)^\alpha &= dA^\alpha + 2iV_+^\alpha\bar{\psi}\Gamma^\alpha\psi^*V^a + 2iV_-^\alpha\bar{\psi}^*\Gamma^\alpha\psi V^a = 0 \\
 R(B) &= dB + \bar{\psi}\Gamma^{abc}\psi V^{abc} + \frac{3}{4}\epsilon_{\alpha\beta}A^\alpha(V_+^\beta\bar{\psi}\Gamma^\alpha\psi^* + V_-^\beta\bar{\psi}^*\Gamma^\alpha\psi)V^a = 0 \\
 \mathcal{D}\lambda &= d\lambda - \frac{1}{4}\omega^{ab}\Gamma^{ab} - \frac{3}{2}iQ\lambda = 0 \\
 \mathcal{D}V_\pm^\alpha &= dV_\pm^\alpha \mp iQV_\pm^\alpha = 0.
 \end{aligned}$$

Table 6. The Lagrangian.

$$\mathcal{L} = \mathcal{L}_\omega + \mathcal{L}_V + \mathcal{L}_A + \mathcal{L}_B + \mathcal{L}_{F_3} + \mathcal{L}_{F_5} + \mathcal{L}_{P_+P_-} + \mathcal{L}_{V_\pm} + \mathcal{L}_\psi + \mathcal{L}_\lambda + \mathcal{L}_{(2,2)} + \mathcal{L}_{(1,1)}$$

$$\mathcal{L}_\omega = R^{a_1 a_2} V^{a_3 - a_{10}} \epsilon_{a_1 - a_{10}}$$

$$\mathcal{L}_V = 336i R^a \bar{\psi} \Gamma_{a_1 - a_5} \psi V^{a_1 - a_5} - \frac{i}{4} R^a \bar{\lambda} \Gamma^{a a_1 a_2} \lambda V^{a_3 - a_{10}} \epsilon_{a_1 - a_{10}}$$

$$\mathcal{L}_A = \left(-84i \bar{\psi} \Gamma^{a_1 - a_5} \psi^* V^{a_1 - a_5} V_+^\beta \epsilon_{\alpha\beta} - \frac{7}{12} \bar{\psi} \Gamma^{a_1 - a_4} \lambda V^{a_5 - a_{10}} \epsilon_{a_1 - a_{10}} V_-^\beta \epsilon_{\alpha\beta} \right. \\ \left. + F_{a_1 - a_3}^{\beta_1} V_-^{\beta_2} \epsilon_{\beta_1 \beta_2} V^{a_4 - a_{10}} \epsilon_{a_1 - a_{10}} V_+^\beta \epsilon_{\alpha\beta} + \text{c.c.} \right) R(A)^\alpha$$

$$\mathcal{L}_B = \left(-560 \bar{\psi} \Gamma^{a_1 - a_3} \psi V^{a_1 - a_3} - \frac{14}{3} F^{a_1 - a_5} V^{a_6 - a_{10}} \epsilon_{a_1 - a_{10}} \right. \\ \left. + \frac{7}{60} \bar{\lambda} \Gamma^{a_1 - a_5} \lambda V^{a_1 - a_5} \right) \left(R(B) + \frac{3i}{8} \epsilon_{\alpha\beta} A^\alpha R(A)^\beta \right) \\ - 210i \epsilon_{\alpha\beta} dA^\alpha dA^\beta B$$

$$\mathcal{L}_{F_3} = \frac{1}{120} \epsilon_{\beta_1 \beta_2} V_-^{\beta_1} F_{b_1 - b_3}^{\beta_2} \epsilon_{\beta_3 \beta_4} V_+^{\beta_3} F_{b_1 - b_3}^{\beta_4} V^{a_1 - a_{10}} \epsilon_{a_1 - a_{10}} \\ + \frac{1}{12} (\epsilon_{\beta_1 \beta_2} V_-^{\beta_1} F_{b_1 b_2 a}^{\beta_2} \bar{\psi} \Gamma^{b_1 b_2} \lambda + \text{c.c.}) V^{a_2 - a_{10}} \epsilon_{a a_2 - a_{10}}$$

$$\mathcal{L}_{F_5} = \frac{1}{108} F_{b_1 - b_5} F_{b_1 - b_5} V^{a_1 - a_{10}} \epsilon_{a_1 - a_{10}}$$

$$\mathcal{L}_{P_+P_-} = \frac{2}{90} P_{a+}^{\alpha_1} V_+^{\alpha_2} \epsilon_{\alpha_1 \alpha_2} P_{a-}^{\alpha_3} V_-^{\alpha_4} \epsilon_{\alpha_3 \alpha_4} V^{a_1 - a_{10}} \epsilon_{a_1 - a_{10}}$$

$$\mathcal{L}_{V_\pm} = \bar{\psi} \Gamma_{a_1 a_2} \lambda^* (\mathcal{D} V_+^{\beta_1} - V_-^{\beta_1} \bar{\psi}^* \lambda) V_+^{\beta_2} \epsilon_{\beta_1 \beta_2} V^{a_3 - a_{10}} \epsilon_{a_1 - a_{10}} \\ - \frac{2}{9} (\mathcal{D} V_+^{\beta_1} - V_-^{\beta_1} \bar{\psi}^* \lambda) V_+^{\beta_2} \epsilon_{\beta_1 \beta_2} P_{a_1 -}^{\beta_3} V^{\beta_4} \epsilon_{\beta_3 \beta_4} V^{a_2 - a_{10}} \epsilon_{a_1 - a_{10}} + \text{c.c.}$$

$$\mathcal{L}_\psi = \frac{8i}{3} \bar{\psi} \Gamma^{a_1 - a_3} \mathcal{D} \psi V^{a_4 - a_{10}} \epsilon_{a_1 - a_{10}} + \text{c.c.}$$

$$\mathcal{L}_\lambda = \frac{i}{9} \bar{\lambda} \Gamma^{a_1} \mathcal{D} \lambda V^{a_2 - a_{10}} \epsilon_{a_1 - a_{10}} + \text{c.c.}$$

$$\mathcal{L}_{(2,2)} = -42 \bar{\psi}^* \Gamma^b \psi \bar{\psi} \Gamma^{a_1 - a_5} \psi^* V^b V^{a_1 - a_5} + \text{c.c.}$$

$$\mathcal{L}_{(1,1)} = \frac{5}{32} \bar{\psi} \Gamma^b \psi \bar{\lambda} \Gamma^{b a_1 a_2} \lambda V^{a_3 - a_{10}} \epsilon_{a_1 - a_{10}} + \frac{9}{32} \bar{\psi} \Gamma^{b a_1 a_2} \psi \bar{\lambda} \Gamma^b \lambda V^{a_3 - a_{10}} \epsilon_{a_1 - a_{10}} \\ - \frac{7}{2} \bar{\psi} \Gamma^{a_1 - a_3} \psi \bar{\lambda} \Gamma^{a_4 - a_8} \lambda V^{a_1 - a_8} - \frac{21}{2} \bar{\psi} \Gamma^{a_4 - a_8} \psi \bar{\lambda} \Gamma^{a_1 - a_3} \lambda V^{a_1 - a_8}$$

The group manifold lagrangian
of 2b supergravity

Pesando, LC 1993

D = 6 supergravity

D'Auria, Fré, Regge 1983

FDA

$$R^{ab} = d\omega^{ab} - \omega^a_c \wedge \omega^{cb}$$

$$R^a = \mathcal{D}V^a - \frac{i}{2} \bar{\psi} \wedge \Gamma^a \psi$$

$$\rho = \mathcal{D}\psi$$

$$R^\otimes = dB - \frac{i}{2} \bar{\psi} \wedge \Gamma^a \psi \wedge V_a$$

Action

$$\begin{aligned} \mathcal{A} = \int & \left\{ \frac{1}{4} R^{ab} \wedge \varepsilon_{abc_1 \dots c_4} V^{c_1} \wedge V^{c_2} \wedge V^{c_3} \wedge V^{c_4} + \right. \\ & + 3i R^a \wedge \bar{\psi} \wedge \Gamma^b \psi \wedge V_a \wedge V_b - 3i R^\otimes \wedge \bar{\psi} \wedge \Gamma^a \psi \wedge V_a \\ & - i(\bar{\rho} \wedge \Gamma_{abc} \psi - \bar{\psi} \wedge \Gamma_{abc} \rho) \wedge V^a \wedge V^b \wedge V^c + \\ & \left. + 2\sqrt{3} R^a \wedge R^\otimes \wedge V_a \right\} \end{aligned}$$

Some advantages of the group geometric approach

- all fields have a **group-geometric origin**
- all (super)symmetries originate from **diff.s** on G
- **systematic procedure** to construct locally supersymmetric actions
- **closer contact** with usual component approach (wrt. superfield formalism)
- the group manifold action **interpolates** between the component and the superfield actions of the same supergravity theory **Catenacci, Grassi, LC 2014**
- in **$d=6$ and $d=10$ SG** the **selfdual** conditions on the 3-form and 5-form curvatures are a yield of the e.o.m. on G
D'Auria, Fré, Regge 1984; LC 1987; Pesando, LC 1993

Thank you !



Supplementary slides

Note on the closure of (super) symmetry algebra

- **algebra** of Lie derivatives **closes**:

$$[\ell_{\varepsilon_1^A} t_A, \ell_{\varepsilon_2^B} t_B] = \ell_{\varepsilon_3^C} t_C$$

with

$$\varepsilon_3^C = \varepsilon_1^A \partial_A \varepsilon_2^C - \varepsilon_2^A \partial_A \varepsilon_1^C + \varepsilon_1^A \varepsilon_2^B (C_{AB}^C - 2R_{AB}^C)$$

provided R^A satisfies **Bianchi identities**.

- Thus if curvatures are unconstrained, Bianchi id.s hold because of R^A definition. If the curvatures are **constrained** (for ex. their outer components are given in terms of their inner components), Bianchi id.s may imply **further constraints** on the inner components $\longrightarrow$ e.o.m

- In this case symmetry transformations on the fields close only **on shell**.
- This can be **exploited to find the propagation eq.s** by postulating constraints on the outer components of R^A , which is equivalent to **postulate the field variations**. The **closure** of these variations then **imply the propagation equations** for the inner components. In this case even if L is not known, the e.o.m. can be deduced from the symmetry algebra, see for ex **2b supergravity in d=10**
Green, Schwarz 1982.
- To have **off shell** closure we need **auxiliary fields**, thus enlarging the set of Cartan-Maurer eq.s and curvatures.
- For ex. auxiliary fields for $N=1$, $d=4$ SG
Ferrara, van Nieuwenhuizen 1978; Sohnius, West 1981

6. Free differential algebras (FDA)

Sullivan, 1977
D'Auria, Fré 1982

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- generalize Cartan-Maurer eq.s of group manifold G
1-form vielbeins σ^A , by including ***p*-forms** B^i
- **example** : ordinary Cartan-Maurer 1-forms σ^A supplemented by a **single *p*-form** B^i in a representation D^i_j of G

$$d\sigma^A + \frac{1}{2}C^A_{BC} \sigma^B \sigma^C = 0$$

$$dB^i + C^i_{Aj} \sigma^A B^j + \frac{1}{(p+1)!} C^i_{A_1 \dots A_{p+1}} \sigma^{A_1} \dots \sigma^{A_{p+1}} = 0$$

- taking d of l.h.s. and requiring $d^2 = 0$



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$$d\sigma^A + \frac{1}{2}C^A_{BC} \sigma^B \sigma^C = 0$$

$$\underbrace{dB^i + C^i_{Aj} \sigma^A B^j}_{\nabla B^i} + \frac{1}{(p+1)!} C^i_{A_1 \dots A_{p+1}} \sigma^{A_1} \dots \sigma^{A_{p+1}} = 0$$

***p+1* cocycle in G**

Generalized Jacobi identities

$$C^A_{B[C} C^B_{DE]} = 0$$

usual Jacobi id.s

$$C^i_{Aj} C^j_{Bk} - C^i_{Bj} C^j_{Ak} = C^C_{AB} C^i_{Ck}$$

representation condition

$$2 C^i_{A_1 j} C^j_{A_2 \dots A_{p+2}] - (p+1) C^i_{B[A_1 \dots A_p} C^B_{A_{p+1} A_{p+2}]} = 0$$

cocycle condition

- representation condition: the constants C^i_{Aj} must be a **representation of the Lie algebra**
- then $dB^i + C^i_{Aj} \sigma^A B^j = \nabla_G B^i$ **is the G-covariant derivative**
- cocycle condition: expresses the closure under ∇_G of $C^i = C^i_{A_1 \dots A_{p+1}} \sigma^{A_1} \dots \sigma^{A_{p+1}}$ **(p+1 cocycle in G)**

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- **example** : ordinary Cartan-Maurer 1-forms σ^A supplemented by a **single *p*-form** B^i in a representation D^i_j of G

suggests a multibracket

$$\begin{aligned}
 d\sigma^A + \frac{1}{2}C^A_{BC} \sigma^B \sigma^C &= 0 \\
 \underbrace{dB^i + C^i_{Aj} \sigma^A B^j}_{\nabla B^i} + \frac{1}{(p+1)!} \underbrace{C^i_{A_1 \dots A_{p+1}} \sigma^{A_1} \dots \sigma^{A_{p+1}}}_{C^i} &= 0
 \end{aligned}$$

Construction of FDA's

- To extend a Lie algebra to a **FDA**: need a **covariantly closed (p+1)-form** C^i
- given such a form C^i , $C^i +$ covariantly closed (p+1)-form still yields a FDA. But if this cov. closed form is cov. exact ($= \nabla \Phi^i$) $C^i + \nabla \Phi^i$ leads to an **equivalent FDA** via the redefinition

$$B^i \rightarrow B^i + \Phi^i$$

- Thus **inequivalent FDA's** are classified by nontrivial cohomology classes of the covariant derivative ∇ , i.e. by **Chevalley cohomology**
- given a **FDA**, construct a **Lagrangian** with the **p-forms** and **G vielbeins** as fundamental fields, using the building rules for group manifold actions

Example: FDA of d=11 supergravity

D'Auria, Fré 1982

$$d\omega^{ab} - \omega^a_c \omega^{cb} = 0 \quad [= R^{ab}]$$

$$dV^a - \omega^a_b V^b - \frac{i}{2} \bar{\psi} \Gamma^a \psi = 0 \quad [= R^a]$$

$$d\psi - \frac{1}{4} \omega^{ab} \Gamma_{ab} \psi = 0 \quad [= \rho]$$

$$dA - \frac{1}{2} \bar{\psi} \Gamma_{ab} \psi V^a V^b = 0 \quad [= R(A)]$$

nontrivial 4-cocycle

- the d=11 **Fierz identity** $\bar{\psi} \Gamma_{ab} \psi \bar{\psi} \Gamma^a \psi = 0$ ensures FDA closure ($d^2 = 0$)
- extends the superPoincaré Lie algebra in d=11 with a **3-form A** in the identity representation
- $C^i_{A_1 \dots A_{p+1}} \longrightarrow C_{\alpha\beta ab} = -12(C\Gamma_{ab})_{\alpha\beta}$

- The **lagrangian of d =11 supergravity** can be written as a **11- form**, made out of (exterior) products of fields and curvatures, therefore invariant by construction under diff.s. **D'Auria, Fré 1982**
- Original construction (in components) by **Cremmer, Julia, Scherk 1978**

d=11 (first order) supergravity Lagrangian

$$\begin{aligned}
\mathcal{L} = & -\frac{1}{9}R^{a_1a_2}V^{a_3} \wedge \dots \wedge V^{a_{11}}\epsilon_{a_1\dots a_{11}} + \frac{7}{30}T^a \wedge V_a \wedge \bar{\psi}\Gamma^{b_1\dots b_5} \wedge \psi V^{b_6\dots b_{11}}\epsilon_{b_1\dots b_{11}} \\
& + 2\bar{\rho}\Gamma_{c_1\dots c_8}\psi \wedge V^{c_1\dots c_8} - 84F^{(4)} \wedge (i\bar{\psi}\Gamma_{b_1\dots b_5} \wedge \psi V^{b_1\dots b_5} - 10A^{(3)} \wedge \bar{\psi}\gamma_{ab}\psi V^{ab}) \\
& + \frac{1}{4}\bar{\psi}\Gamma^{a_1a_2}\psi \wedge \bar{\psi}\Gamma^{a_3a_4}\psi \wedge V^{a_5\dots a_{11}}\epsilon_{a_1\dots a_{11}} - 210\bar{\psi}\Gamma^{a_1a_2}\psi \wedge \bar{\psi}\Gamma^{a_3a_4}\psi \wedge V^{a_1\dots a_4} \wedge A^{(3)} \\
& - 840 F^{(4)} \wedge F^{(4)} \wedge A^{(3)} - \frac{1}{330}F_{a_1\dots a_4}F^{a_1\dots a_4}V^{c_1\dots c_{11}}\epsilon_{c_1\dots c_{11}} + 2F_{a_1\dots a_4}F^{(4)}V^{a_5\dots a_{11}}\epsilon_{a_1\dots a_{11}}.
\end{aligned}$$