Third-way extensions of massive gravity theories and nonrelativistic limits of higher-spin theories

Jamal Hammoud

Based on work done in collaboration with
Maro Cvitan, Nihat S. Deger and Jan Rosseel

University of Zagreb Faculty of Science
Department of Physics

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PART 1: Third-way extension of massive gravity in AdS_3

Massive modes in 3D gravity

Three-dimensional gravity provides a tractable laboratory for quantum gravity and holography, notably through AdS_3/CFT_2[1] ...

...with the caveat:

Pure Einstein gravity in three dimensions has no local propagating bulk degrees of freedom.

If one wants local gravitational excitations, mimicking higher-dimensional setups, Einstein gravity must be modified.

Local dynamics without breaking diffeomorphism invariance

Fully nonlinear theories such as TMG, NMG, GMG and **MMG** have linearized spectra containing **massive spin-2 modes.**

[1] [Witten, 2007]

The massive gravity theories

The relevant massive-gravity theories are obtained by supplementing the Einstein–Hilbert action with topological or higher-derivative terms:

$$\mathcal{L}_{\mathrm{GMG}} = \underbrace{\underbrace{\frac{1}{\mu}\left(\Gamma \mathrm{d}\Gamma + \frac{2}{3}\Gamma^3\right) + \overbrace{\mathcal{L}_{\mathrm{EH}}}^{}}_{\mathrm{TMG}} + \frac{1}{m^2}\left(R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2\right)}_{\mathrm{GMG}} .$$

with $\overbrace{\mathcal{L}_{\mathrm{EH}} + \frac{1}{m^2}\left(R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2\right)}^{\mathrm{NMG}}$.

theory	modification	derivative order
TMG[1], (**MMG***[2])	Lorentz–Chern–Simons*	3
NMG[3]	curvature-squared	4
GMG [4]	TMG + NMG	4

[1] [Deser et al., 1982]
[2] [Bergshoeff et al., 2014]
[3] [Bergshoeff et al., 2009a]
[4] [Bergshoeff et al., 2009b]

What makes massive gravity theories unitary?

bulk unitarity (one for each massive mode)	**boundary unitarity** (two conditions)
no-ghost no-tachyon	positive central charges

- Most known massive gravity theories can be made unitary only either in the bulk or at boundary.
- This applies to TMG, NMG and GMG and is commonly referred to as the *bulk-boundary clash.*
- **MMG is the only known fully (bulk+boundary) unitary theory!**

What sets MMG apart?

MMG has a special property – **third-way consistency.**

The nonlinear structure of MMG – third-way consistency

MMG's equation of motion $\mathcal{E}_{\mu\nu}$ is third-way consistent[1], meaning that it is only covariantly-conserved on-shell:

$$\nabla^{\mu}\mathcal{E}_{\mu\nu} = A_{\nu}{}^{\rho\sigma}\mathcal{E}_{\rho\sigma}\,.$$

Such equations cannot arise from metric-only actions.

MMG can be understood as a **third-way extension** of Einstein's theory:

$$\mathcal{L}_{\mathrm{MMG}} = \mathcal{L}_{\text{Einstein Gravity}}[e] + \mathcal{L}_{\text{3rd-way}}[e,\varpi]\,.$$

[1] [Bergshoeff et al., 2015a]

The nonlinear structure of MMG – third-way extension

Similarly, one can third-way extend any metric-only theory $\mathcal{L}_0[e]$ [1]:

$$\mathcal{L}_{\text{third-way extended}} = \mathcal{L}_0[e] + \underbrace{\tau\varepsilon^{\mu\nu\rho} e_\mu{}^a D[\varpi]_\nu e_{\rho a} + \kappa \mathcal{L}_{\text{CS}}[\varpi]}_{\equiv \mathcal{L}_{\text{3rd-way}}[e,\varpi]} .$$

- The extension introduces two coupling constants: τ and κ.
- The additional connection $\varpi_\mu{}^a$ cannot be integrated out of the Lagrangian
- Increases the derivative order of the theory by one, adding an additional propagating mode.

This is a mechanism to generate third-way consistent theories.

[1] [Deger et al., 2024]

Why third-way extend?

The third-way extension enlarges both the parameter space and the propagating spectrum:

new couplings: $(\tau, \kappa) \longrightarrow$ **larger parameter space**, new field: $\varpi_\mu{}^a \longrightarrow$ **additional bulk DOF**

... and what theory do we third-way extend?

$$\mathrm{TMG} \xleftarrow{m^2 \to \infty} \mathrm{GMG}(\Lambda_0, \sigma, \mu, m^2) \xrightarrow{\mu \to \infty} \mathrm{NMG}.$$

theory	massive modes	unitarity conditions	parameter space	Unitary?
MMG	1	2+(1+1)	4	✓
GMG	2	2+(2+2)	4	bulk only
3rd-way GMG	3	2+(3+3)	6	?

GMG in first-order formalism

Third-way extended GMG is best formulated in first-order form:

$$\mathcal{L}_{\text{3rd-GMG}}(e,\omega,h,f,\varpi) = \mathcal{L}_{\text{GMG}}(e,\omega,h,f) + \mathcal{L}_{\text{3rd-way}}(e,\varpi)\,,$$

with ω, h and f as auxiliary fields.
Using the auxiliary fields, the Lagrangian can be cast into Chern-Simons-like form[1]

$$\mathcal{L}^{\text{diag}}_{\text{3rd-GMG}} = \mathcal{K}_{AB} X^A \cdot \mathrm{d}X^B + \mathcal{M}_{AB\mathring{e}} \cdot X^B \times X^C\,, \qquad (X_\mu{}^a)^A = (e_\mu{}^a, \omega_\mu{}^a, h_\mu{}^a, f_\mu{}^a, \varpi_\mu{}^a)\,.$$

Diagonalization then yields the masses

$$\{\frac{1}{\ell}, -\frac{1}{\ell}, m_1, m_2, m_3\}\,,$$

with $\pm\frac{1}{\ell}$ as the two massless Einstein modes, and m_i the three real solutions of a cubic polynomial–**the massive propagating bulk modes.**

[1] [Bergshoeff et al., 2015b]

Unitarity conditions for 3rd-GMG

The diagonalized linearized Lagrangian reads:

$$\mathcal{L}^{\text{diag}}_{\text{3rd-GMG}} = a_+ \mathcal{L}_{+1/\ell} + a_- \mathcal{L}_{-1/\ell} + \sum_{i=1}^{3} b_i \mathcal{L}_{m_i}$$

Bulk unitarity		Boundary unitarity
No tachyons	**No ghosts**	**Positive central charges**
$m_i^2 > \frac{1}{\ell^2}, \qquad i = 1, 2, 3$	$b_i m_i < 0, \qquad i = 1, 2, 3$	$c^\pm > 0 \iff \mp a_\pm > 0$

All three sets of conditions must be satisfied simultaneously.

The unitarity conditions cannot be satisfied simultaneously

Performing the parameter analysis of the full 3rd-GMG theory then leads to the conclusions:

Result

- 3rd-GMG admits a unitary linearized bulk spectrum **and is the first fifth-order three-dimensional gravity model to do so.**
- Throughout the bulk-unitary region, the two boundary central charges **cannot both be positive.** Third-way extension does not improve the unitarity properties of GMG.

Let's stay positive though, part 2 is coming up!

PART 2: Nonrelativistic limits of self-dual theories

What is a self-dual theory, and why nonrelativistic?

Self-dual modes appear precisely in the generic linearized massive gravity Lagrangian:

$$\mathcal{L}^{\text{diag}}_{\text{3rd-GMG}} = a_+ \mathcal{L}_{+1/\ell} + a_- \mathcal{L}_{-1/\ell} + \sum_{i=1}^{3} b_i \boxed{\mathcal{L}_{m_i}},$$

$\boxed{\mathcal{L}_{\mathsf{SD},2} \equiv \mathcal{L}_{m_i}}$ $\longleftrightarrow$ one massive spin-2 mode of definite helicity.

Nonrelativistic motivation: Certain solid-state fQHE setups effectively described by planar Schrödinger equations, satisfied by higher-spin fields, containing **exactly one degree of freedom**. This effect has been experimentally verified for spin-2, in the form of the Girvin-MacDonald-Plazman mode[1], with higher-spin variants (p-atic Hall phases) predicted theoretically[2].

Driving question: How to get spin-s Schrödinger from a Lagrangian formulation?

[1] [Girvin et al., 1986]

[2] [Bergshoeff et al., 2018, Bergshoeff et al., 2024]

The bottom-up approach

The **target** is clearly defined:

Start from a self-dual spin-s Lagrangian, and obtain:

- A Schrödinger equation,
- acting on a nonrelativistic spin-s field
- propagating exactly *one degree of freedom*.

To begin, let's warm up with spin-2!

Spin-2: The nonrelativistic limit

Anticipating the Schrödinger equation, a complexified self-dual Lagrangian is taken:

$$\mathcal{L}_{\mathrm{SD},2} = \varepsilon^{\mu\nu\rho}\left(\bar{E}_\mu{}^\sigma \partial_\nu E_{\rho\sigma} + M\varepsilon_{\mu\alpha\beta}\bar{E}_\nu{}^\alpha E_\rho{}^\beta\right) .$$

Upon splitting the spacetime indices $\mu \to \{0, i\}$, the field content becomes:

$$\{E_{00}, E_{0i}, E_{i0}, E_{ij}\} .$$

To proceed with the nonrelaivistic limit $c \to \infty$, powers of c need to be reinstated by **prescribing scalings**:

$$E_{00} \to \frac{1}{c^2}\mathrm{e}^{-\mathrm{i}mc^2 t} e_{00} , \qquad E_{i0} \to \frac{1}{c}\mathrm{e}^{-\mathrm{i}mc^2 t} e_{i0} , \qquad E_{0i} \to \frac{1}{c}\mathrm{e}^{-\mathrm{i}mc^2 t} e_{0i} ,$$

$$E_{ij} \to \mathrm{e}^{-\mathrm{i}mc^2 t} e_{ij} , \qquad M \to mc , \qquad \partial_0 \to \frac{1}{c}\partial_0$$

... although there is some freedom here.

Taking the $c \to \infty$ and computing equations of motion eliminates all field components, in favor of the symmetric-traceless field $e_{\{ij\}}$, which satisfies:

$$\mathrm{i}\dot{e}_{\{ij\}} + \frac{1}{2m}\nabla^2 e_{\{ij\}} = 0\,,$$

giving **exactly one spin-2 DOF, satisfying a Schrödinger equation.**

How does this generalize to higher spins?

Spin-3 self-dual theory

For higher spins, **auxiliary fields are necessary** for relativistic consistency. Concretely, for spin-3: U_μ and Φ[1] must appear:

$$\mathcal{L}_{\mathrm{SD},3} = \mathcal{L}_3[E] + \mathcal{L}_{31}[E,U] + \mathcal{L}_{10}[U,\Phi]\,, \qquad \text{with } E_{\mu\alpha\beta}$$

How to assign scalings to the auxiliary fields?

$$U_i \longrightarrow c^{-N_i} e^{-imc^2 t} u_i\,, \quad U_0 \longrightarrow c^{-(1+N_0)} e^{-imc^2 t} u_0\,, \quad \Phi \longrightarrow c^{-(1+N_\phi)} e^{-imc^2 t} \phi.$$

Default scaling $N_i, N_0, N_\phi = 0$

The auxiliary fields remain in the nonrelativistic Lagrangian and **participate in the limit.**

Maximal scaling $N_i, N_0, N_\phi \gg$

The auxiliary fields are suppressed by powers of c and **disappear in the nonrelativistic limit.**

[1] [Aragone and Khoudeir, 1993]

The auxiliary fields can be scaled out

Both choices lead to the same outcome in the nonrelativistic limit:

$$\mathrm{i}\dot{e}_{\{abc\}} + \frac{1}{2m}\nabla^2 e_{\{abc\}} = 0,$$

after performing a full equation of motion analysis, with the caveat of *differing gauge-symmetry structures*.

The key takeaway:

$$\mathcal{L}^{(3)}_{\mathrm{NR}}\Big|_{\mathrm{default}} \simeq \mathcal{L}^{(3)}_{\mathrm{NR}}\Big|_{\mathrm{maximal}},$$

The self-dual spin-3 nonrelativistic limit is insensitive to the scalings of auxiliary fields.

Auxiliary field structures are not known for arbitrary spin self-dual theories.

Arbitrary spin gives one Schrödinger mode

The lower-spin examples justify scaling out:

- the *unknown* auxiliary-field tower;
- the trace and mixed-symmetry components ,

giving the simple Lagrangian form:

$$\mathcal{L}_{\mathrm{SD}s} = \varepsilon^{\mu\nu\rho}\left(\bar{E}_{\mu\sigma_1\cdots\sigma_{s-1}}\partial_\nu E_\rho{}^{\sigma_1\cdots\sigma_{s-1}} + M\varepsilon_{\mu\alpha\beta}\bar{E}_\nu{}^{\alpha\sigma_1\ldots\sigma_{s-2}}E_\rho{}^\beta{}_{\sigma_2\ldots\sigma_{s-2}}\right) .$$

The calling calculation then proceeds analogously to the lower-spin cases, and ...

... only a single degree of freedom survives, obeying:

$$\mathrm{i}\dot{e}_{\{i_1\ldots i_s\}} + \frac{1}{2m}\nabla^2 e_{\{i_1\ldots i_s\}} = 0.$$

Conclusions

Third-way massive gravity

- 3rd-GMG propagates three massive modes.
- It admits bulk-unitary regions, but not simultaneous bulk and boundary unitarity.
- *TODO: Critical points in the parameter space, where some of the mass eigenvalues happen to be equal, lead to logarithmic CFTs* [1]*, which remain to be explored.*

Nonrelativistic self-dual theories

- Nonrelativistic limits of self-dual theories are insensitive to the content of the auxiliary field sector.
- The nonrelativistic limit yields a planar Schrödinger mode for the highest-spin field, for any spin s.
- *TODO: Explore more direct contact with p-atic and W_∞ descriptions of higher-spin collective modes in fQHE.*

[1] [Li et al., 2008, Skenderis et al., 2009, Grumiller et al., 2011]

Thank you for your attention!

Spin-2: The nonrelativistic limit

Step 1: The spacetime indices $\mu, \nu, \rho ...$ are split into timelike and spacelike components $\{0, i\}$, giving the Lagrangian:

$$\mathcal{L}_{\mathrm{SD},2} = \varepsilon_{ij}\left(\bar{E}_{00}\partial_i E_{j0} + E_{00}\partial_i \bar{E}_{j0} - \bar{E}_{0k}\partial_i E_{jk} - E_{0k}\partial_i \bar{E}_{jk} - \bar{E}_{i0}\partial_0 E_{j0} + \bar{E}_{ik}\partial_0 E_{jk}\right)$$
$$+ M\left(\bar{E}_{0i}E_{i0} + \bar{E}_{i0}E_{0i} - \bar{E}_{00}E_{ii} - \bar{E}_{ii}E_{00} - \bar{E}_{ij}E_{ji} + \bar{E}_{ii}E_{jj}\right) .$$

Step 2: To take the $c \to \infty$ limit, powers of c should be reinstated in the Lagrangian, according to dimensional analysis:

$$E_{00} \to \frac{1}{c^2}\mathrm{e}^{-\mathrm{i}mc^2 t} e_{00} , \qquad E_{a0} \to \frac{1}{c}\mathrm{e}^{-\mathrm{i}mc^2 t} e_{a0} , \qquad E_{0a} \to \frac{1}{c}\mathrm{e}^{-\mathrm{i}mc^2 t} e_{0a} ,$$
$$E_{ab} \to \mathrm{e}^{-\mathrm{i}mc^2 t} e_{ab} , \qquad M \to mc , \qquad \partial_0 f \to \frac{1}{c}\partial_0 f$$

The Lagrangian then becomes:

$$c\mathcal{L}_{SD,2} = \overbrace{mc^2\bar{e}_{ij}\left[i\varepsilon_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl}\right]e_{kl}}^{mc^2\bar{e}Oe \equiv c\mathcal{L}^{(2)}_{SD,2}} + c\mathcal{L}^{(0)}_{SD,2} + \mathcal{O}(c^{-2})\,.$$

Step 3: The divergent c^2 algebraic term can be transported to subleading order:

$$mc^2\bar{e}\,O\,e \quad\Longleftrightarrow\quad \bar{\lambda}Oe + \bar{e}O\lambda - \frac{1}{mc^2}\bar{\lambda}O\lambda$$

using a Hubbard-Stratonovich (HS) transformation, and permitting a dynamical $c \to \infty$ limit.

The Lagrangian, after performing the HS transformation, and taking $c \to \infty$ reads:

$$\mathcal{L}_{\mathrm{NR},2} = \bar{\lambda}_{ij} O_{ijkl} e_{kl} + \bar{e}_{ij} O_{ijkl} \lambda_{kl} + \mathcal{L}^{(0)}_{SD,2} + \mathcal{O}(c^{-2})$$

Step 4: The total field content present in the Lagrangian reads:

$$\{e_{00}, e_{i0}, e_{0i}, e_{ij}, \lambda_{ij}\}\,.$$

After computing and using all equations of motion, **all field components** are algebraically set to zero, or determined as functions of

$$e_{\{ij\}} \equiv e_{(ij)} - \frac{1}{2}\delta_{ij} e_{kk}\,.$$

Step 5: The only independent, dynamical field component appearing in the theory is then $e_{\{ij\}}$, satisfying the planar Schrödinger equation:

$$i\dot{e}_{\{ij\}} + \frac{1}{2m}\partial^2 e_{\{ij\}} = 0$$

with a proper counting giving **exactly one complex degree of freedom.**

To summarize the success of spin-2:

1. Split indices
2. Assign scalings to fields and constants
3. Apply the HS transformation and take $c \to \infty$
4. Account for all field components and use equations of motion to eliminate all but the highest-spin symmetric-traceless one
5. Obtain the Schrödinger equation for the highest-spin field

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