RG=GR: A Common Cardinal Resolution of the Cosmological Constant and Higgs Hierarchy Problems

Jerzy Kowalski-Glikman

work with Laurent Freidel, Rob Leigh, and Djordje Minic

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Dedicated to the memory of

Rob Leigh (1964-2026) and Jerzy Lukierski (1936-2026)

Two hierarchy problems — one common origin?

COSMOLOGICAL CONSTANT

- The one-loop correction to vacuum energy:

$$\rho_{vac}^{(1)} \sim \Lambda_{UV}^4$$
$$\rho_{vac}^{obs} \ll \Lambda_{UV}^4$$

HIGGS MASS

- The one-loop correction to Higgs mass:

$$\delta M_H^2 \sim \Lambda_{UV}^2$$
$$M_H^2 \ll \Lambda_{UV}^2$$

The problem arises because the UV momentum scale is treated as independent from IR spacetime geometry

$$\text{Wilsonian EFT:} \quad \underbrace{L}_{\text{IR geometry}} \quad \text{fixed independently of} \quad \underbrace{\Lambda_{\text{UV}}}_{\text{UV scale}}$$

A physical UV scale requires a physical ruler

- **Ordinary QFT**: A UV cutoff refers to momentum measured in a local physical frame, and for an observer with four-velocity u^μ,

$$E_u = -u^\mu\, p_\mu\,, \quad |P|_u^2 = (g^{\mu\nu} + u^\mu u^\nu)\, p_\mu p_\nu$$

- A condition such as $|P|_u < \Lambda_{UV}$ uses both the metric and a physical observer/frame.
- **(Quantum) gravity**: A physical fixed UV scale does not specify a fixed set of degrees of freedom when geometry is dynamical.
- Defining the cutoff with a fixed reference metric introduces background structure; defining it with the dynamical metric makes the cutoff subspace geometry dependent.

Instead of fixing a scale,
fix the cardinality (number) of accessible modes.

Cardinal number

- What can be specified without choosing a background geometry is a pure number: the **cardinal number** $\mathcal{N}$ that counts the phase-space cells accessible to the system:

$$\boxed{\mathcal{N} = \mathrm{Tr} P_\Omega}$$

- where P_Ω is the projector onto the quantum states accessible to the physical system/process. $\mathcal{N}$ is the dimension of the accessible spectral subspace (the number of independent field modes retained by the projector). It is a pure number. Different observers may assign different coordinates, energies and momenta to the same states, but they agree on how many states there are.
- Weyl theorem says that semiclassically

$$\boxed{\mathcal{N} = \frac{1}{(2\pi\hbar)^4}\int_\Omega d^4x d^4p}$$

- **No spacetime metric or observer frame is needed to count the cells.** The Liouville measure on the phase space T^*M is metric independent; the physical context determines the domain Ω.

Momentum values are observer/geometry dependent; cardinal number is not.

Similar ideas were proposed in papers by Becker &Reuter, Ferrero & Percacci, Branchina et.al., …

Cardinal number II

$\mathcal{N}$ *is fixed, but* $\mathcal{V}$ **and** $\tilde{\mathcal{V}}$ **could vary**

- In what follows we will choose the spacetime to be an 4-sphere (Euclidean de Sitter) of radius L, and momentum space to be a 4-ball of radius $\tilde{L}$. Then the Weyl theorem gives

$$\boxed{\mathcal{N} = \frac{\mathscr{V}_{S^4_L} \tilde{\mathscr{V}}_{B^4_{\tilde{L}}}}{(2\pi\hbar)^4} = \frac{L^4 \tilde{L}^4}{12\hbar^4}}$$

- Since $\mathcal{N}$ is fixed, when $\tilde{L}$ grows, L shrinks and vice versa. L and $\tilde{L}$ should be determined dynamically.

One-loop effective action

- The starting point for our calculation is one-loop effective action

$$\Gamma_1 := \frac{1}{2} \ln \det\left[\varepsilon^{-2}(\Box_g + m^2)\right] , \quad \varepsilon^2 \equiv \mu^2 + m^2$$

- It can be computed to be

$$\boxed{\Gamma_1 = \frac{\mathcal{N}}{2} \left[\ln \left(\frac{\tilde{L}^2 + m^2}{\mu^2 + m^2} \right) - \frac{1}{2} + \frac{m^2}{\tilde{L}^2} - \frac{m^4}{\tilde{L}^4} \ln \left(\frac{\tilde{L}^2 + m^2}{m^2} \right) \right]}$$

- At fixed $\mathcal{N}$ the power-law UV sensitivity disappears: the large-$\tilde{L}$ dependence is only logarithmic.
- In standard EFT the spacetime volume is held fixed and factored out. The corresponding vacuum-energy density is

$$\frac{\Gamma_1}{\mathscr{V}} = \frac{\Lambda_{UV}^4}{64\pi^2} \left[\ln \left(\frac{\Lambda_{UV}^2 + m^2}{\mu^2 + m^2} \right) - \frac{1}{2} + \frac{m^2}{\Lambda_{UV}^2} - \frac{m^4}{\Lambda_{UV}^4} \ln \left(\frac{\Lambda_{UV}^2 + m^2}{m^2} \right) \right]$$

$$\boxed{\begin{array}{rll} \text{Cardinal} & \Gamma_1 \sim \mathcal{N} \ln \tilde{L} , & \mathcal{N} \text{ fixed} \\ \text{EFT} & \Gamma_1/\mathscr{V} \sim \Lambda_{UV}^4 \ln \Lambda_{UV} , & \mathscr{V} \text{ fixed} \end{array}}$$

RG=GR

- For the phase space $S^4_L \times B^4_{\tilde{L}}$

$$L^4 \tilde{L}^4 = 12\pi\hbar^4 \mathcal{N}_G$$

- Consider now (Euclidean) Einstein-Hilbert action

$$\Gamma_0(L) = \frac{\pi}{\hbar G}\left(\frac{\Lambda}{3}L^4 - 2L^2\right) = \frac{\pi m_{\mathrm{P}}^2}{m_{\mathrm{c}}^2}\left(\frac{12\mathcal{N}_G m_{\mathrm{c}}^4}{\tilde{L}^4} - 2\frac{\sqrt{12\mathcal{N}_G}m_{\mathrm{c}}^2}{\tilde{L}^2}\right)$$

$$m_{\mathrm{c}}^2 = \frac{\hbar\Lambda}{3} \approx \left(1.19 \times 10^{-33}\mathrm{eV}\right)^2$$

- Here $\mathcal{N}_G$ denote the cardinality associated with the gravitational sector. Since L = L($\tilde{L}$; $\mathcal{N}_G$) at fixed $\mathcal{N}_G$

$$\left.\frac{\partial\Gamma_0}{\partial\tilde{L}}\right|_{\mathcal{N}_G} = 0 \qquad \Longleftrightarrow \qquad \left.\frac{\partial\Gamma_0}{\partial L}\right|_{\mathcal{N}_G} = 0$$

$$\boxed{\underbrace{\text{scale stationarity}}_{\text{RG}} \qquad \Longleftrightarrow \qquad \underbrace{\text{Einstein equation}}_{\text{GR}}}$$

What is $\mathcal{N}_G$?

- First consider pure gravity. There are two solutions of Einstein (or RG) equations

$$\tilde{L}_*^2 = \sqrt{12\mathcal{N}_G} m_c^2 \quad \text{or} \quad \tilde{L}_*^2 = \infty$$

- According to Gibbons and Hawking the on-shell value of Euclidean de Sitter action is (minus) the entropy, and these two solutions correspond to

$$S_{GH} = \frac{\pi m_P^2}{m_c^2} \quad \text{or} \quad S_{GH} = 0$$

- Maximal entropy selects the finite-$\tilde{L}$ branch.

- The gravitational cardinality $\mathcal{N}_0$ is identified with gravitational entropy

$$\boxed{\mathcal{N}_G \equiv S_{GH} = \frac{\pi m_P^2}{m_c^2} \sim 10^{122}\,, \qquad \Lambda = \frac{2\pi m_P^2}{\hbar^2}\frac{1}{\mathcal{N}_G}}$$

$$\boxed{\tilde{L}_*^2 \equiv \mu^2 = \sqrt{12\pi}\, m_P m_c = \simeq (10^{-2}\,\mathrm{eV})^2}$$

The cardinal bound and equilibrium

- The effective action is

$$\Gamma(\mathscr{N}_G, \mathscr{N}; \tilde{L}) = \Gamma_0 + \Gamma_1$$

- where $\mathcal{N}$ counts the independent one-particle field modes accessible to the system, retained by the cardinal regulator (Weyl law)

$$\boxed{\mathscr{N} = \mathrm{Tr}\, P_{\mathscr{N}} \simeq \frac{\mathscr{V}\tilde{\mathscr{V}}}{(2\pi\hbar)^4}}$$

- We assume that $0 < \mathcal{N} \leq \mathcal{N}_G$. In analogy with Gibbons-Hawking, we define entropy as minus the on-shell value of effective action.

$$\left.\frac{\partial \Gamma}{\partial \tilde{L}}\right|_{\mathscr{N}} = 0 \implies \tilde{L} = \tilde{L}_* \,, \qquad S \equiv -\Gamma(\mathscr{N}_G, \mathscr{N}; \tilde{L}_*)$$

- Then the equilibrium condition of maximal entropy gives $\mathcal{N} = \mathcal{N}_G$.

$$\frac{\partial S}{\partial \mathscr{N}} > 0 \,, \quad 0 < \mathscr{N} < \mathscr{N}_G \implies S = S_{max} : \mathscr{N} = \mathscr{N}_G$$

Radiative stability of cosmological constant

$$\Gamma(\widetilde{L}) = \Gamma_0(\widetilde{L}) + \Gamma_1(\widetilde{L})$$

$$\frac{\Gamma_0}{\mathcal{N}} = \frac{\mu^4}{\widetilde{L}^4} - 2\frac{\mu^2}{\widetilde{L}^2}, \qquad \frac{\Gamma_1}{\mathcal{N}} = \frac{1}{2}\left[\ln\left(\frac{\widetilde{L}^2+m^2}{\mu^2+m^2}\right) - \frac{1}{2} + \frac{m^2}{\widetilde{L}^2} - \frac{m^4}{\widetilde{L}^4}\ln\left(1+\frac{\widetilde{L}^2}{m^2}\right)\right].$$

- We introduce dimensionless variables

$$u := \frac{\widetilde{L}^2}{\mu^2}, \qquad a := \frac{\mu^2}{m^2}.$$

- For m $\gg \mu \cong 10^{-2}$ eV, a $\ll$ 1 and the field equations gives

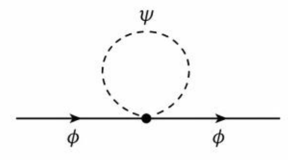
$$0 = \frac{1}{\mathcal{N}}\frac{\partial \Gamma}{\partial u} = \frac{2(u-1)}{u^3} + \frac{a}{3} + O(a^2), \qquad \boxed{u_* = 1 - \frac{a}{6} + \frac{5}{24}a^2 + \cdots}$$

- The one-loop matter contribution is suppressed by μ^2/m^2, and therefore only weakly deforms the classical solution.

$$\boxed{\frac{\delta\Lambda}{\Lambda} = -\frac{1}{6}\frac{\mu^2}{m^2} + O\left(\frac{\mu^4}{m^4}\right), \qquad \left.\frac{\delta\Lambda}{\Lambda}\right|_{\mathrm{EFT}} \sim \frac{m_P^2}{m_c^2} \sim 10^{122}}$$

Radiative stability of Higgs mass

$$\mathscr{L}_{\text{int}} = \frac{\lambda}{2}\,\varphi^2\psi^2$$



Ψ-loop contribution to the mass of ϕ.

- Let M_H be the classical mass of φ, while m be the mass of the field ψ running in the loop. Integrating out ψ

$$\Gamma_1[\varphi] = \frac{1}{2}\,\mathrm{Tr}\ln\left[\epsilon^{-2}\left(-\Box + m^2 + \lambda\varphi^2\right)\right] \approx \Gamma_1[0] + \frac{\lambda}{2}\,\mathrm{Tr}\left[\frac{\varphi^2}{-\Box + m^2}\right] + O(\lambda^2)$$

- The second term gives the mass correction; for constant φ it can be written as

$$\frac{\lambda}{2}\mathscr{V}_{S^4}\,\delta M_H^2\,\varphi^2$$

- Therefore

$$\delta M_H^2 = \frac{\lambda}{\mathscr{V}_{S^4}}\,\mathrm{Tr}\,\frac{1}{-\Box + m^2} = \frac{2\lambda}{\mathscr{V}_{S^4}}\frac{\partial\Gamma_1}{\partial m^2}$$

$$\boxed{\delta M_H^2 = \frac{\lambda}{16\pi^2}\widetilde{L}^2\left[1 - \frac{m^2}{\widetilde{L}^2}\ln\left(1 + \frac{\widetilde{L}^2}{m^2}\right)\right]}$$

Radiative stability of Higgs mass

- To one-loop accuracy it is sufficient to evaluate the correction at the classical solution $\widetilde{L}_* = \mu$. Then

$$\delta M_H^2 = \frac{\lambda m^2}{16\pi^2}\left[\frac{\mu^2}{m^2} - \ln\left(1 + \frac{\mu^2}{m^2}\right)\right] \simeq \frac{\lambda}{32\pi^2}\frac{\mu^4}{m^2}\left[1 + O\left(\frac{\mu^2}{m^2}\right)\right]$$

- which is negligibly small. For Higgs self-interaction, with the identification $m \sim M_H$,

$$\boxed{\frac{\delta M_H^2}{M_H^2} \sim \frac{\lambda}{32\pi^2}\left(\frac{\mu}{M_H}\right)^4 \ll 1.}$$

- This should be contrasted with the EFT result

$$\text{EFT:} \quad \delta M_H^2 \sim \frac{\lambda}{16\pi^2}\Lambda_{\text{UV}}^2 \quad \longrightarrow \quad \text{Cardinal RG=GR:} \quad \delta M_H^2 \sim \frac{\lambda}{32\pi^2}\frac{\mu^4}{m^2}$$

Comment: And what about curvature?

- In our computation of one-loop effective action, we assumed that spacetime is flat ($p^2 = \delta^{\mu\nu} p_\mu p_\nu$) in spite of the fact that the (Euclidean) spacetime S^4 is obviously curved. One can check however that the curvature correction are subleading at large $\mathcal{N}$: for maximally symmetric space

$$\Gamma_1(\tilde{L};\mathcal{N}) \sim \mathcal{N}[\ldots] + \sqrt{\mathcal{N}}[\ldots] + \ldots$$

- and in general

$$\Gamma_1(\tilde{L};\mathcal{N}) \sim \mathcal{N}[\ldots] + \sqrt{\mathcal{N}}\frac{\mathscr{R}}{\sqrt{\mathscr{V}}}[\ldots] + \ldots\,, \qquad \mathscr{R} = \int d^4x\sqrt{g}R\,,\ \mathscr{V} = \int d^4x\sqrt{g}$$

$$\boxed{\frac{\Gamma_1^{\text{curvature}}}{\Gamma_1^{\text{leading}}} = O\left(\frac{1}{\sqrt{\mathcal{N}}}\right)}$$

Comment: Contextuality beyond equilibrium

- The cardinal approach **should be of universal use**; it should be applicable to the standard QFT processes and give results that agree with experiments.
- A question is what are **contextual** $\mathcal{V}_{\text{proc}}$ and $\mathcal{N}_{\text{proc}}$ for a process in question. They must be determined by the physics of the process, rather than chosen as arbitrary regulator parameters.

Equilibrium: $\mathcal{N} = \mathcal{N}_{\text{G}}$;	Local process: $\mathcal{N} = \mathcal{N}_{\text{proc}} \leq \mathcal{N}_{\text{G}}$

- $\mathcal{V}_{\text{proc}}$ and $\mathcal{N}_{\text{proc}}$ encode different physical properties of the system: $\mathcal{V}_{\text{proc}}$ is the intrinsic four-volume of the process, $\mathcal{N}_{\text{proc}}$ is the number of independent modes accessible in the process. Then

$$\tilde{\mathscr{V}} = (2\pi\hbar)^4 \frac{\mathscr{N}_{\text{proc}}}{\mathscr{V}_{\text{proc}}}$$

- The finite-cardinality preserves gauge invariance when the accessible subspace is defined by a gauge-covariant spectral projector; this was explicitly demonstrated for scalar-QED vacuum polarization.

Conclusions

- In EFT spacetime and matter are treated separately; the former is an unmovable arena for the dynamics of the latter.
- In quantum gravity spacetime geometry and matter are fundamentally dynamical and universally coupled.

The contextual accessible phase space size instead of an external scale.

$\mathcal{N}$ fixed $\Rightarrow$ UV and IR linked $\Rightarrow$ RG = GR

$\mathcal{N} = \mathcal{N}_G$ in equilibrium

CC and Higgs mass are radiatively stable