The Geometry of Gravitational Radiation

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Based on:

JH, Emil Have, Vijay Nenmeli, Gerben Oling [2025 & 2026]

as well as [JH, today]

Introduction

- It is clear that in AdS/CFT the notion of a boundary energy-momentum tensor is crucial.
- This is usually defined in terms of holographic renormalisation methods (see e.g. [Balasubramanian, Kraus, 1999] and [de Haro, Solodukhin, Skenderis, 2000]).
- The goal is to do the same for asymptotically flat spacetimes near future null infinity.
- Null infinity is a conformal Carroll manifold [Duval, Gibbons, Horvathy, 2014].
- Requires understanding asymptotic solutions with arbitrary Carroll data at $\mathcal{I}^+$ (boundary covariance).
- This work builds on [JH, 2015] where this was done in 3D. For related work see [Freidel, Riello, 2024].

Introduction

- BMS is a symmetry of the S-matrix [Strominger, 2013] and controls the infrared triangle.
- S-matrix: two boundary problem, diagonal BMS symmetry
- Quite a few aspects can be studied as a single-boundary problem: definition of BMS charges, Bondi loss equations etc.
- This talk:
  - Algebraic and covariant description of metric near $\mathcal{I}^+$
  - Boundary covariant definition of radiation. Overlap with [Fiorucci, Pekar, Petropoulos, Vilatte, last week]
  - Boundary energy-momentum-news complex
  - BMS currents from an action formalism

Carroll geometry at $\mathcal{I}^+$

- Minkowski space in spherical coordinates with $u = t - r$

$$ds^2 = -2dudr - du^2 + r^2 d\Omega^2$$

- $\mathcal{I}^+$ is reached for large r keeping u fixed. Coordinates at $\mathcal{I}^+$ are $x^\mu = (u, x^i)$ with x^i coordinates on S^2.
- We use a gauge with ∂_r tangent to a null geodesic ($\Gamma^\mu_{rr} = 0$):

$$ds^2 = -2e^\beta \tau_\mu dx^\mu dr + g_{\mu\nu} dx^\mu dx^\nu$$

  τ_μ is independent of r.
- Boundary conditions:

$$\beta = \mathcal{O}(r^{-1}) \,, \qquad g_{\mu\nu} = r^2 h_{\mu\nu} + \mathcal{O}(r)$$

  where $h_{\mu\nu}$ has signature $(0, 1, 1)$: $h_{\mu\nu} = e^1_\mu e^1_\nu + e^2_\mu e^2_\nu$.

Carroll geometry at $\mathcal{I}^+$

- Complete set of vielbeine at $\mathcal{I}^+$: $\{\tau_\mu, e^1_\mu, e^2_\mu\}$
- Inverse vielbeine: $\{-v^\mu, e^\mu_1, e^\mu_2\}$ from boundary conditions for the inverse metric

  $$g^{r\mu} = v^\mu + \mathcal{O}(r^{-1})\,, \qquad g^{\mu\nu} = r^{-2}h^{\mu\nu} + \mathcal{O}(r^{-3})\,, \qquad g^{rr} = -g^{r\rho}g^{r\sigma}g_{\rho\sigma}$$

  where $h^{\mu\nu} = e^\mu_1 e^\nu_1 + e^\mu_2 e^\nu_2$.
- Carroll vector v^μ obeys: $v^\mu h_{\mu\nu} = 0$ and $v^\mu \tau_\mu = -1$.
- $r =$ cst hypersurfaces have no definite signature: g^{rr} has no definite sign.
- Non-Lorentzian geometry: complete set of vielbeine that do not transform under the Lorentz group. They feature in GR near singularities, null hypersurfaces, infinities, approximations.

Carroll geometry at $\mathcal{I}^+$

$$ds^2 = -2e^\beta \tau_\mu dx^\mu dr + g_{\mu\nu} dx^\mu dx^\nu$$

- Residual bulk diffeomorphisms generated by $\xi^r \partial_r + \xi^\mu \partial_\mu$:

$$\xi^r = r\Lambda_D + \mathcal{O}(1)\,, \qquad \xi^\mu = \chi^\mu + r^{-1} h^{\mu\nu} \lambda_\nu + \mathcal{O}(r^{-2})$$

  act on boundary vielbeine as

$$\delta\tau_\mu = \mathcal{L}_\chi \tau_\mu + \Lambda_D \tau_\mu + \lambda_\mu\,, \qquad \delta h_{\mu\nu} = \mathcal{L}_\chi h_{\mu\nu} + 2\Lambda_D h_{\mu\nu}$$

  - boundary diffeos: χ^μ
  - boundary Weyl: Λ_D
  - Carroll boosts: λ_μ with $v^\mu \lambda_\mu = 0$ (lightcones collapse $\to \mathcal{I}^+$)
- BMS transformations: boundary diffeos for which $\delta\tau_\mu = 0 = \delta h_{\mu\nu}$

- Vacuum Einstein equations: $-\mathcal{L}_v h_{\mu\nu} = K h_{\mu\nu}$
- Boundary geometry constraint solved by:

$$h_{\mu\nu} dx^\mu dx^\nu = e^{2\varphi}\left(dX^1 dX^1 + dX^2 dX^2\right)$$

  where φ, X^1, X^2 are scalars.
- The Einstein equations also tell us

$$h^\rho_\mu h^\sigma_\nu g_{\rho\sigma} = r^2 h_{\mu\nu} + r C_{\mu\nu} + \mathcal{O}(1)$$

  $C_{\mu\nu}$ is the shear (spatial and STF) and arbitrary.
- Residual gauge transformations (bdry diffeos, Weyl and local Carroll boosts):

$$\begin{aligned} \delta C_{\mu\nu} &= \mathcal{L}_\chi C_{\mu\nu} + \Lambda_D C_{\mu\nu} + \Delta_{\mu\nu} \\ \Delta_{\mu\nu} &= 2 h^\rho_{\langle\mu} h^\sigma_{\nu\rangle} \left(\mathcal{D}_\rho + a_\rho\right) \lambda_\sigma \end{aligned}$$

  $\mathcal{D}_\rho$ is some Carroll covariant derivative and $a_\rho = \mathcal{L}_v \tau_\rho$.

Vacuum shear [Compère, Fiorucci, Ruzziconi, 2018]

- Start with Minkowski with a flat slicing of $\mathcal{I}^+$:

$$ds^2 = -2dU dR + 2R^2 dw\bar{w}$$

- Construct the most general coordinate transformation such that

$$ds^2 = -2\tau_\mu dx^\mu dr + 2r^2 e^{2\varphi} dZ d\bar{Z} + rC_{\mu\nu} dx^\mu dx^\nu + \cdots$$

- Then we find

$$R = re^{\Phi} + \mathcal{O}(1)\,, \qquad U = e^{-\Phi} F + \mathcal{O}(r^{-1})\,, \qquad w = f(Z) + \mathcal{O}(r^{-1})$$

with $\Phi = \varphi - \frac{1}{2}\log f'(Z) +$ c.c. and φ regular.

- Vacuum shear (depends on 2 scalars F, Φ):

$$C^{\text{vac}}_{\mu\nu} = 2h^{\rho}_{\langle\mu} h^{\sigma}_{\nu\rangle} \left(-D_\rho \partial_\sigma F + F\left(D_\sigma \partial_\rho \Phi + \partial_\rho \Phi \partial_\sigma \Phi\right)\right)$$

‘Gauging’ the conformal Carroll algebra

- The conformal Carroll algebra in (2+1)D is (left out rotations)

$$
\begin{array}{rclrclrcl}
[P_a\,,B_b] &=& \delta_{ab}H & [D\,,H] &=& -H & [D\,,P_a] &=& -P_a \\
[D\,,K] &=& K & [D\,,K_a] &=& K_a & [K\,,P_a] &=& -B_a \\
[K_a\,,H] &=& -B_a & [K_a\,,P_b] &=& -\delta_{ab}D+J_{ab} & [K_a\,,B_b] &=& -\delta_{ab}K
\end{array}
$$

- Define the Lie algebra-valued connection

$$
\mathcal{A}_\mu = H\tau_\mu + P_a e^a_\mu + B_a {\omega_\mu}^a + \frac{1}{2}J_{ab}{\omega_\mu}^{ab} + Db_\mu + Kf_\mu + K_a {f_\mu}^a
$$

whose curvature is

$$
\begin{aligned}
\mathcal{F}_{\mu\nu} &= \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + [\mathcal{A}_\mu\,,\mathcal{A}_\nu] \\
&= HR(H)_{\mu\nu} + P_a {R(P)_{\mu\nu}}^a + B_a {R(B)_{\mu\nu}}^a + \frac{1}{2}J_{ab}{R(J)_{\mu\nu}}^{ab} \\
&\quad + DR(D)_{\mu\nu} + KR(K)_{\mu\nu} + K_a {R(K)_{\mu\nu}}^a
\end{aligned}
$$

- Goal: impose curvature constraints such that the field content is reduced to τ_μ, e_μ^a and $C_{\mu\nu}$.
- The curvature constraints must be gauge invariant and consistent with the Bianchi identities.
- Split algebra $\mathfrak{m} \oplus \mathfrak{h}$ where $\mathfrak{m} = \{H, P_a\}$. Gauge transformations

$$\begin{aligned} \delta\mathcal{A}_\mu &= \mathcal{L}_\chi A_\mu + \partial_\mu \Sigma + [A_\mu\,, \Sigma] \\ \delta\mathcal{F}_{\mu\nu} &= \mathcal{L}_\chi F_{\mu\nu} + [F_{\mu\nu}\,, \Sigma] \\ \Sigma &= B_a \lambda^a + \frac{1}{2} J_{ab} \lambda^{ab} + D\Lambda_D + K\sigma + K_a \sigma^a \end{aligned}$$

- The appropriate curvature constraints are to set all curvatures to zero except the K-curvatures:

$$\mathcal{F}_{\mu\nu} = K R(K)_{\mu\nu} + K_a R(K)_{\mu\nu}{}^a$$

- The shear is contained in the Carroll boost connection $\omega_\mu{}^a$.

- We now solve the curvature constraints for the connections $b_\mu, \omega_\mu{}^a, \omega_\mu{}^{ab}, f_\mu, f_\mu{}^a$ and observe that:

$$ds^2 = -2dr\left(\tau_\mu + \mathcal{O}(r^{-2})\right)dx^\mu + \left(r^2 h_{\mu\nu} + r g^{(-1)}_{\mu\nu} + g^{(0)}_{\mu\nu} + \mathcal{O}(r^{-1})\right)dx^\mu dx^\nu$$

$$\begin{aligned} g^{(-1)}_{\mu\nu} &= -\tau_\mu b_\nu - \tau_\nu b_\mu + 2e^a_{(\mu}\omega_{\nu)}{}^a \\ v^\mu g^{(0)}_{\mu\nu} &= v^\mu\left(-\tau_\mu f_\nu - \tau_\nu f_\mu\right) \end{aligned}$$

- The K-curvatures measure the deviation from the vacuum shear:

$$\begin{aligned} 2v^\mu R(K)_{\mu\nu}{}^a e^a_\rho &= -\mathcal{L}_v\left(N_{\rho\nu} - N^{\text{vac}}_{\rho\nu}\right) \\ 2v^\mu R(K)_{\mu\nu} &= h^{\rho\sigma}\mathcal{D}_\rho\left(N_{\sigma\nu} - N^{\text{vac}}_{\sigma\nu}\right) \\ 2h^\mu_\rho h^\nu_\sigma R(K)_{\mu\nu} &= h^\mu_{[\rho}h^\nu_{\sigma]}\left[-2h^{\rho\sigma}\left(\mathcal{D}_\mu + a_\mu\right)\left(\mathcal{D}_\rho - a_\rho\right)\left(C_{\sigma\nu} - C^{\text{vac}}_{\sigma\nu}\right)\right. \\ &\qquad \left. + h^{\rho\sigma}\left(N_{\mu\rho}C_{\nu\sigma} - N^{\text{vac}}_{\mu\rho}C^{\text{vac}}_{\nu\sigma}\right)\right] \end{aligned}$$

- News tensor $N_{\mu\nu} = -\left(\mathcal{L}_v + \frac{1}{2}K\right)C_{\mu\nu}$ and vacuum news

$$N^{\text{vac}}_{\mu\nu} = 2h^\rho_{\langle\mu}h^\sigma_{\nu\rangle}\left(\mathcal{D}_\rho\partial_\sigma\Phi + \partial_\rho\Phi\partial_\sigma\Phi - \mathcal{D}_\rho a_\sigma - a_\rho a_\sigma\right)$$

Hierarchy

- The K-curvatures are equal to 5 of the 10 leading order Weyl tensor components (Newman–Penrose scalars $\psi_4, \psi_3,$ Reψ_2).
- There is a hierarchy under gauge transformations. 4 possibilities:

$$
\begin{array}{lrclrclrcl}
i). & h^\mu_\rho h^\nu_\sigma R(K)_{\mu\nu} & = & 0 & v^\mu R(K)_{\mu\nu} & = & 0 & v^\mu e^a_\sigma R(K)_{\mu\nu}{}^a & = & 0 \\
ii). & h^\mu_\rho h^\nu_\sigma R(K)_{\mu\nu} & \neq & 0 & v^\mu R(K)_{\mu\nu} & = & 0 & v^\mu e^a_\sigma R(K)_{\mu\nu}{}^a & = & 0 \\
iii). & h^\mu_\rho h^\nu_\sigma R(K)_{\mu\nu} & \neq & 0 & v^\mu R(K)_{\mu\nu} & \neq & 0 & v^\mu e^a_\sigma R(K)_{\mu\nu}{}^a & = & 0 \\
iv). & h^\mu_\rho h^\nu_\sigma R(K)_{\mu\nu} & \neq & 0 & v^\mu R(K)_{\mu\nu} & \neq & 0 & v^\mu e^a_\sigma R(K)_{\mu\nu}{}^a & \neq & 0
\end{array}
$$

- Similar classification of (non-)radiative spaces in [Freidel, Pranzetti, 2021].
- Three types of non-radiative spacetimes: vacuum, strongly and weakly non-radiative.

$$\delta C^{\text{vac}}_{\mu\nu} = \mathcal{L}_\chi C^{\text{vac}}_{\mu\nu} + \Lambda_D C^{\text{vac}}_{\mu\nu} + 2h^\rho_{\langle\mu} h^\sigma_{\nu\rangle} \left(\mathcal{D}_\rho \lambda_\sigma + a_\rho \lambda_\sigma\right)$$

- The radiative shear $C_{\mu\nu} - C^{\text{vac}}_{\mu\nu}$ is Carroll boost invariant.

Boundary energy-momentum tensor

- Bulk metric

$$ds^2 = -2drV_\mu dx^\mu + (\Pi_{\mu\nu} - SV_\mu V_\nu)\, dx^\mu dx^\nu$$

  where $\Pi_{\mu\nu} = \delta_{ab} E^a_\mu E^b_\nu$ has signature $(0, 1, 1)$.
- Vielbeine on $r =$ cst cutoff surface: V_μ, E^a_μ.
- Inverse metric $g^{\mu\nu} = \Pi^{\mu\nu}$, $g^{r\mu} = U^\mu$, $g^{rr} = S$ (norm of normal 1-form dr).
- Variation of the bulk action:

$$\delta S_{\text{EH}} = \cdots + \int_{r=\Lambda} d^3x\, E\, J^M N_M\,, \qquad N_M = \partial_M r$$

- IN BS gauge: $E :=$ det $\left(-V_\mu, E^a_\mu\right) = er^2 e^\beta$.
- Cutoff surface $r = \Lambda$ has no definite character.

- The generalisation of the GHY extrinsic counterterm is

$$S_{\text{ext}} = 2\int d^3xE\left(\delta^M_P + V^M N_P\right)\nabla_M N^P$$

$V^M = -(\partial_r)^M$ and $N_P = \partial_P r$ [Parattu, Chakraborty, Padmanabhan, 2016].

- This removes normal derivatives of variations and leads to:

$$\delta\left(S_{\text{EH}} + S_{\text{ext}}\right) = \cdots + \int_{r=\Lambda} d^3x\, E\left(\mathcal{T}^\mu \delta V_\mu + \frac{1}{2}\mathcal{T}^{\mu\nu}\delta\Pi_{\mu\nu} + 2r^{-1}E^{-1}\delta\left(ES\right)\right)$$

- $2r^{-1}E^{-1}\delta\left(ES\right)$ dominates at LO in r and leads to a variation of the Bondi mass at $\mathcal{O}(1)$. We cancel it by adding

$$S_{\text{norm}} = -2\int_{r=\Lambda} d^3xEr^{-1}S$$

- Divergence at order r removed by adding intrinsic counterterm

$$S_{\mathsf{int}} = -\int d^3x Er \left(R[C] - \frac{1}{4}\Pi^{\mu\nu}\mathcal{L}_U V_\mu \mathcal{L}_U V_\nu \right)$$

- We obtain

$$\delta \left(S_{\mathsf{EH}} + S_{\mathsf{ext}} + S_{\mathsf{norm}} + S_{\mathsf{int}} \right) \Big|_{\mathsf{os}} = \int d^3x e \left(T^\mu \delta\tau_\mu + \frac{1}{2} T^{\mu\nu} \delta h_{\mu\nu} + \frac{1}{2} S^{\mu\nu} \delta C_{\mu\nu} \right)$$

Here, T^μ is the energy current, $T^{\mu\nu}$ the momentum-stress, and $S^{\mu\nu}$ the news tensor.

$$S^{\mu\nu} = \frac{1}{2} N^{\mu\nu} = -\frac{1}{2} h^{\mu\rho} h^{\nu\sigma} \left(\mathcal{L}_v + \frac{1}{2} K \right) C_{\rho\sigma}$$

- Finite counterterms make the on shell action Weyl invariant.

- The constraint $-\mathcal{L}_v h_{\mu\nu} = K h_{\mu\nu}$ is solved by

$$h_{\mu\nu} dx^\mu dx^\nu = e^{2\varphi}\left(dX^1 dX^1 + dX^2 dX^2\right)$$

- We vary φ, X^1, X^2. Fundamental ambiguity: $T^{\mu\nu}$ or $T^{\mu\nu} + t^{\mu\nu}$

$$t^{\mu\nu} = \frac{1}{2} h^{\mu\rho} h^{\nu\sigma} \mathcal{L}_v \chi_{\rho\sigma} - v^{(\mu} h^{\nu)\sigma} h^{\rho\kappa} \mathcal{D}_\rho \chi_{\kappa\sigma}$$

- The components of the energy-momentum tensor (EMT) are:

$$\begin{aligned}
\tau_\mu T^\mu &= -v^\rho v^\sigma g^{(1)}_{\rho\sigma} + \cdots \\
P_\rho = h_{\rho\mu} \tau_\nu T^{\mu\nu} &= \frac{3}{2} v^\mu h^\nu_\rho g^{(1)}_{\mu\nu} + \frac{1}{2} h^{\sigma\kappa} \mathcal{D}_\sigma \chi_{\kappa\rho} + \cdots \\
\tilde{T}^{\rho\sigma} = T^{\mu\nu} h^{\langle\rho}_\mu h^{\sigma\rangle}_\nu &= \frac{1}{2} h^{\mu\rho} h^{\nu\sigma} \mathcal{L}_v \chi_{\mu\nu} + \cdots
\end{aligned}$$

- Ambiguity in Bondi momentum aspect P_ρ.
- The dots are derivatives of boundary Carroll data and shear.

- Weyl invariance and anomalous Carroll boosts:

$$
\begin{aligned}
T^{\mu}\tau_{\mu} + T^{\mu\nu}h_{\mu\nu} + \frac{1}{4}N^{\mu\nu}C_{\mu\nu} &= 0 \\
h^{\mu}_{\rho}T^{\rho} - \frac{1}{2}\left(\mathcal{D}_{\rho} - a_{\rho}\right)N^{\rho\mu} &= \mathcal{A}^{\mu}_{\mathsf{B}} = v^{\rho}h^{\mu\sigma}R(K)_{\rho\sigma}
\end{aligned}
$$

- Diffeo invariance leads to the Bondi loss equations

$$
-e^{-1}\partial_{\mu}\left(eT'^{\mu}{}_{\nu}\right) + T'^{\mu}\partial_{\nu}\tau_{\mu} + \frac{1}{2}T'^{\mu\rho}\partial_{\nu}h_{\mu\rho} = \text{flux in } K\text{-curvatures}
$$

- Primed EMT $T'^{\mu}{}_{\nu} = T'^{\mu}\tau_{\nu} + T'^{\mu\rho}h_{\rho\nu}$ is a standard Carroll EMT

$$
\begin{aligned}
T'^{\mu} &= -\tau \cdot T' v^{\mu} \\
T'^{\mu\nu} &= -2v^{(\mu}h^{\nu)\kappa}P'_{\kappa} - \frac{1}{2}h^{\mu\nu}\tau \cdot T' + t^{\mu\nu}
\end{aligned}
$$

$$
\tau \cdot T' = U^{\rho}U^{\sigma}C_{r\rho r\sigma}\Big|_{\mathcal{O}(r^{-3})}, \qquad P'_{\kappa} = -U^{\rho}\Pi^{\sigma}_{\kappa}C_{r\rho r\sigma}\Big|_{\mathcal{O}(r^{-3})}
$$

- See [Fiorucci, Pekar, Petropoulos, Vilatte, 2025] for a Carrollian field theory description of Bondi loss.

- Consider a bulk diffeo generated by an asymptotic Killing vector

$$\begin{aligned} 0 = \delta\tau_\mu &= \mathcal{L}_K \tau_\mu + \Lambda_D \tau_\mu + \lambda_\mu \\ 0 = \delta h_{\mu\nu} &= \mathcal{L}_K h_{\mu\nu} + 2\Lambda_D h_{\mu\nu} \end{aligned}$$

- Solutions for K^μ are super translations and super rotations.
- Contract diffeo Ward identity with K^ν:

$$\partial_\mu \left(e T'^\mu{}_\nu K^\nu \right) = -e K^\nu \mathcal{F}_\nu$$

- BMS current is $eT'^\mu{}_\nu K^\nu$.
- Primed EMT from a variational argument? Vary $C_{\mu\nu} - C^{\text{vac}}_{\mu\nu}$ (WIP)
- Extend to global boundary consisting of $\mathcal{I}^-$ and spatial infinity?
- Boundary effective theory