On the Extended Geometry of Kac-Moody Symmetries of Quantum Gravity

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Based on: [1] WIP (K.G.); [2] 2607.01999, [3] 2102.02152 (K.G., Peter West).

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Origins of 11D SUGRA

On the light cone, $i, j = 3, \ldots, 11$, find graviton, three-form of $\mathrm{SO}(9)$

$$h_{ij} = h_{ji} \qquad (44), \tag{1}$$

$$A_{i_1 i_2 i_3} = A_{[i_1 i_2 i_3]} \qquad (84), \tag{2}$$

Here $h^i{}_i = 0$; $44 + 84 = 128$ dof. Instead of seeking non-linear theory:

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Introduce 'dual' formulations of above fields

$$A_{i_1 .. i_6} = A_{[i_1 .. i_6]} \qquad (84) \tag{3}$$

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Propose **Duality Relations:**

$$D^{1|2}_{ijk} = A_{ijk} + \tfrac{1}{6!}\epsilon_{ijk}{}^{l_1..l_6} A_{l_1..l_6} = 0 \tag{5}$$

$$D^{0|3}_{ij} = h_{ij} + \frac{1}{8!}\epsilon_i{}^{k_1\ldots k_8} h_{k_1\ldots k_8,j} = 0 \tag{6}$$

$h^i{}_i = 0$ gives $h_{[i_1..i_8,j]} = 0$ $(81 \to 80)$; $D^{0|3}_{ij} = 0$ gives $D_{[ij]} = 0$ $(80 \to 44)$.

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Why Stop Dualizing Here?

Continue the Dual Descriptions

Find ∞ no. of duals by appending blocks of 9 indices on previous via

$$X_{i_1 \ldots i_9, J} = \epsilon_{i_1 \ldots i_9} X_J. \tag{7}$$

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Introduce the set of fields

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Subject these to duality relations, for example:

$$D^{1|4}_{ijk} = A_{i_1 i_2 i_3} - \frac{1}{9!}\epsilon^{l_1\dots l_9} A_{l_1\dots l_9,i_1 i_2 i_3} = 0, \tag{9}$$

$$D^{0|6}_{ij} = h_{ij} + \frac{1}{9!8!}\epsilon_i{}^{k_1\dots k_8}\epsilon^{l_1\dots l_9} h_{l_1\dots l_9,k_1\dots k_8,j} = 0. \tag{10}$$

All higher tensors are therefore constrained to describe the original

$$44 + 84 = 128 \tag{11}$$

physical bosonic degrees of freedom.

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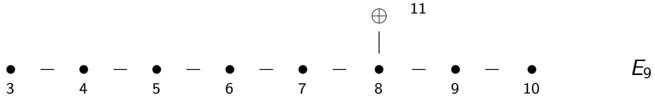
These Fields are in 1-1 Correspondence With Borel Roots of E_9

The E_9 Dynkin Diagram

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & & & & \oplus\ 11 & & & & \\
 & & & & & & & & & & | & & & & \\
\bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet \\
3 & & 4 & & 5 & & 6 & & 7 & & 8 & & 9 & & 10
\end{array}
\qquad E_9
$$

The E_9 Dynkin Diagram



- Nodes $3, \ldots, 10$ give the $A_8 = \mathrm{SL}(9)$ **gravity line**, with $\mathrm{GL}(9)$'s $K^i{}_j$. Simple Roots $E_3 = K^3{}_4,\ E_4 = K^4{}_5,\ ..,\ E_{10} = K^{10}{}_{11}$.
- Node 11 gives the three-form $R^{i_1 i_2 i_3}$, simple root $E_{11} = R^{9\,10\,11}$.
- First few generators are:

ℓ	A_8 representation	generator
0	**80**	$K^i{}_j$
1	**84**	$R^{i_1 i_2 i_3}$
2	$\overline{\mathbf{84}}$	$R^{i_1 \cdots i_6}$
3	**80**	$R^{i_1 \cdots i_8, j}$
4	**84**	$R^{i_1 \cdots i_9, j_1 j_2 j_3}$
5	$\overline{\mathbf{84}}$	$R^{i_1 \cdots i_9, j_1 \cdots j_6}$

Borel Roots of E_9 Come From $E_9/I_c(E_9)$

$E_9/I_c(E_9)$

Schematically,

$$E_9 = \{\widehat{K}^i{}_j,\ R^{i_1 i_2 i_3}, R_{i_1 i_2 i_3},\ R^{i_1\cdots i_6}, R_{i_1\ldots i_6}, R^{i_1\cdots i_8,j}, R_{i_1\ldots i_8,j}, \ldots\}. \qquad (12)$$

Define

$$J^{ij} := \delta^{ik}\widehat{K}^j{}_k - \delta^{jk}\widehat{K}^i{}_k, \qquad S^{ijk} := R^{ijk} - R_{ijk}, \qquad (13)$$

$$T^{ij} := \delta^{ik}\widehat{K}^j{}_k + \delta^{jk}\widehat{K}^i{}_k, \qquad T^{ijk} := R^{ijk} + R_{ijk}. \qquad (14)$$

Then

$$I_c(E_9) = \{J^{ij}, S^{ijk}, \ldots\}, \qquad (15)$$

$$E_9/I_c(E_9) = \{T^{ij}, T^{ijk}, T^{i_1\cdots i_6}, T^{i_1\cdots i_8,j}, \ldots\}. \qquad (16)$$

$E_9/I_c(E_9)$

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Take

$$g = \exp(\mathcal{V}) \in E_9/I_c(E_9), \tag{17}$$

with

$$\mathcal{V} = h_{ij}T^{ij} + A_{ijk}T^{ijk} + A_{i_1\ldots i_6}T^{i_1\cdots i_6} + h_{i_1\ldots i_8,j}T^{i_1\cdots i_8,j} + \ldots. \tag{18}$$

Dualities Built From Fields on $E_9/I_c(E_9)$. Preserved by $I_c(E_9)$?

$I_c(E_9)$ Preserves the Dualities

Under a local $h \in I_c(E_9)$ transformation $g \to gh$

$$\delta\mathcal{V} = [\Lambda_{ijk} S^{ijk}, \mathcal{V}]. \tag{19}$$

The first variations are

$$\delta A_{i_1 i_2 i_3} = -6 h_{[i_1}{}^{k} \Lambda_{i_2 i_3]k} + A_{i_1 i_2 i_3}{}^{k_1 k_2 k_3} \Lambda_{k_1 k_2 k_3}, \tag{20}$$

$$\delta A_{i_1 \dots i_6} = 120 \Lambda_{[i_1 i_2 i_3} A_{i_4 i_5 i_6]} - \frac{2}{3} \Lambda^{k_1 k_2 k_3} \left(h_{i_1 \dots i_6 k_1 k_2, k_3} + h_{[i_1 \dots i_5 | k_1 k_2 k_3 |, i_6]} \right). \tag{21}$$

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The dualities vary as

$$\delta D^{1|2}_{i_1 i_2 i_3} = \frac{1}{6} \Lambda_{j_1 j_2 j_3} \epsilon_{i_1 i_2 i_3}{}^{j_1 \cdots j_6} D^{1|2}_{j_4 j_5 j_6} - 6 D^{0|3}_{k[i_1} \Lambda^{k}{}_{i_2 i_3]}. \tag{22}$$

More generally,

$$\boxed{\delta D^{(r)} = \sum_s M^{(r)}{}_s D^{(s)}} \tag{23}$$

so the complete set $D^{(r)} = 0$ is preserved.

The Fields Transform — Space-Time Must Transform Too

Introducing Space-Time

Under $\mathrm{GL}(9)$, the i, j indices in $h_{ij}(x)$ transform, but so does x.

Introduce the translation representation

$$l_1^{E_9} = \{P_i,\, Z^{ij},\, Z^{i_1\ldots i_5},\, Z^{i_1\ldots i_8},\, Z^{i_1\ldots i_7,j},\ldots\}. \tag{24}$$

Already

$$[R^{ijk}, P_l] = 3\delta_l^{[i} Z^{jk]}, \tag{25}$$

so $\{P_i\}$ is not closed under E_9.
Therefore

$$\boxed{E_9 \otimes_s l_1^{E_9} / I_c(E_9)}. \tag{26}$$

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This contains the usual gravitational $\mathrm{GL}(9) \otimes_s \{P_i\}/\mathrm{SO}(9)$ realization,

$$g \supset e^{x^i P_i} e^{h_i{}^j(x) K^i{}_j}, \tag{27}$$

which gives, at linear order,

$$e_i{}^a(x) = \delta_i{}^a + h_i{}^a(x) + \ldots. \tag{28}$$

So Far We Have Used Only Nine Space-Time Directions

From E_9 to E_{11}

Go from 9D to 11D via: $A_8 = \mathrm{SL}(9) \quad \longrightarrow \quad A_{10} = \mathrm{SL}(11)$

$$
\begin{array}{ccccccccccccccccccccc}
 & & & & & & & & & & & & & & \oplus^{11} & & & & & & \\
 & & & & & & & & & & & & & & | & & & & & & \\
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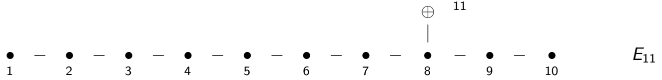

The first fields are

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The nonlinear 3-6 duality relation is

$$F_{a_1 \ldots a_7} = \frac{1}{4!} \epsilon_{a_1 \ldots a_7 b_1 \ldots b_4} F^{b_1 \ldots b_4}, \tag{30}$$

$$F_{a_1 \ldots a_4} = 4 \partial_{[a_1} A_{a_2 a_3 a_4]}, \qquad F_{a_1 \ldots a_7} = 7 \partial_{[a_1} A_{a_2 \ldots a_7]} + \ldots . \tag{31}$$

These dualities arise from the coset model

$$\boxed{E_{11} \otimes_s l_1^{E_{11}} / I_c(E_{11})}. \tag{32}$$

This reverses standard story in E_{11} literature [1], see also [3]. Found KMA from lightcone dof and dualities, bypassing long/detailed historical path starting from non-linear 11D SUGRA.

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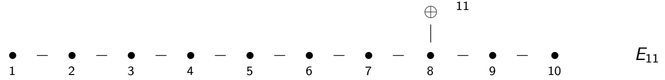

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Historically, the Logic Was Reversed

The Historical Route: Dimensional Reduction

Historically, began from full non-linear 11D SUGRA, then studied toroidal reduction of 11D SUGRA to $D = 11 - n$, found:

- hidden E_n symmetries,
- scalar dynamics organised by $E_n/I_c(E_n)$.

D	Hidden symmetry	Coset / nonlinear realization
10 IIA	$\mathrm{O}(1,1)$	–
9	$\mathrm{GL}(2)$	$\mathrm{GL}(2)/\mathrm{SO}(2)$
8	$E_3 = A_2 \times A_1$	$\dfrac{\mathrm{SL}(3)\times\mathrm{SL}(2)}{\mathrm{SO}(3)\times\mathrm{SO}(2)}$
7	$E_4 = A_4$	$\mathrm{SL}(5)/\mathrm{SO}(5)$
6	$E_5 = D_5$	$\mathrm{SO}(5,5)/(\mathrm{SO}(5)\times\mathrm{SO}(5))$
5	E_6	$E_6/\mathrm{Sp}(8)$
4	E_7	$E_7/\mathrm{SU}(8)$
3	E_8	$E_8/\mathrm{SO}(16)$
2	E_9	$E_9/I_c(E_9)$
1	E_{10}	$E_{10}/I_c(E_{10})$

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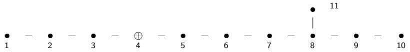
One way to motivate E_{11} is that the E_{11} diagram explains the table visually, e.g. in 4D



$\mathrm{GL}(4) \times E_7 \subset E_{11}$

Same Idea Holds for 4D Gravity

4D Gravity and A_1^{+++}

Start from 4D gravity: $e_\mu{}^a(x)$ comes from $\mathrm{GL}(4) \otimes \{P_a\}/\mathrm{SO}(1,3)$:

$$\underset{1}{\bullet} \quad - \quad \underset{2}{\bullet} \quad - \quad \underset{3}{\bullet} \qquad\qquad \underset{4}{\bar{\bullet}} \qquad\qquad \mathrm{GL}(4) \tag{33}$$

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3	$\mathrm{GL}(3) \times A_1$	$\underset{1}{\bullet} - \underset{2}{\bullet} \quad \underset{3}{\bar{\bullet}} \quad \underset{4}{\bullet}$
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1	$\mathrm{GL}(1) \times A_1^{++}$	$\underset{1}{\bar{\bullet}} \quad \underset{2}{\bullet} - \underset{3}{\bullet} = \underset{4}{\bullet}$

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2	$\mathrm{GL}(2) \times A_1^+$	$\underset{1}{\bullet} \quad \underset{2}{\bar{\bullet}} \quad \underset{3}{\bullet} = \underset{4}{\bullet}$
1	$\mathrm{GL}(1) \times A_1^{++}$	$\underset{1}{\bar{\bullet}} \quad \underset{2}{\bullet} - \underset{3}{\bullet} = \underset{4}{\bullet}$

Suggests we should replace the missing link in 4D and activate node 4:

$$\underset{1}{\bullet} - \underset{2}{\bullet} - \underset{3}{\bullet} = \underset{4}{\oplus} \qquad A_1^{+++} \tag{34}$$

Start Again From the 4D Light Cone

On the light cone, $i, j = 3, 4$, find

$$h^{(0)}_{ij} = h^{(0)}_{ji}, \qquad h^{(0)i}{}_i = 0 \qquad (2). \tag{35}$$

Introduce a 'dual' formulation

$$h^{(1)}_{i,j} = h^{(1)}_{j,i} \qquad (3), \tag{36}$$

with

$$D^{0|1}_{ij} = h^{(0)}_{ij} + \epsilon_i{}^k h^{(1)}_{k,j} = 0. \tag{37}$$

$D^{0|1}_{[ij]} = 0$ gives

$$h^{(1)i}{}_i = 0 \qquad (3 \to 2). \tag{38}$$

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Continue by appending complete blocks of two indices:

$$h^{(0)}_{ij}, \qquad h^{(1)}_{i,j}, \qquad h^{(2)}_{i_1 i_2, j, k}, \qquad h^{(3)}_{i_1 i_2, j_1 j_2, k, l}, \qquad \ldots \tag{39}$$

All are constrained to describe the original two graviton degrees of freedom.

These Fields Again Reproduce the Borel Roots of Affine Algebra A_1^+

The Light-Cone Algebra is A_1^+

$$\underset{3}{\bullet} = \underset{4}{\oplus} \qquad A_1^+$$

- Node 3 gives the $A_1 = \mathrm{SL}(2)$ **gravity line**, with $\mathrm{GL}(2)$'s $K^i{}_j$.
- Node 4 gives the symmetric tensor $R^{i,j} = R^{j,i}$.
  Symmetric indices give the double link.
- First few generators are:

ℓ	A_1 representation	generator
0	**3**	$K^i{}_j$
1	**3**	$R^{i,j}$
2	**3**	$R^{i_1 i_2, j, k}$
3	**3**	$R^{i_1 i_2, j_1 j_2, k, l}$

Restoring the two omitted directions $A_1 = \mathrm{SL}(2) \to A_3 = \mathrm{SL}(4)$:

$$\underset{1}{\bullet} - \underset{2}{\bullet} - \underset{3}{\bullet} = \underset{4}{\oplus} \qquad A_1^{+++}$$

The Light-Cone Construction Has Reconstructed the Same A_1^{+++}

10D Theory on Doubled Space-Time

The 10D CBS graviton, two-form and dilaton arise from coset model

$$\frac{\left[\mathrm{SO}(10,10)\otimes \mathrm{GL}(1)\right]\otimes_s \{P_a, Q^a\}}{\mathrm{SO}(10)\otimes \mathrm{SO}(10)}. \tag{40}$$

The fields

$$g_{\mu\nu}(x,y), \qquad A_{\mu\nu}(x,y)\equiv B_{\mu\nu}(x,y), \qquad \phi(x,y) \tag{41}$$

depend on doubled coordinates

$$z^{\Pi}=(x^{\mu},y_{\mu}), \qquad \partial_{\Pi}=(\partial_{\mu},\bar{\partial}^{\mu}), \qquad \bar{\partial}^{\mu}:=\partial/\partial y_{\mu}. \tag{42}$$

The generalized vielbein is

$$E_{\Pi}{}^{A}=\begin{pmatrix} e_{\mu}{}^{a} & A_{\mu\rho}e_{b}{}^{\rho} \\ 0 & e_{b}{}^{\nu}\end{pmatrix}, \tag{43}$$

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Gauge invariance is imposed via 'generalized diffeomorphism', and uses

$$\partial_{\Pi}\partial^{\Pi}F=0 \qquad \text{weak section condition}, \qquad (44)$$

$$\partial_{\Pi}F\,\partial^{\Pi}G=0 \qquad \text{strong section condition}, \qquad (45)$$

Latter eliminates dependence on doubled coordinate.

Does Gauge Invariance Really Require These Section Conditions

Linearized Alternative to the Section Conditions

[2] studies linearised doubled gauge transformations

$$\begin{aligned} \delta h_{\mu\nu} &= 2\partial_{(\mu}\xi_{\nu)} - 2\bar{\partial}_{(\mu}\Lambda_{\nu)}, \\ \delta A_{\mu\nu} &= 2\partial_{[\mu}\Lambda_{\nu]} - 2\bar{\partial}_{[\mu}\xi_{\nu]}, \qquad \delta\phi = 2\bar{\partial}^{\mu}\Lambda_{\mu}. \end{aligned} \tag{46}$$

The two-form and graviton equations are schematically

$$\begin{aligned} E^{A}_{\mu\nu} &= 3\partial^{\lambda}\partial_{[\lambda}A_{\mu\nu]} + 2\bar{\partial}^{\lambda}\bar{\partial}_{[\lambda}A_{\mu\nu]} + 2\partial_{[\mu}\bar{\partial}^{\lambda}h_{|\lambda|\nu]} + \bar{\partial}_{[\mu}\partial^{\lambda}h_{|\lambda|\nu]} \\ &\quad + 2\partial_{[\mu}\bar{\partial}_{\nu]}h^{\kappa}{}_{\kappa} + 4\partial_{[\mu}\bar{\partial}_{\nu]}\phi = 0, \end{aligned} \tag{47}$$

$$\begin{aligned} E^{h}_{\mu\nu} &= \left(\partial^2 h_{\mu\nu} - 2\partial^{\lambda}\partial_{(\mu}h_{\nu)\lambda} - \eta_{\mu\nu}\partial^2 h^{\kappa}{}_{\kappa} + 2\partial_{(\mu}\bar{\partial}^{\lambda}A_{\lambda|\nu)}\right) \\ &\quad + \left(\bar{\partial}^2 h_{\mu\nu} - 2\bar{\partial}^{\lambda}\bar{\partial}_{(\mu}h_{\nu)\lambda} - \eta_{\mu\nu}\bar{\partial}^2 h^{\kappa}{}_{\kappa} + 2\bar{\partial}_{(\mu}\partial^{\lambda}A_{\lambda|\nu)}\right) + \text{mixed terms} = 0. \end{aligned} \tag{48}$$

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Their variations vanish provided the *parameters*, rather than the fields, obey [2]

$$\partial_{\lambda}\bar{\partial}^{\lambda}\left(\partial_{[\mu}\xi_{\nu]} - \bar{\partial}_{[\mu}\Lambda_{\nu]}\right) = 0, \tag{3.8}$$

$$\partial_{\lambda}\bar{\partial}^{\lambda}\left(\bar{\partial}_{(\mu}\Lambda_{\nu)} - \partial_{(\mu}\xi_{\nu)}\right) = 0, \tag{3.9}$$

$$\partial_{\lambda}\bar{\partial}^{\lambda}\left(\bar{\partial}^{\nu}\xi_{\nu}\right) = 0, \tag{3.10}$$

$$\partial_{\lambda}\bar{\partial}^{\lambda}\left(\partial^{\nu}\Lambda_{\nu}\right) = 0. \tag{3.11}$$

No section condition on the fields: $g_{\mu\nu}, A_{\mu\nu}, \phi$ may retain dependence on (x^{μ}, y_{μ}).

Extended Coordinates Need Not Be Eliminated