Generalized Lie derivative of (s)pinors on Courant algebroids

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Joint work with Fridrich Valach [1]

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Canonical infinitesimal isometry

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$$b^h_{h+\epsilon\delta}(v) := v - \epsilon \frac{1}{2} h^{-1}\delta(v) \quad \forall v \in V$$

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Induced map between orthonormal bases:

$$\{e_i\} \mapsto \{b^h_{h+\epsilon\delta}(e_i)\}$$

Lie derivative as a vector field lift

1-1 correspondence between tensor fields on M and $GL(n, \mathbb{R})$-equivariant functions on $\mathrm{Fr}M$:

$$T : \mathrm{Fr}M \to V$$
$$T : R_A f \mapsto \rho(A^{-1}) T(f)$$

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On (M,g) with g of signature (p,q),
pinor fields are equivariant functions $\psi \colon PM \to \Sigma$.

Goal: define a lift of X to $X^P \in \mathfrak{X}(PM)$.

Vector field lifts via $\pi : PM \to O_g M$

There is a smooth surjection $\pi : PM \to O_g M$:

$$
\begin{tikzcd}
\mathrm{Pin}(p,q) \circlearrowright PM \arrow[d, "\pi"] \\
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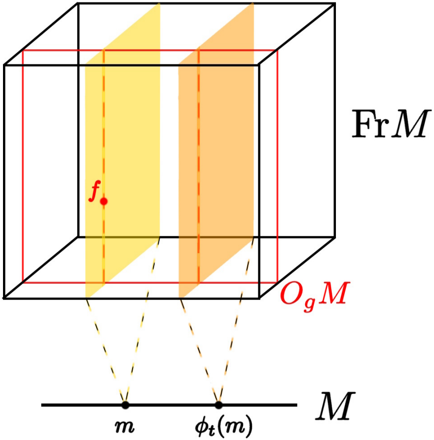
Metric Lie derivative

Defining $X^O \in \mathfrak{X}(O_g M)$ at $f \in O_g M$:



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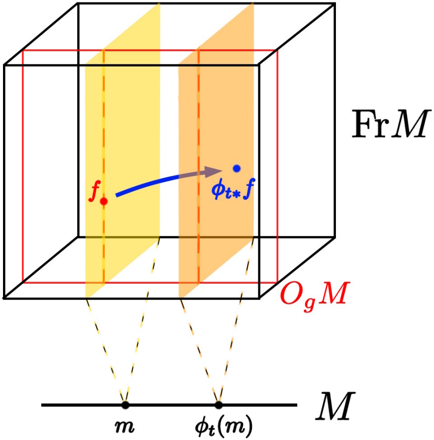

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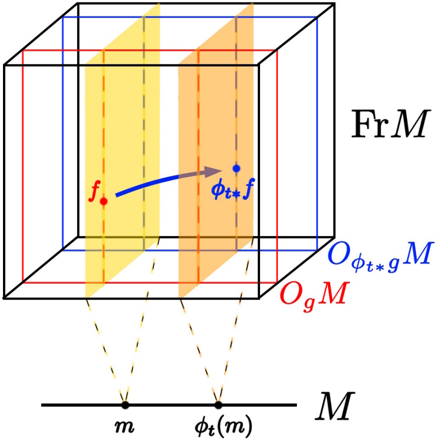

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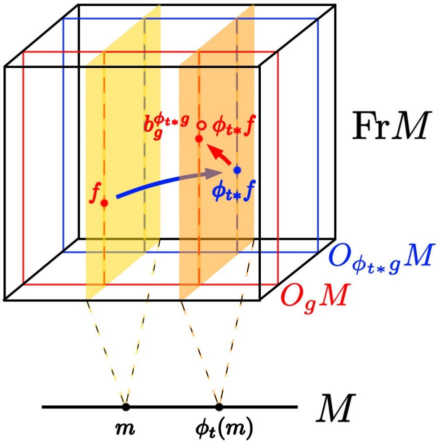

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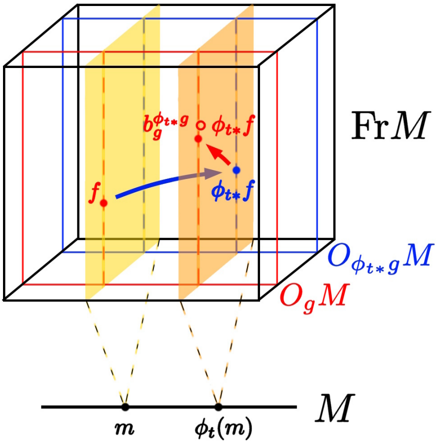

$$X^O_f := \frac{d}{dt}\Big|_{t=0} b_g^{\phi_{t*}g} \circ \phi_{t*}f \quad \in T_f O_g M$$

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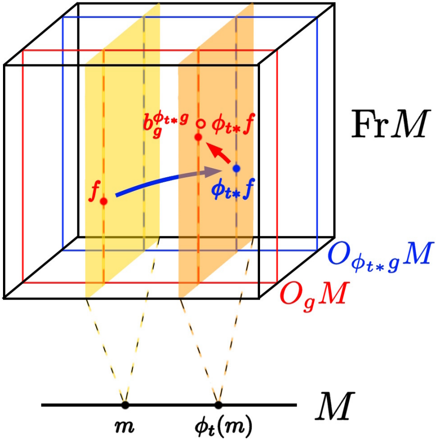

$X_f^O := \frac{d}{dt}|_{t=0}\, b_g^{\phi_{t*}g} \circ \phi_{t*}f \quad \in T_f O_g M$
Then $\pi_* X^P := X^O$ defines $X^P \in \mathfrak{X}(PM)$

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Definition: a CA is $(E, \rho, \langle . , . \rangle_E , [. , .]_E)$:

- $E \to M$
- $\rho : E \to TM$
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- $[. , .]_E : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$

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Courant algebroids

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A generalized LC connection is a map $D : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ s.t. its generalized torsion vanishes and $D_u(V_\pm) \subset V_\pm$

Lift to the Pin bundle

Consider V_+ with $\langle\,,\,\rangle|_{V_+}$ of signature (r, s).

$$O(r,s) \times O(p-r, q-s) \circlearrowright O_{V_\pm}M$$

$$\begin{array}{c} O_{V_\pm}M \\ \big\downarrow \\ M \end{array}$$

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Generalised metric Lie derivative

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Canonical infinitesimal isometry for generalized metrics:

$$b^{V_\pm}_{V'_\pm} : \{\, e_a \,,\ e_{\dot{b}} \,\} \mapsto \{\, p_{V'_+} e_a \,,\ p_{V'_-} e_{\dot{b}} \,\}$$

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Define $u^P \in \mathfrak{X}(P_{V_\pm} M)$ as: $\pi_* u^P := u^O$.

Action of $\mathcal{L}^{\mathcal{G}}$ on $\psi \in S(V_+) \otimes \Gamma(V_-)$:

Metric Lie derivative of $\psi \in S(V_+) \otimes \Gamma(V_-)$

$$(\mathcal{L}^{\mathcal{G}}_u \psi)^{\dot{c}} = (D_u \psi)^{\dot{c}} + \frac{1}{2}(D_a u_b)\gamma^{ab}\psi^{\dot{c}} + 2(D^{[\dot{c}} u^{\dot{d}]})\psi_{\dot{d}}$$

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Thank you.