Quantum chaos and string theory scattering amplitudes

Dorin Weissman

INFN, Naples

Corfu Workshop on Quantum Gravity and Strings, September 2026

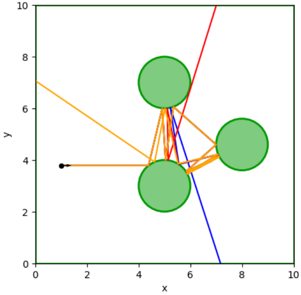
References

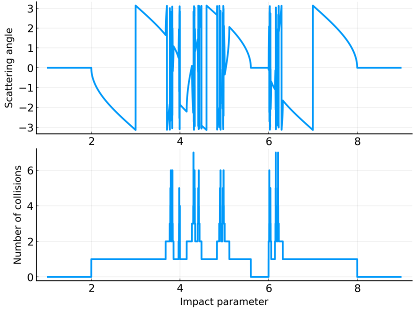
Based on work with M. Bianchi, M. Firrotta, and J. Sonnenschein:

- *Measure for chaotic scattering amplitudes* [**arXiv:2207.13112**]
- *Measuring chaos in string scattering processes* [**arXiv:2303.17233**]
- *From spectral to scattering form factor* [**arXiv:2403.00713**]
- *Multi-dimensional chaos I: Classical and quantum mechanics* [**arXiv:2510.03007**]
- *Multi-dimensional chaos II: String scattering amplitudes, curve repulsion, and RMT* [**arXiv:2606.24490**]

Classical chaos

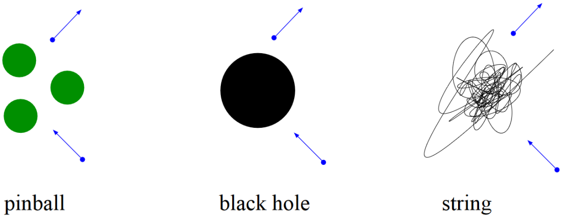
The most intuitive definition of chaos is classical - the “butterfly effect”.
One of the simplest systems where we see it is 3-disk pinball:





Chaos in string scattering

String theory can probe chaos in a similar way, from scattering of highly excited string (HES) states [**Gross-Rosenhaus (2021)**]:



Quantum chaos and random matrices

We will consider one particular indicator of chaos in quantum mechanical systems. If we have a discrete spectrum

$$\hat{H}\,\Psi = E_n\Psi$$

We can define the energy **level spacings**

$$\delta_n \equiv E_n - E_{n-1}$$

In chaotic systems the Hamiltonian can be thought of as a **random matrix**, and then we have a prediction for the **distribution of spacings**. We will use extensively the distribution of **spacing ratios**,

$$r_n \equiv \frac{\delta_n}{\delta_{n-1}}$$

- Typically, the spectrum has to be "**unfolded**" using the average density of states $\rho(E)$ to uncover the universal statistics of δ_n. Spacing ratios generally do not require unfolding.

The Gaussian β-ensembles

The joint probability density function of the eigenvalues of a random matrix in the Gaussian ensembles is:

$$P_L(\lambda_1, \lambda_2, \ldots, \lambda_L) = \mathcal{N}_L(\beta) \times \exp\left(-\frac{\beta}{2}\sum_{i=1}^{L} \lambda_i^2\right) \prod_{1\leq i<j\leq L} |\lambda_i - \lambda_j|^\beta$$

This corresponds, as pointed out by [**Dyson 1962**], to the system of a one-dimensional Coulomb gas, with

$$V(\lambda_i) = \frac{1}{2}\lambda_i^2\,, \qquad V_{int}(\lambda_i, \lambda_j) = -\log|\lambda_i - \lambda_j|$$

at temperature $T = \frac{1}{\beta}$.
The three Gaussian classical ensembles of RMT–orthogonal, unitary, and symplectic–are located at $\beta = 1$, 2, and 4, respectively, but one can define a generalization for any $\beta > 0$ starting from the eigenvalue distribution.

The level spacings are expected to have the distribution (the Wigner–Dyson surmise):

$$p_\beta(s) = \mathcal{N}_\beta s^\beta \exp(-c_\beta s^2)$$

This is the distribution of the spacings of unfolded eigenvalues, $z_n = I(\lambda_n)$, and $s_n = z_{n+1} - z_n$.
The spacing ratios are distributed according to [**Atas-Bogomolny-Giraud-Roux (2013)**]:

$$f_\beta(r) = C(\beta) \frac{(r + r^2)^\beta}{(1 + r + r^2)^{1+\frac{3}{2}\beta}}$$

- One key feature is the level repulsion at small s or r.
- The distribution of ratios is symmetric under $r \to \frac{1}{r}$. It is useful to work with $\tilde{r} = \min(r, \frac{1}{r}) \leq 1$.

Random matrices for chaotic scattering

- Chaotic scattering in quantum mechanics corresponds to having a random unitary S-matrix [**Blümel-Smilansky 1988**].
- These can be defined by the *circular β-ensembles* in RMT. The eigenvalues of unitary matrices are all on the unit circle, and we can talk instead about the *eigenphases* ϕ_i:

$$\lambda_i = e^{i\phi_i}$$

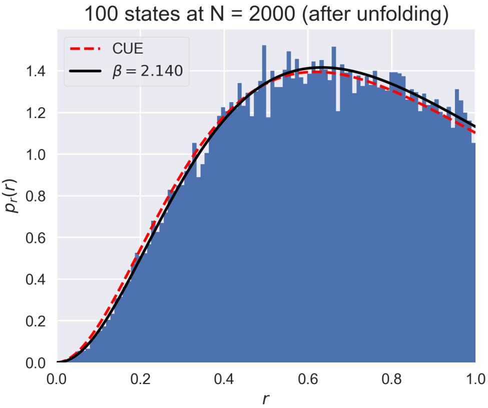
- The probability measure is now

$$P(\phi_1, \ldots, \phi_L) = \mathcal{N}_L(\beta) \prod_{i<j} |e^{i\phi_i} - e^{i\phi_j}|^\beta$$

- For large matrices, the distribution of spacings of eigenphases is the same Wigner-Dyson distribution as in the corresponding Gaussian ensembles. The level repulsion is governed by the same Vandermonde determinant $\prod_{i<j} |\lambda_i - \lambda_j|^\beta$ in both types of ensembles.

Results: spacings in string theory amplitudes

Our main result is that we find the RMT distributions for spacing ratios in scattering amplitudes of highly excited strings by analyzing the spacings between consecutive peaks in the angular dependence of the amplitude:

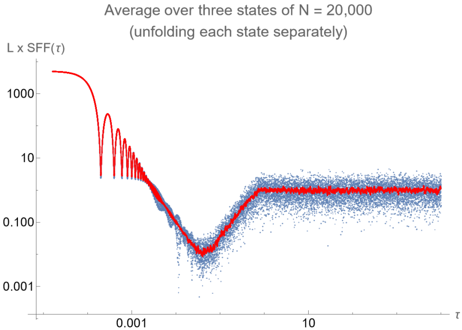


From spectral to scattering form factor

We also compute the (equivalent of the) spectral form factor [**BFSW 2024**],

$$\mathrm{SFF}(t) = \frac{1}{L} \sum_{i,j} e^{i(\lambda_i - \lambda_j)t}$$

and find the characteristic dip–ramp–plateau structure associated with chaotic behavior and expected from RMT (CUE in this case):



Chaotic behavior of string amplitudes

We see the chaotic patterns in the angle dependence of string amplitudes of highly excited states $H_N^{(i)}$. Each state is defined (in part) by a partition $\{g_n\}$ of the level N with helicity h:

$$N = \sum_{n=1}^{\infty} n\, g_n\,, \qquad h = \sum_{n=1}^{\infty} g_n\,.$$

The simplest amplitude is the three-point function of $H_N^{(i)}$ and two tachyons:

$$\mathcal{A}_{H_N^{(i)} \to TT} \propto (\sin\alpha)^h \prod_{n=1}^{N} \left(\frac{(1 - n\cos^2\frac{\alpha}{2})_{n-1}}{\Gamma(n)} \right)^{g_n}$$

The state is constructed in the DDF formalism, as we will review briefly below.

Constructing highly excited string (HES) states

It is tedious to construct vertex operators of physical string states with high excitation number. Reminder: A generic open-string state is built from terms

$$\xi_{\mu_1 \ldots \mu_k} \alpha^{\mu_1}_{-n_1} \ldots \alpha^{\mu_k}_{-n_k} |0; p\rangle$$

corresponding to vertex operators

$$\xi_{\mu_1 \ldots \mu_k} \partial^{n_1} X^{\mu_1} \ldots \partial^{n_k} X^{\mu_k} e^{ip \cdot X}$$

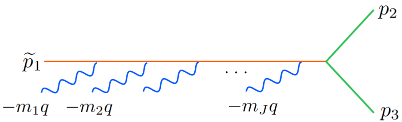
A physical state $|\phi\rangle$ should obey the Virasoro constraints, i.e. for all $n > 0$

$$L_n |\phi\rangle = 0 \,, \qquad L_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_{n-m} \cdot \alpha_m$$

as well as $(L_0 - 1)|\phi\rangle = 0$ which sets the mass, $\alpha' M^2 = -1 + \sum_k n_k \equiv N - 1$.

DDF Construction

Di Vecchia, Del Giudice, and Fubini [**DDF, 1972**] proposed a systematic way for constructing vertex operators of highly excited strings.
Useful reviews are in [**Gross-Rosenhaus 2021, Bianchi-Firrotta 2019**].
Pictorially, the state is created by repeatedly scattering photons off a tachyon,



The tachyon vertex operator is

$$V_0 = e^{ip\cdot X}\,, \qquad \alpha' p^2 = 1$$

We define the DDF operators

$$A_n^i = \frac{i}{\sqrt{2\alpha'}} \oint \frac{dz}{2\pi i} \partial X^i e^{inq\cdot X}$$

where $i = 1, \ldots, D-2$ and $q^2 = 0$. One also imposes $2\alpha' p \cdot q = 1$. One can prove that **all physical states** of the string can be written as

$$\prod_k (\lambda_k \cdot A_{-m_k}) e^{ip\cdot X}$$

where $\lambda_k \cdot q = 0$. The states of this form automatically satisfy the Virasoro constraints, and have

$$p_N = p - Nq$$

.

The decay amplitude

The easiest amplitude that one can compute is $H_N^{(i)} \to TT$, the decay of a highly excited DDF state to two tachyons.
We don't take the most generic state, but one of the form

$$\prod_{n=1}^{\infty} (\lambda \cdot A_{-n})^{g_n} \, e^{ip\cdot X}$$

The polarization λ is fixed to be a circular polarization with $\lambda^2 = 0$, and the number of states of this form at a given level N is equal to the number of partitions of N.

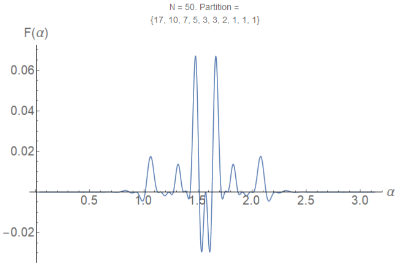
$$N = \sum_{n=1}^{\infty} n \, g_n \, , \qquad h = \sum_{n=1}^{\infty} g_n \, .$$

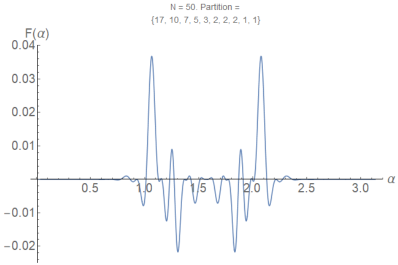
The derivation of the amplitude is too technically involved to be repeated here. The result can be written as

$$\mathcal{A}_{H_N^{(i)}\to TT} \propto (\sin\alpha)^h \prod_{n=1}^{N}\left(\frac{(1-n\cos^2\frac{\alpha}{2})_{n-1}}{\Gamma(n)}\right)^{g_n}$$

where $(a)_n = a(a+1)\dots(a+n-1)$ is the Pochhammer symbol. It is a product of polynomials of $\cos^2\frac{\alpha}{2}$, where α is the angle between the DDF photon and one of the tachyons.

It is a highly oscillating function that depends greatly on the choice of partition:





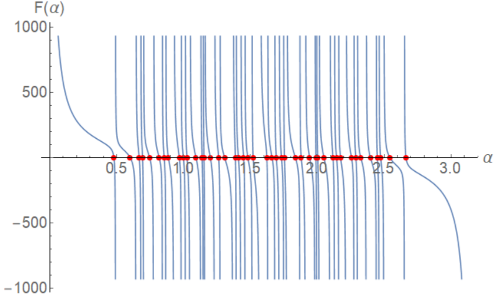
Spacing statistics

We note that the amplitude has erratically spaced peaks. This is better visualized when plotting the logarithmic derivative

$$F(\alpha) = \frac{d\log\mathcal{A}}{d\alpha} = \frac{1}{2\sqrt{x(1-x)}}\left(h(2x-1) - 2x(1-x)\sum_{n=1}^{\infty} g_n \sum_{k=1}^{n-1}\frac{n}{k-nx}\right)$$

where $x(\alpha) = \cos^2\frac{\alpha}{2}$.

N = 100. Partition =
{17, 10, 10, 9, 7, 7, 6, 5, 5, 4, 4, 4, 3, 2, 2, 2, 1, 1, 1}



Integer partitions

We now analyze the spacings of z_n defined by $F(z_n) = 0$.

- The number of zeros for a given state scales linearly with N. For large enough N one has enough points for any single amplitude at that level. At smaller values of N we perform a combined analysis of the amplitudes of many states.

The number of integer partitions of N is (asymptotically)

$$p_N \approx \frac{1}{4\sqrt{3}N} e^{C\sqrt{N}}, \qquad C \equiv \pi\sqrt{\frac{2}{3}}$$

The number of partitions of length h follows approximately a Gumbel distribution

$$d_N(h) = \frac{1}{\sigma} \exp\left(-\frac{h-\mu}{\sigma} - e^{-\frac{h-\mu}{\sigma}}\right)$$

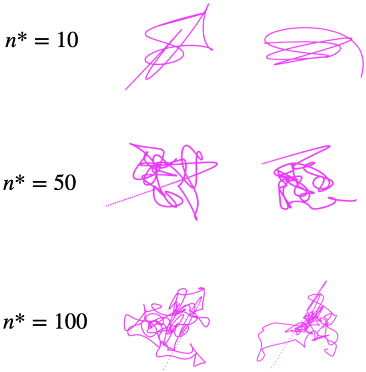
where $\sigma \sim \sqrt{N}$ and the distribution is peaked around

$$\langle h \rangle \approx \frac{1}{C}\sqrt{N}\log N$$

which makes this the “typical helicity” of a random state at level N.

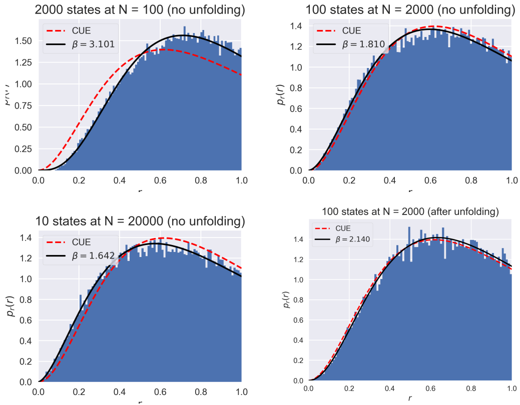
Generic string states

Generic partitions will create **generic** string states:



The results

We plot the distributions of all the values of $\tilde{r}_n \equiv \min(r_n, \frac{1}{r_n})$ accumulated for either many states at intermediate N, or a few states at very large N:



There is a non-trivial dependence of the fitting parameter β on N. We have indications that it may converge to the CUE value of 2.

More general amplitudes

We have also studied the scattering amplitude of one HES and three tachyons, $T+T\to T+H_N^{(i)}$. This amplitude is given by the Veneziano amplitude times a state-dependent dressing factor:

$$\mathcal{A}(s,\theta,\phi)=\mathcal{A}_{\text{Ven.}}(s,\theta)\times\mathcal{D}_{H_N^{(i)}}(s,\theta,\phi)$$

where the dressing factor has a compact expression in the high-energy fixed-angle limit:

$$\mathcal{D}_{H_N^{(i)}}(s,\theta,\phi)=(\alpha' s)^{h/2}\left(\frac{\sin\phi\sin\theta}{\cos^2\frac{\theta}{2}}\right)^h\prod_{n=1}^{N}\left(\mathcal{P}_n(\theta,\phi)\right)^{g_n}$$

The dressing factor is constructed from the Jacobi polynomials

$$\mathcal{P}_n(\theta,\phi)=P_{n-1}^{(-2n+1,-n\sin\theta\cos\phi)}\left(1-\frac{2}{\cos^2\frac{\theta}{2}}\right)$$

For these amplitudes we find a rich structure when we look at the curves where

$$\mathcal{F}_i(x, y) \equiv \partial_i \log \mathcal{D}(x, y) = 0$$

where $x \equiv \cos\theta$, $y \equiv \cos\phi$.

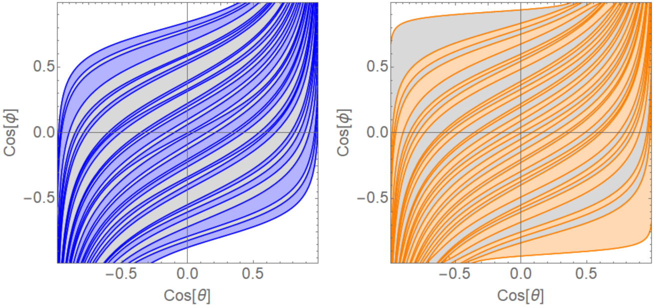


Figure 1: Curves of zeros of the x-derivative (left), and y-derivative (right) of $\log \mathcal{D}(x, y)$, for the state with $N = 50 = 13 + 10 + 9 + 5 + 5 + 3 + 2 + 1 + 1 + 1$.

If we treat the areas under the curves as the "eigenvalues", the areas *between* the curves will be spacings, and we find the distribution of spacing ratios again appears to converge to RMT:

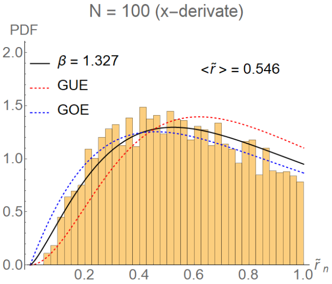


Figure 2: Distributions of ratios of adjacent areas compared to RMT. The distribution is obtained after averaging over 100 states of $N = 100$.

Conclusions

- Instead of energy level spacings, one can look at spacings between solutions $F(z_n) = 0$ for $F(z) = \frac{d}{dz} \log \mathcal{A}$.
- In string theory we can find compact analytic expressions for amplitudes of highly excited string states. These amplitudes have a rich structure and they display RMT statistics, an indicator of quantum chaos.
- The most interesting next step would be to look at scattering processes like $H_N T \to H_N T$ (see e.g. [**Hashimoto-Matsuo-Yoda (2022)**]).
- Our recent work focused on examining chaotic behavior in a more generic setting where the dependence is on more than one kinematic parameter.
- Other than looking at only string amplitudes, we can apply the lessons learned to other systems.