Weyl gauge symmetry and Weyl-Dirac-Born-Infeld action

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- Outline:

- Gauging scale symmetry $\Rightarrow$ Weyl conformal geometry (WG)

- Weyl gauge theory of Quadratic Gravity (WQG) $\Rightarrow$ SSB to Einstein-Hilbert (EH) action

- EH + SM: natural embedding in WG

- Weyl-Dirac-Born-Infeld (WDBI) $\Rightarrow$ WQG $\Rightarrow$ EH. Then add SM.

- WG dual to Riemannian geometry of dressed metric (RG^*)

- Entanglement in quantum nonlocal RG^*

- Conclusions

• Goal: find a gauge theory beyond SM and GR

- SM and Higgs mechanism confirmed @ LHC; origin of EW scale in SSB/Higgs
- Gravity: origin of Planck mass M_P, cc Λ? must go beyond Einstein-Hilbert gravity; Seek alternative to Sugra/Strings. How?
- Noether's gauge principle (as in SM) $\rightarrow$ Gauging a space-time symmetry (beyond Poincaré) $\rightarrow$ geometry

$\rightarrow$ Einstein-Hilbert gravity?

• Scale/dilatation symmetry:

- SM with $m_{\rm Higgs}=0$ scale invariant [Bardeen 1995]; in Early Universe / short distances EFTs are scale invariant.
- Discrete (fractals); Global, Local, Gauged: Weyl gauge symmetry - Weyl group: Poincaré & dilatations

- realised in Weyl geometry - first gauge theory, Weyl 1918!
- Gauging full conformal group: not a true gauge theory! (no physical gauge bosons)$\rightarrow$ Conformal gravity.

• Weyl gauge symmetry: $\Rightarrow$ WG definition: equiv classes of $(g_{\alpha\beta},\, \omega_\mu)$ related by:

$$\omega'_\mu = \omega_\mu - \partial_\mu \ln \Sigma, \qquad g'_{\mu\nu} = \Sigma^2\, g_{\mu\nu}, \qquad \phi' = \Sigma^{-1}\phi, \qquad \psi' = \Sigma^{-3/2}\,\psi.$$

- non-metric (*): $\tilde\nabla_\mu g_{\alpha\beta} = -2\omega_\mu\, g_{\alpha\beta}, \qquad \tilde\nabla_\mu g_{\alpha\beta} = \partial_\mu\, g_{\alpha\beta} - \tilde\Gamma^\rho_{\mu\alpha}\, g_{\rho\beta} - \tilde\Gamma^\rho_{\nu\alpha}\, g_{\rho\alpha}.$

Hermann Weyl, 1885 - 1955

credit photo: ETH

• History: Parallel transport of a vector $(\tilde\nabla_\mu u^\alpha = 0)$: direction & norm change:

$$d|u|^2 = dx^\mu\, \tilde\nabla_\mu (u^\alpha u^\beta g_{\alpha\beta}) = -2\,|u|^2\, dx^\mu\, \omega_\mu \neq 0, \quad \Rightarrow \quad |u|^2 = |u_0|^2 \exp\Big[-2 \int_{\gamma(\tau)} dx^\mu\, \omega_\mu \Big].$$

Weyl 1918: Gravity+EM (ω_μ real photon), idea failed, photon$\sim$ U(1), but WG correct.

Einstein 1918: atomic spectral distances depend on path, WG not physical (unless $\omega_\mu = \partial_\mu \ln \Sigma$). A long lasting (wrong) legacy.

• In this talk: WG: vector-tensor gauge theory of gravity, Weyl gauge group = Poincaré & D(1) dilatations, ω_μ gauge boson

(*): unlike Riemannian geometry: $\nabla_\mu g_{\alpha\beta} = 0.$ $\qquad \Rightarrow \qquad \Gamma^\rho_{\mu\nu} = \frac{1}{2}\, g^{\rho\lambda} \big[\partial_\mu g_{\mu\lambda} + \partial_\nu g_{\mu\lambda} - \partial^\rho g_{\mu\nu} \big]$ (Levi-Civita).

• WG: Historical Formulation: non-metric (no torsion, affine): $\tilde\nabla(\tilde\Gamma)$: $\quad \omega'_\mu=\omega_\mu-\partial_\mu\ln\Sigma, \quad g'_{\mu\nu}=\Sigma^2\,g_{\mu\nu},$ (*)

$$\tilde\nabla_\mu\, g_{\alpha\beta}=-2\omega_\mu\, g_{\alpha\beta}\,, \qquad \tilde\Gamma^\rho_{\mu\nu}=\Gamma^\rho_{\mu\nu}+(1/2)\,\big(\,\delta^\rho_\mu\,\omega_\nu+\delta^\rho_\nu\,\omega_\mu-g_{\mu\nu}\,\omega^\rho\,\big), \qquad \omega_\mu\sim\tilde\Gamma^\alpha_{\alpha\mu}-\Gamma^\alpha_{\alpha\mu}, \quad \tilde\Gamma^\alpha_{\mu\nu}=\tilde\Gamma^\alpha_{\nu\mu}, \quad \text{inv. of } (*)$$

$$[\tilde\nabla_\mu,\tilde\nabla_\nu]\,v^\rho=\tilde R^\rho{}_{\sigma\mu\nu}\,v^\sigma \qquad \Rightarrow \qquad \tilde R^\rho{}_{\sigma\mu\nu}=\partial_\mu\tilde\Gamma^\rho_{\nu\sigma}-\partial_\nu\tilde\Gamma^\rho_{\mu\sigma}+\tilde\Gamma^\rho_{\mu\tau}\tilde\Gamma^\tau_{\nu\sigma}-\tilde\Gamma^\rho_{\nu\tau}\tilde\Gamma^\tau_{\mu\sigma}\,. \qquad F_{\mu\nu}=\tilde\nabla_\mu\,\omega_\nu-\tilde\nabla_\nu\,\omega_\mu=\partial_\mu\omega_\nu-\partial_\nu\omega_\mu$$

$\Rightarrow$ if $\omega_\mu\to 0$, $\tilde\Gamma\to\Gamma$, Weyl geometry $\to$ Riemannian, $\tilde R^\rho{}_{\sigma\mu\nu}\to R^\rho{}_{\sigma\mu\nu}$, etc.

under (*): $\Rightarrow$ $\tilde R'{}^\rho{}_{\sigma\mu\nu}=\tilde R^\rho{}_{\sigma\mu\nu}$, $\tilde R'_{\mu\nu}=\tilde R^\rho_{\mu\nu}$, $\tilde R'=\Sigma^{-2}\tilde R$, But: $(\tilde\nabla_\lambda R)'\neq\Sigma^{-2}\,\tilde\nabla_\lambda\tilde R$: not covariant/not metric.

$\Rightarrow$ Action: $\sqrt{g}\,\tilde R^2+\sqrt{g}\,F^2_{\mu\nu}+\sqrt{g}\,\tilde C^2_{\mu\nu\alpha\beta}+\sqrt{g}\,\tilde G$, invariant under (*).

- non-metric $\to$ for calculatins must go to (metric) Riemannian picture (∇_μ, $R^\rho{}_{\sigma\mu\nu}$, $R_{\mu\nu}$, R): $\quad \tilde R=R-6\,\nabla_\mu\omega^\mu-6\,\omega_\mu\omega^\mu$.

• WG: Weyl gauge covariant, metric (not affine) formulation! [Dirac 1973, D.G. 2309.11372, D.G., Micu, Condeescu 2312.13384]

$(\tilde\nabla_\mu + 2\,\omega_\mu)\, g_{\alpha\beta} = 0. \quad \Rightarrow \quad \widehat\nabla_\mu g_{\alpha\beta} \equiv \tilde\nabla_\mu g_{\alpha\beta}\Big\vert_{\partial_\mu\to\partial_\mu+\text{charge}\times\omega_\mu} = 0,$ Weyl geometry= "covariantised" Riemannian geom wrt D(1)

$\widehat\nabla_\mu$ for Weyl geometry "=" ∇_μ for Riemannian geometry!

In general:

$\Rightarrow \quad \widehat\nabla_\mu X \equiv (\tilde\nabla_\mu + q_X\,\omega_\mu)\, X = 0,$ if: $X' = \Sigma^{q_X}\, X$ then $\widehat\nabla'_\mu X' = \Sigma^{q_X}\, \widehat\nabla_\mu X,$ $X \equiv X^{\mu_1\mu_2\dots}_{\nu_1\nu_2\dots}$ (p,r) tensor.

$\Rightarrow \; \widehat R'^{\rho}{}_{\sigma\mu\nu} = \widehat R^{\rho}{}_{\sigma\mu\nu}, \quad \widehat R'_{\mu\nu} = \widehat R^{\rho}_{\mu\nu}, \quad \widehat R' = \Sigma^{-2}\widehat R,$ $[\widehat\nabla_\mu, \widehat\nabla_\nu]\, X^\rho = \tilde R^{\lambda}{}_{\sigma\mu\nu}\, X^\sigma + \tilde q_X F_{\mu\nu} X^\sigma = \widehat R^{\lambda}{}_{\sigma\mu\nu}\, X^\sigma + q_X F_{\mu\nu} X^\sigma.$

$\Rightarrow \widehat\nabla'_\mu \widehat R'_{\alpha\beta} = \widehat\nabla_\mu \widehat R_{\alpha\beta}; \quad \widehat\nabla'_\mu \widehat R' = \Sigma^{-2}\widehat\nabla_\mu \widehat R,$

$\Rightarrow$ Action: $\sqrt{g}\,\widehat R^2 + \sqrt{g}\,\widehat F^2_{\mu\nu} + \sqrt{g}\,\widehat C^2_{\mu\nu\alpha\beta} + \sqrt{g}\,\widehat G$: Weyl gauge invariant, metric formulation. Avoids Einstein's critique:

Norm of vectors invariant! $d|u|^2 = dx^\mu \widehat\nabla_\mu (u^\alpha u^\beta g_{\alpha\beta}) = 0$

[Dirac 1973, Hobson 2009.06407; DG 2203.05381]

also: $\widehat R^{\rho}{}_{\sigma\mu\nu} = \tilde R^{\rho}{}_{\sigma\mu\nu} - \delta^\rho_\sigma\; F_{\mu\nu}, \quad \widehat R_{\mu\nu} = \tilde R_{\mu\nu} - F_{\mu\nu}, \quad \widehat R = \tilde R, \quad \widehat F_{\mu\nu} = \tilde F_{\mu\nu}$

$\widehat R = R - 6\,\nabla_\mu \omega^\mu - 6\,\omega_\mu\omega^\mu,$ R: Riemann geometry

• Weyl quadratic gravity: Broken phase: Einstein-Hilbert gravity + massive ω_μ [D.G. arXiv:2203.05381, 2104.15118, 1812.08613]

$$\mathcal{L}_0 = \sqrt{g}\,\Big[\frac{1}{4!}\,\frac{1}{\xi^2}\,\widehat{R}^2 - \frac{1}{4\,\alpha^2}\,F_{\mu\nu}^2\Big] = \sqrt{g}\,\Big[\frac{1}{4!}\,\frac{1}{\xi^2}\big(-2\phi^2\,\widehat{R} - \phi^4\big) - \frac{1}{4\,\alpha^2}\,F_{\mu\nu}^2\Big], \qquad \text{eq. motion } \phi^2 = -\widehat{R}.$$

Go to Riemannian notation: $$\mathcal{L}_0 = \sqrt{g}\,\Big\{\frac{-1}{2\,\xi^2}\Big[\frac{1}{6}\,\phi^2\,R + (\partial_\mu\phi)^2\Big] - \frac{\phi^4}{4!\,\xi^2} + \frac{1}{8\,\xi^2}\,\phi^2\,\Big[\omega_\mu - \partial_\mu \ln\phi^2\Big]^2 - \frac{1}{4\,\alpha^2}\,F_{\mu\nu}^2\Big\}.$$

$\ln\phi \to \ln\phi - \Sigma$ (would-be-Goldstone)

If $\langle\phi\rangle \neq 0$, "gauge fixing": $\Sigma = \phi/\langle\phi\rangle \quad\Rightarrow\quad g'_{\mu\nu} = \Sigma^2 g_{\mu\nu}, \quad \phi' = \Sigma^{-1}\,\phi = \langle\phi\rangle, \quad \omega'_\mu = \omega_\mu - \partial_\mu \ln\phi^2$

$$\mathcal{L}_0 = \sqrt{g'}\,\Big[-\frac{1}{2}M_p^2\,R' + \frac{3}{4}M_p^2\,\alpha^2\,\omega'_\mu\omega'^{\,\mu} - \frac{1}{4}F'^{\,2}_{\mu\nu} - \Lambda\,M_p^2\Big], \qquad M_p^2 \equiv \frac{\langle\phi^2\rangle}{6\,\xi^2}, \quad \Lambda \equiv \frac{1}{4}\langle\phi\rangle^2.$$

$\Rightarrow$ Einstein-Proca action; M_p, Λ, m_ω by Stueckelberg mechanism; ω_μ massive. All masses Λ, M_p, m_ω of geometric origin.

$\Rightarrow$ when ω_μ decouples $\to \tilde{\Gamma} \to \Gamma$ then WG $\to$ Riemannian. Spontaneous breaking (# dof const), non-trivial.

$\Rightarrow$ Einstein gravity from the broken phase of Weyl action. $\Lambda \ll M_p^2$ because gravity is ultraweak: $\xi \ll 1$.

Adding to $\mathcal{L}_0$: $(1/\eta^2)\,\sqrt{g}\,\widehat{C}^2_{\mu\nu\rho\sigma}$: no change to above mechanism. Spin-2 ghost $m_\eta \propto \eta\,M_p \sim M_p$ if $\eta \sim \mathcal{O}(1)$ [Hawking hep-th/0107088]

Alternatively: removed by b.c.'s: Maldacena 1105.5632; Hell, Lüst, Zoupanos [2311.08216, 2306.13714]. $(\widehat{C}_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma})$

• WQG: Weyl quadratic gravity $\Rightarrow$ Einstein-Hilbert gravity + massive ω_μ

$\Rightarrow$ we have: $\nabla^\mu J_\mu = 0$; $J_\mu = \frac{\alpha}{2\xi^2}\,\phi\,[\partial_\mu - \alpha\,\omega_\mu]\,\phi \sim \widehat{\nabla}_\mu \widehat{R}$; if $\phi \to \langle\phi\rangle$, $\nabla_\mu \omega^\mu = 0$ gauge fixing.

- J_μ extends a global scale symmetry current $\propto \phi\,\partial_\mu\phi$ [G.G. Ross et al 1801.07676; M. Shaposhnikov et al]

$\Rightarrow$ if ω_μ not dynamical: $\omega_\mu \sim (1/\alpha)\,\partial_\mu(\ln\phi^2)$, $J_\mu = 0$ "integrable" case $\Rightarrow$ Weyl conformal gravity $C^2_{\mu\nu\rho\sigma}$! [Mannheim et al]

- $M_p^2 \sim \langle\phi\rangle^2/\xi^2$, $\Lambda \sim \langle\phi\rangle^2$, $m_\omega^2 \sim \alpha^2 M_p$: geometric origin $\Rightarrow$ Geometry as the origin of mass!

- Parameters in the theory: dim-less ξ, α and $\langle\phi\rangle$ $\Rightarrow$ we fix Λ, M_P and m_ω. [D.G. 2203.05381 [hep-th]]

$\Rightarrow$ Weyl's theory: gauge theory of scale inv, an embedding of Einstein gravity. Power counting renormalizable [Stelle 1979]

- Conservation laws: $T_{\mu\nu}$ and J_μ: wrt to Weyl geom $\widehat{\nabla}^\mu J_\mu = 0$ and wrt Riemannian $\nabla^\mu J_\mu = 0$!

- Eqs of motion $\sqrt{g}\,\widehat{R}^2 + \sqrt{g}\,\widehat{F}^2_{\mu\nu} + \sqrt{g}\,\widehat{C}^2_{\mu\nu\rho\sigma} + \sqrt{g}\,\widehat{G}$: in Weyl gauge covariant formalism (with $\widehat{\nabla}$) "simple" form/available.

[C. Condeescu, D.G., A. Micu, 2412.16548 EPJC]

• Adding matter: SM Higgs in Weyl gravity/geometry h=neutral/singlet higgs [Higgs doublet, see page 19]

$$\mathcal{L} = \sqrt{g}\,\Big\{\frac{1}{4!\,\xi^2}\,\widehat{R}^2 - \frac{1}{4}\,F_{\mu\nu}^2\ - \frac{1}{12}\,\xi_1\,h^2\,\widehat{R}\ + \frac{1}{2}\,(\widehat{D}_\mu h)^2 - \frac{\lambda}{4!}\,h^4\Big\},\qquad \widehat{R}^2 \to -2\phi^2\,\widehat{R} - \phi^4$$

$$= \sqrt{g}\,\Big\{-\frac{1}{12}\underbrace{\Big[\frac{1}{\xi^2}\phi^2 + \xi_1\,h^2\Big]}_{=6\,\rho^2}\widehat{R} - \frac{1}{4}\,F_{\mu\nu}^2 + \frac{1}{2}\,(\widehat{D}_\mu h)^2 - \frac{1}{4!}\Big[\lambda\,h^4 + \frac{1}{\xi^2}\phi^4\Big]\Big\},\qquad \widehat{D}_\mu h = (\partial_\mu - \alpha\,\omega_\mu)h.$$

$\ln\rho \to \ln\rho - \Sigma$ ("new" would-be-Goldstone) [DG, 2104.15118]

Go to Riemannian notation:

$$\mathcal{L} = \sqrt{g}\,\Big\{-\frac{1}{2}\Big[\frac{1}{6}\rho^2\,R + (\partial_\mu\rho)^2\Big] + \frac{3}{4}\,\rho^2\,\Big(\omega_\mu - \frac{1}{\alpha^2}\,\partial_\mu\ln\rho^2\Big)^2 - \frac{1}{4}\,F_{\mu\nu}^2 + \frac{1}{2}(\widehat{D}_\mu h)^2 - V(h,\rho)\Big\}.$$

Stueckelberg: radial direction $\ln\rho$ eaten by ω_μ. Next: $\Sigma = \dfrac{\rho}{\langle\rho'\rangle}$, $\langle\rho'\rangle = M_P$, $\omega'_\mu = \omega_\mu - \dfrac{1}{\alpha^2}\partial_\mu\ln\rho^2$

$\Rightarrow$ Einstein-Proca: (ω_μ) :
$$\mathcal{L} = \sqrt{g}\,\Big\{-\frac{1}{2}\,M_P^2\,R' + \frac{3}{4}\,\alpha^2\,M_p^2\,\omega'_\mu\hat{\omega}'^\mu - \frac{1}{4}\,F'^{\,2}_{\mu\nu} + \frac{1}{2}\,(\widehat{D}'_\mu h')^2 - V\Big]\Big\},$$

Unitarity gauge: $\quad h' \to M_p\sqrt{6}\sinh\frac{\sigma}{M_p\sqrt{6}}, \qquad \omega'_\mu \to \omega'_\mu + \partial_\mu \ln\cosh^2\frac{\sigma}{M_p\sqrt{6}}$

$$\Rightarrow \quad \mathcal{L} = \sqrt{\hat{g}}\,\Big\{ -\frac{1}{2}\,M_p^2\,R' + \frac{3}{4}\,\alpha^2\,M_p^2\;\omega'_\mu\,\omega'^\mu\cosh^2\frac{\sigma}{M_p\sqrt{6}} \;-\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}\,(\partial_\mu\sigma)^2 - V(\sigma)\Big\},$$

where
$$V(\sigma) \;=\; V_0\,\Big\{\,\xi^2\Big[1-\xi_1\sinh^2\frac{\sigma}{M_p\sqrt{6}}\Big]^2 + \lambda\sinh^4\frac{\sigma}{M_p\sqrt{6}}\Big\},$$

$\Rightarrow$ Higgs coupling to ω_μ: $\qquad \mathcal{L} \sim \sqrt{g'}\;\,\alpha^2\,\sigma^2\,\omega'_\mu\,\omega'^{\,\mu} + \ldots.$

- (neutral/singlet) higgs from Weyl vector fusion, early Universe: $\omega_\mu\,\omega_\mu \to \sigma\,\sigma$

• SM Fermions in Weyl gravity/geometry:

$$\mathcal{L}_\psi = \frac{i}{2}\sqrt{g}\,\overline{\psi}\,\gamma^a\,e_a^\mu\,\underbrace{\Big[\partial_\mu + \frac{1}{2}\,s_\mu^{ab}\,\sigma_{ab}\Big]}_{\nabla_\mu}\psi + \text{h.c.}$$

spin connection (Riemann): $s_\mu^{ab} = -e^{\lambda b}\,(\partial_\mu e_\lambda^a - \Gamma_{\mu\lambda}^\nu\,e_\nu^a)$.

- Weyl spin connection: $\tilde{s}_\mu^{ab} = s_\mu^{ab}\Big|_{\partial_\lambda\to\partial_\lambda+(\text{charge})\,\alpha\,\omega_\lambda}$,

$$s_\mu^{ab} \to \tilde{s}_\mu^{ab} = s_\mu^{ab} + (1/2)\,\alpha\,(e_\mu^a\,e^{\nu b} - e_\mu^b\,e^{\nu a})\;\omega_\nu.$$

$\Rightarrow$ $\tilde{s}_\mu^{ab}$ is Weyl gauge invariant (like $\tilde{\Gamma}$). Lagrangian: replace $\partial_\lambda\psi \to \partial_\lambda + q_\psi\,\alpha\,\omega_\lambda$. Then ω_μ cancels out. $\sigma_{ab} = 1/4\,[\gamma_a, \gamma_b]$.

$$L_\psi = \frac{i}{2}\sqrt{g}\,\overline{\psi}\,\gamma^a\,e_a^\mu\Big[\partial_\mu + q_\psi\,\alpha\,\omega_\mu + \frac{1}{2}\,\tilde{s}_\mu^{ab}\,\sigma_{ab}\Big]\psi = \frac{i}{2}\sqrt{g}\,\overline{\psi}\,\gamma^a\,e_a^\mu\Big[\partial_\mu + \frac{1}{2}\,s_\mu^{ab}\,\sigma_{ab}\Big]\psi + \text{h.c.}$$

In SM: $$\mathcal{L}_\psi = \frac{i}{2}\sqrt{g}\,\overline{\psi}\,\gamma^a\,e_a^\mu\Big[\partial_\mu - i g\,\vec{T}\,\vec{\hat{A}}_\mu - i\,Y\,g'\,\hat{B}_\mu + \frac{1}{2}\,s_\mu^{ab}\,\sigma_{ab}\Big]\psi + \text{h.c.}$$

$\Rightarrow$ $\mathcal{L}_\psi$ as in SM in Riemannian geometry, no coupling to ω_μ! Yukawa interactions: invariant [Kugo 1977]

- Note: if U(1)$_Y\times$D(1) kinetic mixing: $\hat{B}_\mu = B'_\mu - \omega'_\mu\tan\tilde{\chi}$; coupling $\propto Y$. ω_μ anomaly free & massive!

• SM Gauge bosons:

$$\mathcal{L}_b = -\frac{1}{4}\sqrt{g}\, F_{\mu\nu}\, F_{\rho\sigma}\, g^{\mu\rho}\, g^{\nu\sigma},$$

with $F_{\mu\nu} = \tilde{\nabla}_\mu A_\nu - \tilde{\nabla}_\nu A_\mu ... = \partial_\mu A_\nu - \partial_\nu A_\mu ...$, symmetric $\tilde{\Gamma}$. $\mathcal{L}_b$ invariant under (*) for $\hat{A}_\mu = A_\mu$.

$\Rightarrow$ Action similar to (pseudo)Riemannian case. $\Rightarrow$ only SM Higgs sector changes!

$\Rightarrow$ SM in Weyl geometry: minimal embedding, no new dof's beyond SM & WG. Higgs coupling to ω_μ.

- $m_\omega \sim \alpha M_p$ so usually heavy $\alpha \sim \mathcal{O}(1)$, in UV region.

But it can be ultralight, for (allowed) ultraweak $\alpha \ll 1$. Current bound: $> 10^{-14}$ eV!

[Saridakis 2601.19407, E. Radu 2009.05376]

- Higgs mass quantum corrections: $\delta m_\sigma^2 \propto m_\omega^2$. Light m_ω: solution to mass hierarchy ?

- above m_ω symmetry restored; no scale, no counterterm!

[D.G. 2203.05381, 2104.15118]

• Weyl geometry vs Weyl anomaly: $d=4-2\epsilon$ [DG 2309.11372; 2408.07160]

$$W_c = \int d^dx\sqrt{g}\,\Big\{a_1\widehat R^2 + b_1\widehat F_{\mu\nu}^2 + c_1\widehat C_{\mu\nu\rho\sigma}^2 + d_1\widehat G\Big\}\ \widehat R^{(d-4)/2} \qquad (\phi^2=-\widehat R), \qquad \text{no}\quad \mu^{(d-4)}!!$$

$T_\mu^\mu=\nabla_\mu J^\mu=0$, $J^\mu\propto\widehat\nabla^\mu\widehat R$. Renormalized action, $d=4$, Weyl geometry: $\hat\Box/\phi^2$: gauge invariant!

$$W_r = \int d^4x\sqrt{g}\,\Big\{\hat R\Big[a_0+a_1\ln\frac{\hat\Box}{\phi^2}\Big]\hat R + \widehat F_{\mu\nu}\Big[b_0+b_1\ln\frac{\hat\Box}{\phi^2}\Big]\widehat F^{\mu\nu} + \widehat C_{\mu\nu\rho\sigma}\Big[c_0+c_1\ln\frac{\hat\Box}{\phi^2}\Big]\widehat C^{\mu\nu\rho\sigma} \qquad \text{Weyl term}$$
$$+\ \widehat R_{\mu\nu\rho\sigma}\Big[d_0+d_1\ln\frac{\hat\Box}{\phi^2}\Big]\widehat R^{\rho\sigma\mu\nu} - 4\widehat R_{\mu\nu}\Big[d_0+d_1\ln\frac{\hat\Box}{\phi^2}\Big]\widehat R^{\nu\mu} + \widehat R\Big[d_0+d_1\ln\frac{\hat\Box}{\phi^2}\Big]\widehat R\ \Big\}, \qquad \text{Euler term.}$$

$\Rightarrow$ Weyl gauge invariant $\Rightarrow$ no Weyl anomaly $\Rightarrow$ WQG: consistent (quantum) gauge theory of Weyl group.

$\Rightarrow$ In broken phase, Weyl anomaly recovered: $\phi^2\to\langle\phi\rangle^2=\Lambda$; ω_μ massive; $\widehat R\to R=-4\,\Lambda$, etc

$\Rightarrow$ brackets [....]: one-loop (log) running. No UV scale required (!)

• More general: Weyl-DBI gauge theory action (d dimensions) [DMG 2407.18173]

$$S_d = \int d^d\sigma \Big\{ -\det \Big[a_0\, \widehat{R}\, g_{\mu\nu} + a_1\, \widehat{R}_{\mu\nu} + a_2\, \widehat{F}_{\mu\nu}\Big] \Big\}^{1/2} = \int d^d\sigma \sqrt{g}\; a_0^{\frac{d}{2}}\; \hat{R}^{\frac{d}{2}} \Big\{ \det \big[\delta^\lambda_\nu + X^\lambda{}_\nu\big] \Big\}^{1/2}, \quad X^\lambda{}_\nu = \frac{g^{\lambda\rho}}{a_0\, \widehat{R}} \big[a_1\, \widehat{R}_{\rho\nu} + a_2\, \widehat{F}_{\rho\nu}\big].$$

- S_d is Weyl gauge invariant in d dimensions, with dimensionless couplings $a_{0,1,2}$, no need for a UV regulator/scale! $(d = 4 - 2\epsilon)$
- Taylor expand, leading term = Weyl quadratic gravity in d dim's; with gauge invariant regularisation $\widehat{R}^{(d-4)/2}$

$$S_d = \int d^d\sigma \sqrt{g}\; \underbrace{\widehat{R}^{(d-4)/2}}_{R^{-\epsilon}} \Big[\frac{1}{4!\,\xi^2}\, \widehat{R}^2 - \frac{1}{\eta^2} \widehat{C}_{\mu\nu\rho\sigma} \widehat{C}^{\mu\nu\rho\sigma} - \frac{1}{\alpha^2} \widehat{F}_{\mu\nu}\, \widehat{F}^{\mu\nu} + \frac{1}{\eta^2}\, \widehat{G}\Big] + \mathcal{O}(X^3)$$

where $\xi = f_1(a_0, a_1) \ll 1$, $\eta = f_2(a_0, a_1) < 1$, $\alpha = f_3(a_2, a_1) < 1$, for suitable a_0, a_1, a_2.
- Endorses/proves mathematically the regularisation of Weyl quadratic gravity!

- $\mathcal{O}(X^3)$: non-perturbative structure: $\displaystyle\int d^4x \sqrt{g}\, \frac{1}{\widehat{R}^2} \big[C_{\mu\nu\rho\sigma}\, C^{\mu\nu\rho\sigma}\big]^2 + \cdots$, ...some are quantum corections to WQG

⇒ Weyl-DBI action = Regularised (!) Weyl quadratic gravity + non-perturbative terms;

⇒ Weyl-DBI more general than WQG, The only theory that does not need a UV regulator!

• Adding matter: SM + Weyl-DBI [DG, arXiv: 2508.10959]

$A_{\mu\nu}$: all SM & Weyl gauge invariant operators, from WG + SM, of mass dimension 2 in $d=4-2\epsilon$:

$$S_{\mathbf{d}} = \int d^dx \Big[-\det A_{\mu\nu}\Big]^{\frac{1}{2}}, \qquad \begin{aligned} A_{\mu\nu} \;=\;& a_0\, \widehat{R}\, g_{\mu\nu} + a_1\, \widehat{R}_{\mu\nu} + a_2\, \widehat{F}_{\mu\nu} + a_3\, F^{(1)}_{\mu\nu} + a_4^{(i)}\, F^{(i)}_{\alpha\beta} F^{(i)\,\alpha\beta}\, g_{\mu\nu}\, \hat{R}^{-1} \\ +\;& a_5\, |\hat\nabla_\alpha H|^2\, \hat{R}^{1-d/2}\, g_{\mu\nu} + a_6\, |H|^2 \hat{R}^{2-d/2} g_{\mu\nu} + a_7 |H|^4\, \hat{R}^{3-d}\, g_{\mu\nu} \\ +\;& a_8\, \big(i\, \overline{\psi}\gamma^a\, e_a^\alpha \hat\nabla_\alpha\psi + \text{h.c.}\big)\, \hat{R}^{1-d/2}\, g_{\mu\nu} + a_9\, \big(\overline{\psi}_L\, Y_\psi\, H\psi_R + \overline{\psi}_L Y'_\psi \tilde{H}\, \psi'_R + \text{ h.c.}\big)\, \hat{R}^{2-3\,d/4}\, g_{\mu\nu}, \\ +\;& a_{10}\, \hat{F}_{\alpha\beta} \hat{F}^{\alpha\beta} \hat{R}^{-1} g_{\mu\nu} + a_{11}\, \hat{F}_{\alpha\beta} F^{(1)\,\alpha\beta}\, \hat{R}^{-1} g_{\mu\nu}, \end{aligned}$$

Then expand for $a_0 \gg a_k \sim 1$, where $a_0 \sim 1/\xi$, $\xi \sim \Lambda/M_p \ll 1$:

$$\begin{aligned} S_{\mathbf{d}} \;=\;& \int d^dx\, \sqrt{g}\, \Big\{\, \widehat{R}^{(d-4)/2} \Big[\frac{1}{4!\,\xi^2}\, \widehat{R}^2 - \frac{1}{\eta^2}\, \widehat{C}^2_{\mu\nu\rho\sigma} - \frac{1}{4\alpha^2}\, \widehat{F}^2_{\mu\nu} - \frac{1}{\eta^2} \hat{G}\Big] \\ -\;& \frac{1}{4\alpha_j^2}\, F^{(j)}_{\mu\nu} F^{(j)\,\mu\nu} + |\hat\nabla_\mu H|^2 - \frac{\xi_H}{6} |H|^2\, \hat{R} - \lambda\, |H|^4\, \widehat{R}^{2-d/2} + \Big(\frac{i}{2}\, \overline{\psi}_L \gamma^a e_a^\alpha \nabla_\alpha \psi_R + \text{h.c.}\Big) \\ +\;& \big(\overline{\psi}_L Y_\psi H\psi_R + \overline{\psi}_L\, Y'_\psi \tilde{H}\, \psi'_R + \text{h.c.}\big)\; \widehat{R}^{1-d/4}\, \Big\} + a_0^{d/2} \mathcal{O}\Big(\frac{a_i}{a_0}\Big)^3. \end{aligned}$$

⇒ in leading order: regularised (!) SM + WQG in $d=4-2\epsilon$ with manifest Weyl gauge invariance. Then $d\to 4$.

⇒ No UV regulator needed, GEOMETRY does it ($\widehat{R}^\epsilon$). WDBI action: more fundamental.

$A_{\mu\nu}$ new space-time metric; no SM brane ("actors") vs d=10 bulk/gravity ("stage"); removes distinction matter-geometry;

⇒ difference from strings! Physics is relational: no actors vs stage: only "actors".

• Riemannian view (RG) of Weyl gauge symmetry: WG as RG of invariant $g^*_{\mu\nu}$: to avoid gauge fixing artefacts in UV
Wilson line of dilatations: [Dirac 1955] (BH singularities, etc)

$$V(x) = \exp\Big\{\int_{-\infty,\,\gamma(\lambda)}^{x} \omega_\mu(y)\,dy^\mu\Big\}, \qquad \hat\nabla_\mu V(x) = 0, \qquad \gamma(\lambda) : \text{geodesic}$$

⇒ Non-local dressed fields (1D objects!)

$$g^*_{\mu\nu}(x) \equiv \exp\Big\{2\int_{-\infty}^{x} dy^\mu\,\omega_\mu(y)\Big\}\,g_{\mu\nu}, \quad \phi(x)^* \equiv \exp\Big\{-\frac{d-2}{2}\int_{-\infty}^{x} dy^\mu\,\omega_\mu(y)\Big\}\,\phi, \quad \text{Gauge invariant:} \quad g^{*\prime}_{\mu\nu} = g^*_{\mu\nu}, \quad \phi^{*\prime} = \phi^*.$$

⇒ Weyl-gauge symmetric phase in UV: Non-commutativity (NC)

$$[\partial_\mu, \partial_\nu]\, g^*_{\alpha\beta} = 2\, F_{\mu\nu} g^*_{\alpha\beta}, \qquad [\partial_\mu, \partial_\nu]\,\phi^* = -\frac{d-2}{2}\, F_{\mu\nu}\,\phi^*.,$$

$$\tilde\Gamma^\rho_{\mu\nu}(g,\omega) = \Gamma^\rho_{\mu\nu}(g^*), \qquad \Rightarrow \quad \hat\nabla_\mu g^*_{\alpha\beta} = \tilde\nabla_\mu g^*_{\mu\nu} = 0, \qquad \text{where} \qquad \tilde\nabla_\mu(\tilde\Gamma) = \nabla_\mu(\Gamma(g^*)) \equiv \nabla^*_\mu.$$

⇒ Weyl gauge invariance in UV enforces non-commutativity and non-locality ! emergent non-commutativity (NC)
⇒ zero-sized points don't exist, only ω_μ-flux threaded areas! Minimal area/scale holonomy. Softens UV; strings w/o strings!
⇒ quantum (!) non-locality QNL: $\langle\omega_\mu\rangle = 0$. At low scales ω_μ decouples, dressing suppressed $\phi^* \to \phi$, $g^*_{\mu\nu} \to g_{\mu\nu}$, locality and commutativity restored!
If $\omega_\mu = \partial_\mu \log\phi$ pure gauge: $F_{\mu\nu} = 0$, $V(x) \sim \phi(x)$, local dressing (!), $g^*_{\mu\nu} = g_{\mu\nu}\,\phi^2$.

• Dual actions in WG versus RG of dressed metric $g^*_{\mu\nu}$

$$\begin{aligned} S_{\rm w} &= \int d^dx\,\sqrt{g}\,\Big[\alpha_1 \widehat R^2 + \alpha_2\,\widehat C^2_{\mu\nu\rho\sigma} + \alpha_3\,\widehat F^2_{\mu\nu} + \alpha_4\,\widehat G\Big]\,(\widehat R^2)^{d-4/2}. \\ &= \int d^dx\,\sqrt{g^*}\,\Big[\alpha_1 R(g^*)^2 + \alpha_2\,C_{\mu\nu\rho\sigma}(g^*)^2 + \tilde\alpha_3\,F^2_{\mu\nu}(g^*) + \alpha_4\,G(g^*)\Big]\,(R^2(g^*))^{(d-4)/4}, \qquad d=4-2\epsilon,\ \text{etc.} \end{aligned}$$

WDBI action in Riemann geometry defined by $g^*_{\mu\nu}$:

$$\begin{aligned} S'_{\rm w} &= \int d^dx\,\big\{-\det\big[a_0\,\widehat R\,g_{\mu\nu} + a_1 \widehat R_{\mu\nu} + a_2 \widehat F_{\mu\nu}\big]\big\}^{1/2} \\ &= \int d^dx\,\big\{-\det\big[a_0\,R(g^*)\,g^*_{\mu\nu} + a_1 R_{\mu\nu}(g^*) + \tilde a_2 F_{\mu\nu}\big]\big\}^{1/2}; \qquad S'_{\rm w} = S_{\rm w} + \text{non-pert. terms} \end{aligned}$$

WG vs RG (g^*) quantum relation.

$\Rightarrow$ At fundamental level: local, Weyl gauge invariant quantum theory, no masses, with underlying WG. Physics: broken phase.

$\Rightarrow$ Dual view: use RG of $g^*_{\mu\nu}$ with QNL + NC: Avoids gauge fixing artefacts (BH singularities,..) in UV, uses dressed fields

$\Rightarrow$ QNL & NC: evidence of Weyl gauge symmetry in RG; physical artefacts of translating WG into our real-world RG of $g^*_{\mu\nu}$.

$\Rightarrow$ Topological inequivalence: seen @ eqs of motion: $\Big[\text{dressing } g_{\mu\nu}, \text{ eq motion of } g_{\mu\nu}\Big]=0$, $\Big[\text{dressing } g_{\mu\nu}, \text{ eq motion of } \omega_\mu\Big]\neq 0$.

• Example of QNL: Entanglement in quantum non-local RG of $g^*_{\mu\nu}$:

Two photons entangled: common origin at x_c, a non-factorisable polarisation entangl matrix $\mathcal{E}_{\rho\sigma}(x_c) = 1/\sqrt{2}\,(\epsilon^{(1)}_\rho \epsilon^{(2)}_\sigma - \epsilon^{(1)}_\sigma \epsilon^{(2)}_\rho)$.

Their correlation is then

$$C^*_{\alpha\beta}(x_1, x_2) = \int d^4x_c \sqrt{g^*}\, \mathcal{A}^*_{\alpha\rho}(x_1, x_c)\, g^{*\,\rho\sigma} \mathcal{E}_{\sigma\lambda}(x_c)\, g^{*\,\lambda\mu} \mathcal{A}^*_{\mu\beta}(x_c; x_2)$$

$$\left[g^*_{\alpha\beta}\, \Box^* - R^*_{(\alpha\beta)} \right] \mathcal{A}^{*\,\beta\nu}(x_1, x_c) = \frac{\delta^\nu_\alpha\, \delta^{(4)}(x_1 - x_c)}{g^{*\,1/4}(x_1)\ g^{*\,1/4}(x_c)},$$

propagator $\mathcal{A}^*_{\mu\nu}(x_1, x_c) = \mathcal{A}_{\mu\nu}(x_1, x_c)$, when no boundaries, photon action is invariant under dressing.

Photon action is modified by the two boundaries where detectors are placed

$$\Delta S_k = \int d^4x \sqrt{g^*}\, g^{*\,\mu\nu} J_\mu(x)\, A_\nu(x)\, \delta^{(4)}(x - x_k), \qquad k = 1, 2.$$

where J_μ boundary currents that break Weyl gauge symmetry, then Weyl charge of the action is non-vanishing, given by that of $\sqrt{g^*(x_k)}\, g^{*\,\mu\nu}(x_k) = V^2(x_k)\sqrt{g(x_k)}\, g^{\mu\nu}(x_k)$, - symmetry broken explicitly. Quantum Correlation of boundaries:

$$C^*_{\mu\nu}(x_1, x_2) \propto V(x_1, x_2)\ C_{\mu\nu}(x_1, x_2), \qquad V(x_1, x_2) = V^2(x_1) V^{-2}(x_2) = \exp\left\{ 2 \int_{x_2}^{x_1} \omega_\alpha dz^\alpha \right\},$$

$$\langle V(x_1, x_2) \rangle_\omega \equiv \int \mathcal{D}\omega_\mu\, V(x_1, x_2)\ e^{iS[\omega]} = 2 \int_{x_2}^{x_1} dy^\mu \int_{x_2}^{x_1} dz^\nu D_{\mu\nu}(y - z), \qquad D_{\mu\nu} : \text{propagator of } \omega_\mu.$$

What this means:

- quantum connected nodes by $g^*_{\mu\nu}$ Wilson line.

- Measurement of polarisation(s) is a boundary condition(s); global solution $C^*_{\mu\nu}(x_1, x_2)$ must satisfy additional constraints b.c.

- "update" at x_2 following a measurement (b.c.) at x_1 is an algebraic update/solution for global $\Box^* = g^{*\,\mu\nu}\,\nabla^*_\mu \nabla^*_\nu$ to satisfy global b.c. of dressed propagators.

- not a spooky action through flat space-time as in QM, but a result of quantum non-local $g^*_{\mu\nu}$ by Wilson line.

- what happens when ω_μ massive? WL dressing suppressed (decoherence). $D_{\mu\nu} \sim \exp(-m_\omega |x_1 - x_2|)/|x_1 - x_2|$.

- for $d \sim 1200$ km of Micius satellite, $m_\omega = 2\pi\hbar/\lambda \sim 10^{-13}$ eV, for $d \sim \lambda$. https:/doi.org/10.1103, 10.1126/science.aan6972

$m_\omega \sim 10^{-13}$ eV for dark matter [Radu et al, PRL126(2021)]. Bounds from atomic clocks $\sim 10^{-14}$ eV: Saridakis [2601.19407].

-implies QNL and NC at lower scales, but negligible: $\alpha \sim m_\omega/M_p \sim 10^{-40}$, $[\partial_\mu, \partial_\nu] \sim \alpha F_{\mu\nu}$.

- Weyl gauge invariance $\Rightarrow$ quantum non-locality of $g^*_{\mu\nu}$ $\Rightarrow$ etanglement as evidence of Weyl gauge symmetry.

- "spooky action" replaced by: global boundary-value problem of $\Box^*$ required by Weyl gauge invariance (Einstein).

- measurement device as global b.c. and part of problem (Bohr, exp indivisibility observer-quantum system).

- quantum non-local hidden variables ($\delta\omega_\mu$)

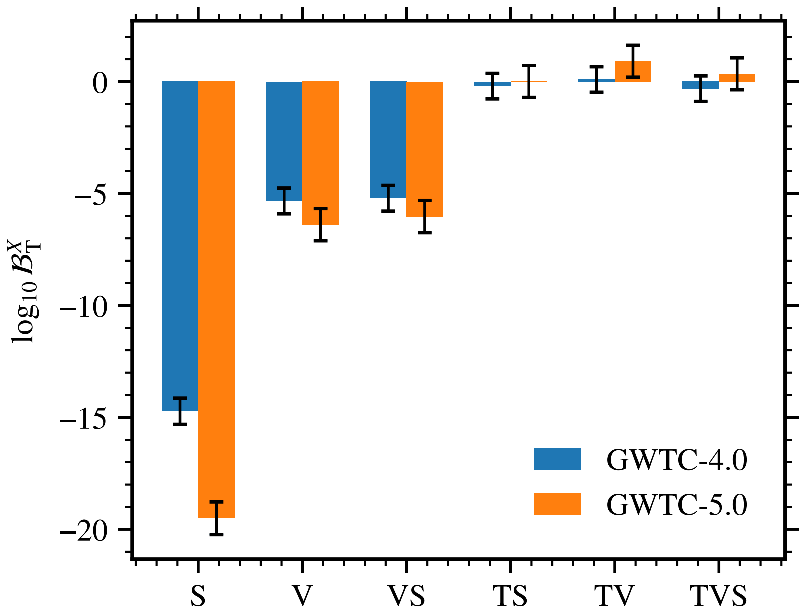
• Weyl gauge symmetry at LIGO-Virgo-KAGRA DG, V. Mandric: [2607.13146]

$g_{\mu\nu} = \overline{g}_{\mu\nu} + \delta g_{\mu\nu}$, $\omega_\mu = 0 + \delta\omega_\mu$,

$\overline{R} = -4\Lambda$. dS background with $\Lambda > 0$:

$$\begin{aligned}\Big(\partial_0\partial_0 - \partial_m\partial_m + \mathcal{H}^2\, m_\omega^2/H^2\Big)\rho_i &= 0.\\ \Big(\partial_0\partial_0 - \partial_m\partial_m + 2\mathcal{H}\partial_0\Big)h_{ij}^{TT} &= 0.\end{aligned}$$

ρ_i: vector polarisation modes.



From LVK Collaboration arxiv:2607.19293.

Comparison of likelihoods for different polarisation hypotheses against the purely tensor hypothesis. The left columns (blue) are previous results (GWTC-4.0), while right columns (orange) include the most recently analysed data (GWTC-5.0).

• Conclusions:

- Weyl geometry: admits a Weyl gauge covariant & metric formulation. Gauge theory.
- WQG, WDBI: the only true gauge theories of a space-time symmetry (beyond Poincaré).

- Stueckelberg mechanism to Einstein-Proca action; ω_μ massive, decouples, $\rightarrow$ Riemannian geometry.
- Einstein-Hilbert action: broken phase of Weyl quadratic gravity / WDBI. Mass of geometric origin $(M_p,\, m_\omega,\, \Lambda > 0)$

- Weyl-DBI: most general Weyl gauge theory (d dim's!) $=$ regularised WQG $+$ non-pert corrections. No UV regularisation!
- SM in Weyl geometry: no new dof's! both geometry and action have this symmetry.

- Riemannian view on Weyl gauge symmetry: non-local dressed metric/fields, soft UV, 1D strings, non-local interactions.
- Dual map and description WG $=$ RG ($g^*_{\mu\nu}$) $+$ QNL $+$ NC.
- UV quantum non-locality (QNL) and non-commutativity (NC) in RG: from Weyl gauge invariance.

- Tests?

a) Starobinsky-like inflation $0.00257 \leq r \leq 0.00303.$, G. Ross et al [1906.03415], DG & Lee [1906.11572], [1809.09174]

b) Gravitational waves, [2607.13146].

c) quantum non-locality (entanglement): evidence for Weyl gauge symmetry?

d) ω_μ as dark matter (geometric solution): Harko et al [2311.16893]

• Weyl group algebra (Poincaré & dilatations) : [DG et al, 2312.13384; Condeescu+Micu 2408.07159]

- generators: translations P_a, Lorentz transformations (rotations) M^{ab} and dilatations D:
- associated gauge fields: vielbein e_μ^a, spin connection $\tilde{s}_\mu^{ab}$ and Weyl gauge boson ω_μ

$$[P_a, M_{bc}] = \eta_{ab}P_c - \eta_{ac}P_a, \qquad [D, P_a] = P_a, \qquad [P_a, P_b] = 0, \qquad [D, M_{ab}] = 0,$$
$$[M_{ab}, M_{cd}] = \eta_{ac}M_{db} - \eta_{bc}M_{da} - \eta_{ad}M_{cb} + \eta_{bd}M_{ca},$$

- infinitesimal transformation: $\delta \equiv \epsilon^A T_A = \xi^a P_a + (1/2)\,\lambda^{ab} M_{ab} + \lambda_D\, D.$
- To construct an action invariant under the symmetries of the theory: build covariant objects (curvatures) out of the fields.
- field strengths of local translations (P_a), rotations (M^{ab}) and dilatations (D)

$$R_{\mu\nu}(P^a) = 2D_{[\mu}e^a_{\nu]} + 2\omega_{[\mu}e^a_{\nu]}, \qquad D_\mu e^a_\nu = \partial_\mu e^a_\nu + \tilde{s}_\mu{}^a{}_b e^b_\nu,$$
$$R_{\mu\nu}(M^{ab}) = \partial_\mu \tilde{s}_\nu{}^{ab} - \partial_\nu \tilde{s}_\mu{}^{ab} + \tilde{s}_\mu{}^a{}_c \tilde{s}_\nu{}^{cb} - \tilde{s}_\nu{}^a{}_c \tilde{s}_\mu{}^{cb} \equiv R^{ab}{}_{\mu\nu}\,,$$
$$R_{\mu\nu}(D) = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu \equiv F_{\mu\nu}\,,$$

which transform covariantly under dilatations: $\delta R_{\mu\nu}(P^a) = \lambda_D R_{\mu\nu}(P^a)$, $\delta R_{\mu\nu}(M^{ab}) = 0$, $\delta R_{\mu\nu}(D) = 0 \Rightarrow$ the action.

$R_{\mu\nu}(P_a) = 0$, spin connection: $\tilde{s}_\mu^{ab} = s_\mu^{ab} + 2(e_\mu^a e^{b\,\nu} - e_\mu^b\, e^{a\,\nu})\omega_\nu = s_\mu^{ab}\Big|_{\partial_\mu \to \partial_\mu + \text{charge}\,\omega_\mu}$ covariantised version of Riemannian s_μ^{ab}

Also $\Gamma^\rho_{\mu\nu} \equiv e^\rho_a\, D_\mu\, e^a_\nu$, then $\widehat{\nabla}_\mu V^\nu = e^\nu_a\, D_\mu V^a$ where $D_\mu V^a = \partial_\mu V^a + q_V\,\omega_\mu\, V^a + \tilde{s}_\mu^a{}_b\, V^b$, Weyl gauge covariant, etc.

• Weyl "photon" - photon mixing ($\omega_\mu - B_\mu$): adding $U(1)_Y$ symmetry to $\mathcal{L}_0$; [source: SM fermions action]

[D.G. arxiv: 2104.15118]

$$\mathcal{L}_0 \to \mathcal{L}_1 = \sqrt{g}\,\Big\{\frac{1}{4!\,\xi^2}\,\widehat{R}^2 - \frac{1}{4}\Big[\hat{F}_{\mu\nu}^2 + 2\sin\chi\,\hat{F}_{\mu\nu}\,\hat{F}_Y^{\mu\nu} + \hat{F}_{Y\,\mu\nu}^2\Big]\Big\}.$$

Re-do calculation, diagonalize mixing by:

$$\hat{\omega}_\mu \;=\; \gamma\,\omega'_\mu \sec\chi, \qquad \hat{B}_\mu = B'_\mu - \omega'_\mu\,\tan\chi,$$

Then

$$\mathcal{L}_1 = \sqrt{\hat{g}}\,\Big\{-\frac{1}{2}\,M_p^2\widehat{R} + \frac{3}{4}\,M_p^2\,\alpha^2\,\gamma^2\,\sec\chi^2\;\omega'_\mu\omega'^\mu - \frac{1}{4}\,(F'^2_{\mu\nu} + F'^2_{Y\,\mu\nu})\Big\},$$

- the photon after EWSB:

$$A_\mu = B'_\mu\cos\theta_w + A^3_\mu\,\sin\theta_w = \big[\hat{B}_\mu + \hat{\omega}_\mu\sin\chi\big]\,\cos\theta_w + \sin\theta_w A^3_\mu.$$

- gauge kinetic mixing $\to$ photon includes a small 'piece' of ω_μ, suppressed by $\sin\chi$ (!)

- Weyl was not completely wrong in trying to relate ω_μ to the photon - they can mix;

- mixing not forbidden by Coleman-Mandula: symmetry direct product $U(1)_Y \times D(1)$; both broken spontaneously.

• Higgs sector in Weyl geometry: (χ: mixing U(1)$_Y$ - D(1) (ω_μ)) $\qquad \widehat{D}_\mu H = \big[\partial_\mu - i\mathcal{A}_\mu - \alpha\,\omega_\mu\big]\,H,$

$$\mathcal{L}_H = \sqrt{g}\,\Big\{\frac{\widehat{R}^2}{4!\,\xi^2} - \frac{\xi_1}{6}\,|H|^2\widehat{R} + |\widehat{D}_\mu H\,|^2 - \lambda\,|H|^4 - \frac{1}{4}\Big[F_{\mu\nu}^2 + 2\sin\chi\,F_{\mu\nu}\,F_Y^{\mu\nu} + F_{y\,\mu\nu}^2\Big]\Big\}.$$

where $\mathcal{A}_\mu = (g/2)\,\vec{\sigma}.\vec{A}_\mu + (g'/2)\,B_\mu$; $\vec{A}_\mu$ is the $SU(2)_L$ boson, B_μ is the $U(1)_Y$ boson.

- Potential:

$$\hat{V}(\sigma) = V_0\Big\{6\lambda\,\sinh^4\frac{\sigma}{M_p\sqrt{6}} + \xi^2\Big[1 - \xi_1\sinh^2\frac{\sigma}{M_p\sqrt{6}}\Big]^2\Big\}, \quad V_0 \equiv (3/4)M_p^4.$$

$$= \frac{1}{4}\Big[\lambda - \frac{1}{9}\,\xi_1\,\xi^2 + \frac{1}{6}\xi_1^2\,\xi^2\Big]\,\sigma^4 - \frac{1}{2}\,\xi_1\xi^2 M_p^2\,\sigma^2 + \frac{3}{2}\,\xi^2 M_p^4 + \mathcal{O}(\sigma^6/M_p^2).$$

if $\xi_1\xi^2 \ll 1$: $\quad \langle\sigma\rangle^2 = (\xi_1\,\xi^2/\lambda)\,M_p^2, \quad m_\sigma^2 = 2\,\xi_1\,\xi^2\,M_p^2,$

- Hierarchy using $\xi\sqrt{\xi_1} \sim 3.5\times 10^{-17}$, $\lambda \sim 0.12$ (SM). Hierarchy controlled by ξ of $\widehat{R}^2$ term!

$$m_Z^2 = \frac{1}{4}\,(g^2 + g'^2)\,\langle\sigma\rangle^2\Big\{1 + \frac{\langle\sigma\rangle^2}{18M_p^2} - \frac{\langle\sigma\rangle^2}{4\,m_\omega^2}\,g'^2\sin^2\chi + \mathcal{O}(\langle\sigma\rangle^4/M_p^4)\Big\}.$$

$\Rightarrow$ Part of Z mass due to Weyl geometry: ω_μ (via mixing χ), beyond Einstein gravity/Riemannian geometry part ($\sim \frac{1}{M_p^2}$)

[D.G. arxiv: 2104.15118]

• Precision constraints (Z mass): [D.G. arxiv: 2104.15118]

$$\varepsilon \equiv \frac{\Delta m_Z}{m_{Z^0}} = -\frac{g'^2 \langle\sigma\rangle^2}{12\, M_p^2} \frac{\sin^2\chi}{\alpha^2} + \mathcal{O}\Big(\frac{\langle\sigma\rangle^4}{M_p^4}\Big) = -\frac{1}{8}\Big(\frac{\langle\sigma\rangle}{m_\omega}\Big)^2 (g'\,\tan\chi)^2 + \mathcal{O}\Big(\frac{\langle\sigma\rangle^4}{m_w^4}\Big).$$

- $\langle\sigma\rangle = 246.22$ GeV; at 68% CL, $\varepsilon = 2.3\times 10^{-5}$, then: $\alpha \geq 2.17\times 10^{-15}\sin\chi.$

- in terms of the mass: $\frac{m_\omega}{\text{TeV}} \geq 6.35\times\tan\chi$

- correlation mass of ω_μ vs $\tan\chi$ mixing.

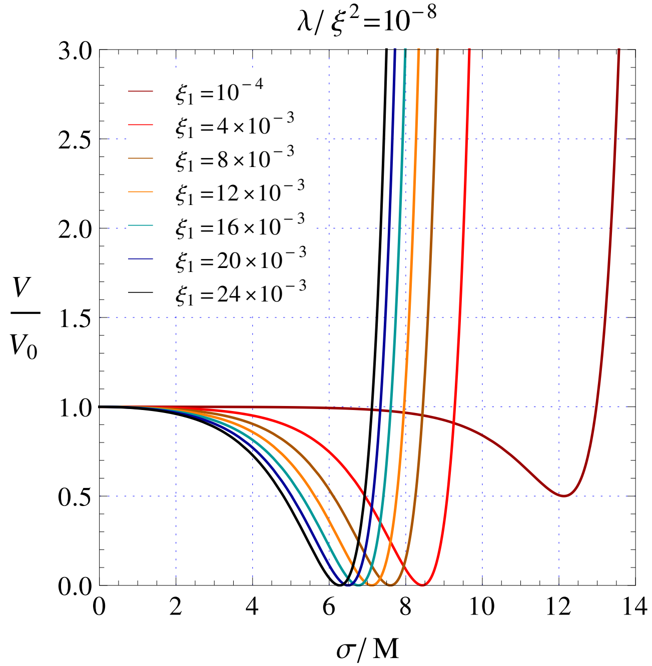
⇒ the constraint from Z-mass is v. strong: e.g. effect of ω_μ to Δa_μ of muon magnetic moment:

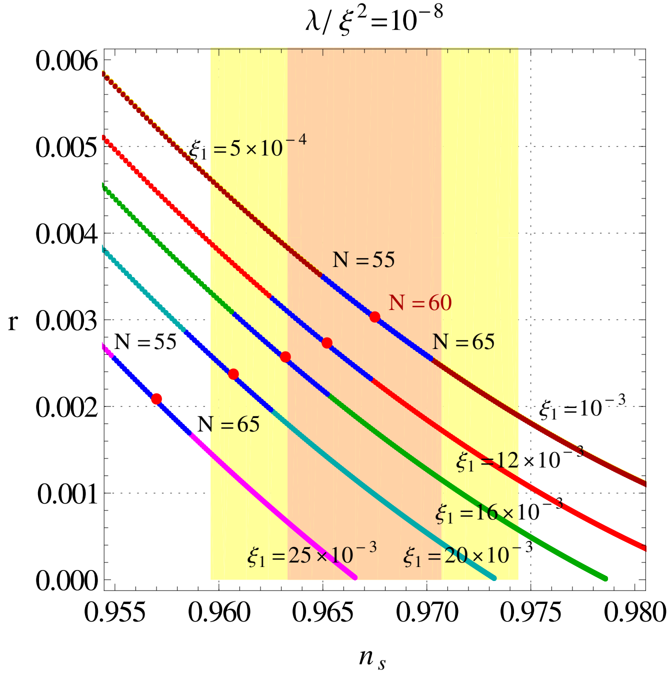
$$\Delta a_\mu \sim \frac{1}{12\pi^2}\frac{m_\mu^2}{m_\omega^2}(g'\tan\chi)^2 = 2.56\times 10^{-13},$$

which is very small (cannot match the ongoing discrepancy).

- Weyl-Starobinski-Higgs R^2-inflation

$$V = V_0 \left\{ \left[1 - \xi_1 \sinh^2 \frac{\sigma}{2\, M_p\sqrt{6}} \right]^2 + (\lambda/\xi^2)\ \sinh^4 \frac{\sigma}{2 M_p \sqrt{6}} \right\}$$





$$\lambda/\xi^2 \ll \xi_1^2 \ll 1: \quad \epsilon = \frac{M_p^2}{2}\frac{V'^2}{V^2} = \frac{\xi_1^2}{3}\sinh^2\frac{2\sigma}{M_p\sqrt{6}} + \mathcal{O}(\xi_1^3); \quad \eta = M_p^2\frac{V''}{V} = -\frac{2\xi_1}{3}\cosh\frac{2\sigma}{M_p\sqrt{6}} + \mathcal{O}(\xi_1^2)$$

$$\Rightarrow \qquad r = 3\,(1-n_s)^2 - (16/3)\,\xi_1^2 + \mathcal{O}(\xi_1^3) \qquad \text{saturates Starobinsky limit for } \xi_1 \to 0$$

- Starobinsky: $R^2 + M_p R$. Weyl: $(1/\xi^2)\,\widehat{R}^2 + \xi_1\, h^2\,\widehat{R}$ $\Rightarrow$ similarity of $r(n_s)$. Gauged version of Starobinsky model!

$0.002567 \leq r \leq 0.00303$ if $n_s = 0.9670 \pm 0.0037$; $(N=60)$ upper limit on r: Starobinsky: $(n_s \approx 0.968)$

D.G. arxiv:2007.14733, 1906.11572; G. Ross, C. Hill, P, Ferreira, J. Noller 1906.

• Weyl anomaly - review in Riemannian geometry:

- Weyl-invariant action $W(\sigma)$ of scalar field (σ) coupled to gravity; $g_{\mu\nu}$ external field. At quantum level, integrating σ generates:

$$W_d = \frac{1}{d-4}\int d^dx\sqrt{g}\,A(d), \qquad d = 4-2\epsilon,$$

$A(d)$ = higher derivative ops: $R\,\Box^{(d-4)/2}R,\qquad R_{\mu\nu}\Box^{(d-4)/2}R^{\mu\nu},\qquad R_{\mu\nu\rho\sigma}\Box^{(d-4)/2}R^{\mu\nu\rho\sigma}$, ... Counterterms:

$$W_c = -\frac{\mu^{d-4}}{(d-4)}\int d^dx\sqrt{g}\,\big(b\,C^2_{\alpha\beta\gamma\delta}+b'\,G\big), \qquad G\equiv R^2_{\mu\nu\rho\sigma}-4R^2_{\mu\nu}+R^2\to\nabla_\mu V^\mu,\ d=4,$$

$$\frac{2}{\sqrt{g}}\,g_{\mu\nu}\,\frac{\delta}{\delta g_{\mu\nu}}\int d^dx\sqrt{g}\,C^2_{\alpha\beta_\gamma\delta} \;=\; (d-4)\,\big(C^2_{\alpha\beta\gamma\delta}+\frac{2}{3}\,\Box R\big), \qquad \text{Weyl term}$$

$$\frac{2}{\sqrt{g}}\,g_{\mu\nu}\,\frac{\delta}{\delta g_{\mu\nu}}\int d^dx\sqrt{g}\,G \;=\; (d-4)\,G, \qquad \text{Euler term}$$

$$T_\mu^\mu = \frac{-2}{\sqrt{g}}g_{\mu\nu}\frac{\delta(W_\sigma+W_d+W_c)}{\delta g_{\mu\nu}}\Big|_{d=4} = \frac{-2}{\sqrt{g}}g_{\mu\nu}\frac{\delta W_c}{\delta g_{\mu\nu}}\Big|_{d\to 4} = b\,[\,C^2_{\mu\nu\rho\sigma}+(2/3)\Box R\,]+b'G\neq 0$$

T_μ^μ non-zero: $\Rightarrow$ Weyl + Euler anomaly from Weyl and Euler terms.

$\Rightarrow$ Weyl anomaly induced action:

$$d \to 4 \qquad W_r = \frac{b}{2} \int d^4x \sqrt{g}\, C_{\mu\nu\rho\sigma} \ln \frac{\Box}{\mu^2} C^{\mu\nu\rho\sigma} + ..., \qquad T^{\mu}_{\mu} \propto -g'_{\mu\nu} \frac{\delta W_r}{\delta g'_{\mu\nu}} \propto b\, C^2_{\mu\nu\rho\sigma}.$$

• Weyl gauge-invariant regularisation: $\mu \to \phi$ [ϕ dilaton] Englert, Gastmans,Truffin 1976]

$$W_c = -\frac{b}{d-4} \int d^dx \sqrt{g}\; \phi^{2(d-4)/(d-2)}\, C^2_{\mu\nu\rho\sigma}, \qquad W_r = \int d^4x \sqrt{g}\; \hat{C}_{\mu\nu\rho\sigma} \Big(c_0 + c_1 \ln \frac{\hat{\Box}}{\phi^2} \Big) \hat{C}^{\mu\nu\rho\sigma}$$

$\Rightarrow$ Weyl invariance maintained for the Weyl term alone, Weyl anomaly recovered for $\phi \to \langle \phi \rangle = \mu$.

$\Rightarrow$ But Euler anomaly (topological) from G remains, since it is independent of μ; [$G \propto \nabla_\mu(...)$ in $d = 4$ only].
Not Weyl invaraint in d dimensions.

• In Weyl geometry: $G \to \widehat{G}$ is indeed Weyl gauge covariant in d dimensions!. Then Euler anomaly can be avoided, as above.

• Weyl geometry vs Weyl anomaly: In Weyl geometry: $\hat{\nabla}_\mu \hat{R}$, $\hat{\nabla}_\mu \hat{R}_{\alpha\beta}$, $\hat{\Box}\hat{R}$, etc: Weyl covariant! [DG 2309.11372; 2408.07160]

$$W_d = \frac{1}{d-4}\int d^dx\sqrt{g}\,A(d), \qquad d = 4-2\epsilon,$$

$A(d)$ Weyl gauge invariant: $\hat{R}\,(\hat{\nabla}_\mu\hat{\nabla}^\mu)^{(d-4)/2}\hat{R}, \qquad \hat{R}_{\mu\nu}\,(\hat{\nabla}_\mu\hat{\nabla}^\mu)^{(d-4)/2}\hat{R}^{\mu\nu}, \quad \hat{R}_{\mu\nu\rho\sigma}\,(\hat{\nabla}_\mu\hat{\nabla}^\mu)^{(d-4)/2}\hat{R}^{\mu\nu\rho\sigma}$, etc.

$$W_c = -\frac{1}{d-4}\int d^dx\sqrt{g}\left\{a_1\hat{R}^2 + b_1\hat{F}^2_{\mu\nu} + c_1\hat{C}^2_{\mu\nu\rho\sigma} + d_1\hat{G}\right\}\,\hat{R}^{(d-4)/2} \qquad (\phi^2 = -\hat{R})$$

$T^\mu_\mu = \nabla_\mu J^\mu = 0$, $J^\mu \propto \hat{\nabla}^\mu\hat{R}$. Renormalized action, $d=4$, Weyl geometry:

$$W_r = \int d^4x\sqrt{g}\,\Big\{\hat{R}\left[a_0 + a_1\ln\frac{\hat{\Box}}{\phi^2}\right]\hat{R} + \hat{F}_{\mu\nu}\left[b_0 + b_1\ln\frac{\hat{\Box}}{\phi^2}\right]\hat{F}^{\mu\nu} + \hat{C}_{\mu\nu\rho\sigma}\left[c_0 + c_1\ln\frac{\hat{\Box}}{\phi^2}\right]\hat{C}^{\mu\nu\rho\sigma} \qquad \text{Weyl term}$$

$$+\ \hat{R}_{\mu\nu\rho\sigma}\left[d_0 + d_1\ln\frac{\hat{\Box}}{\phi^2}\right]\hat{R}^{\rho\sigma\mu\nu} - 4\hat{R}_{\mu\nu}\left[d_0 + d_1\ln\frac{\hat{\Box}}{\phi^2}\right]\hat{R}^{\nu\mu} + \hat{R}\left[d_0 + d_1\ln\frac{\hat{\Box}}{\phi^2}\right]\hat{R}\,\Big\}, \qquad \text{Euler term.}$$

⇒ Weyl gauge invariant! no Weyl anomaly ⇒ WQG: consistent (quantum) gauge theory of Weyl group.

⇒ In broken phase, Weyl anomaly recovered: $\phi^2 \to \langle\phi\rangle^2 = \Lambda$; ω_μ massive; $\hat{R} \to R = -4\,\Lambda$, etc

⇒ brackets [....]: one-loop (log) running. No UV scale required (!)