Chaos, black hole microstates, and momentum fractionation

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Based on 2606.03434, 2508.09669, and work in progress

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Black holes, chaos and all that

- Black hole states are expected to be chaotic
- Fast scrambling paradigm: semiclassical black hole horizons imply maximal chaos (MSS bound: $\lambda \leq 2\pi T$) in *holographic QFTs*
- The diagnostic: out-of-time-ordered correlator (OTOC) in QFT $\langle A^\dagger(t) B^\dagger(0) A(t) B(0) \rangle$

Black holes, chaos and all that

- What about *bulk* chaos?
- Geodesic and wave equations in black hole backgrounds are usually separable (Killing-Yano tensor, AdS_2 algebra etc)
- But whatever happens in the CFT has to happen in the bulk too! How do we see it? – PUZZLE 1

Black holes, microstates and fuzzballs

- Black hole solutions (horizonful) vs microstate solutions (horizonless)
- As a rule, microstates/fuzzballs have some supersymmetry for better control
- Can we diagnose the “degree of black-holishness“ directly from *bulk chaos* probes? – PUZZLE 2

BPS and chaos – an unholy combination

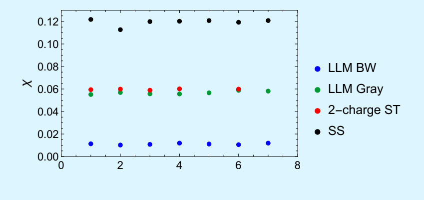
- BPS and chaos do not mix easily!
- Standard quantum chaos diagnostics do not apply to BPS systems
- Huge degeneracy of BPS states $\Rightarrow$ no RMT level statistics
- BPS states are frozen in time with $H = 0$ $\Rightarrow$ no OTOC

BPS chaos – ways out

- “BPS chaos“ by Yiming et al: LMRS projection – project out the degeneracy, look for RMT in the projected spectrum
- Our way: forget eigenvalues, look for eigenvector chaos
- In the classical SUGRA regime this reduces to the study of wave equations
- Berry random wave conjecture: a strongly chaotic wavefunction is statistically equivalent to a sum of waves with random phases $\sim$ Gaussian random field

In a nutshell

- 2606.03434, 2508.09669, and work in progress
- Hierarchy of wave chaos in smooth SUGRA backgrounds
- Chaos becomes stronger with:
  - Decreasing the number of supercharges
  - Increasing the AdS throat length
  - Forming a singularity



- Special chaotic properties (*the strongest and the weakest – in different regimes*) of microstates with *momentum fractionation*. Fortuity?

1. Berry wave chaos in supergravity
2. Berry wave chaos – CFT picture?
3. Fractionated superstrata – fractionation $\leftrightarrow$ fortuity?

1 Berry wave chaos in supergravity

Idea: BPS chaos in supergravity from Berry random waves

- Normalizable waves in the bulk – a probe that also encapsulates the bulk dual of a CFT state:
$$\nabla^2\Phi = 0$$
- Berry random wave conjecture ~ a chaotic wavefunction behaves statistically as a sum of waves with fixed wavenumber k, equidistant wavevectors $\hat{n}_j$ and random *uniform* phases ϕ_j:

$$\Phi(\mathrm{r}) \mapsto \lim_{L\to\infty} \sum_{j=1}^{L} \exp\left[\imath k \left(x\cos\left(\frac{2\pi j}{L}\right) + y \sin\left(\frac{2\pi j}{L}\right)\right) + \phi_j\right]$$

- Wavenumber $k(\mathrm{r}) = \sqrt{2E - V_{\mathrm{r}}(\mathrm{r})} \;\mapsto\; \bar{k} = \sqrt{2E - \bar{V}_{\mathrm{r}}}$
- Outcome: Φ is a random *Gaussian* field $\Rightarrow$ can compute everything (2-point correlators, intensity distributions...)

Berry vs. BPS chaos

- Our way: forget eigenvalues, look for eigenvector chaos
- This is also how the eigenstate thermalization (ETH) works:

$$\langle E_i|O|E_j\rangle = f(E)\delta_{ij} + e^{-S/2}R_{ij}$$

  The RMT statistics of R_{ij} are produced by $|E_i\rangle$, not by O!

- Eigenvector chaos is the most direct extension of ETH to BPS
- Classical regime: eigenvectors become wavefunctions

Berry vs. BPS chaos

- Eigenvector chaos is also secretly involved in the LMRS projection – project onto BPS sectors as

$$O_P = \hat{P}_{\mathrm{BPS}} O \hat{P}_{\mathrm{BPS}}, \quad P_{\mathrm{BPS}} = \sum_{n=1}^{e^S} |\mathrm{BPS}\rangle_n \langle \mathrm{BPS}|_n$$

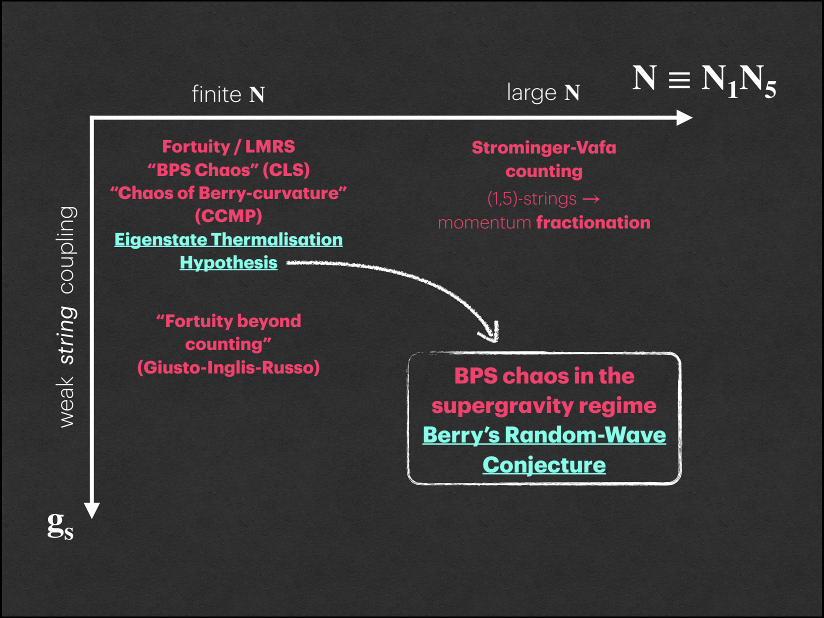
- Chaos is again sourced by the vectors $|\mathrm{BPS}\rangle_n$, not by O which can be simple

- Difference from our approach: need to *mix* different states. We test the effective randomness of *individual* states

Little idiosyncratic book of microstate geometries

- 1/2 BPS: Lin-Lunin-Maldacena (LLM) bubbling solutions – Z-words in $\mathcal{N} = 4$ SYM
- 1/4 BPS: Lunin-Mathur (LM) 2-charge supertubes – 1-5 strings in D1-D5 CFT
- 1/8 BPS: 3-charge D1-D5-P supertubes with momentum on the circle
- 1/8 BPS: superstrata (Bena, Warner, Giusto, Shigemori) – same charges but deform also $\mathbb{T}^4$

Regimes of Berry/BPS chaos



LLM solutions

- Lin, Lunin & Maldacena 2004: “bubbling AdS“ – 1/2 BPS solution from invasion of AdS by giant gravitons
- Super Yang-Mills: only 2 out of 6 scalars excited $\Rightarrow$ all operators are words consisting of Z and $\bar{Z}$ ($Z = \Phi_1 + \imath\Phi_2$)
- Dual matrix model (Berenstein 2004): free fermion in 2D with a constraint, solution specified by the Fermi surface shape:
  – Black disk – AdS
  – Small deformations (rings, droplets etc) – small fluctuations

LLM solutions

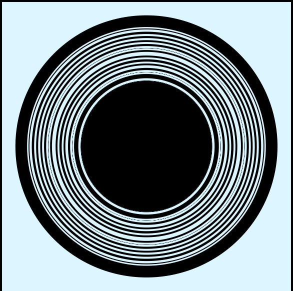
- SUGRA solution: two 3-spheres (Ω_3 and $\tilde{\Omega}_3$) times the (t, x_1, x_2, ξ) manifold:

$$
\begin{aligned}
ds^2 &= \frac{1}{h^2}\Big[-(dt + V_a dx^a)^2 + h^4\left(d\xi^2 + dx_a dx^a\right) + \\
&\quad + \left(\frac{1}{2} - z\right) d\tilde{\Omega}_3^2 + \left(\frac{1}{2} + z\right) d\Omega_3^2\Big] \\
h^2 &= \frac{1}{\xi}\sqrt{\frac{1}{4} - z^2}, \quad \partial_\xi V_a = \frac{\epsilon_{ab}\partial_b z}{\xi}
\end{aligned}
$$

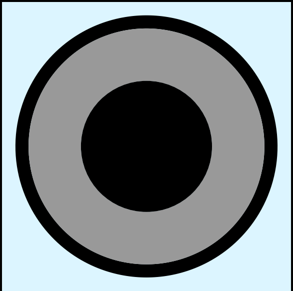
- LLM plane - (x_1, x_2) plane at $\xi = 0$
- $z = \pm 1/2$ – black and white areas
- $-1/2 < z < 1/2$ – grayscale, naked singularity in the LLM plane – the most black-holish it can possibly get at 1/2 BPS

LLM solutions – black&white vs. gray

- $z = \pm 1/2$ – black and white areas
- $-1/2 < z < 1/2$ – grayscale, naked singularity in the LLM plane – the most black-holish it can possibly get at 1/2 BPS
- Our example: disk+rings (Schur polynomials) vs. disk+gray ring (coarse-grained)



Microscopic: N thin rings



Coarse-grained: 1 gray ring

Massless scalar waves in LLM

- Massless scalar:

$$\Phi = \exp(-\imath\omega t + \imath\phi k + \imath\ell\psi + \imath\tilde{\ell}\tilde{\psi})\phi(\rho,\theta)$$

- Introduce θ so LLM plane is at $\theta = 0$ and the AdS boundary is $\theta \;=\; \pi/2 \;\cup\; \theta = -\pi/2 \;\cup\; \rho \to \infty$
- Klein-Gordon in LLM:

$$\partial_r^2\phi + \frac{6\partial_\theta h(1-4z^2) + h\tan\theta\left((1-4z^2) + 12hz\partial_\theta z\right)}{r^2 h(1-4z^2)}\partial_r\phi +$$
$$+\frac{1}{r^2}\partial_\theta^2\phi + \frac{6\partial_r h(1-4z^2) + 2h\left(6rz\partial_r z + 1 - 4z^2\right)}{rh(1-4z^2)}\partial_\theta\phi +$$
$$+\left[\frac{\omega^2 r^2 h^4 - \sec\theta^2\left(k+\omega V_\phi\right)^2}{r^2} + \frac{2\ell(\ell+2)h^4}{1+2z} + \frac{2\tilde{\ell}(\tilde{\ell}+2)h^4}{1-2z}\right]\phi = 0.$$

Berry random wave conjecture: practical tests

- Universal two-point correlator $C_k(\mathrm{r}_1 - \mathrm{r}_2) \equiv \langle \psi_k(\mathrm{r}_1)\psi_k(\mathrm{r}_2) \rangle$
- Averaging over ϕ_j uniform gives (in d spatial dimensions):

$$C_k(\mathrm{r}_1 - \mathrm{r}_2) \mapsto C_k(\Delta r) = \frac{1}{\mathcal{N}} \left(\frac{kr}{2} \right)^{1-\frac{d}{2}} J_{\frac{d}{2}-1}\left(k|\mathrm{r}_1 - \mathrm{r}_2|\right)$$

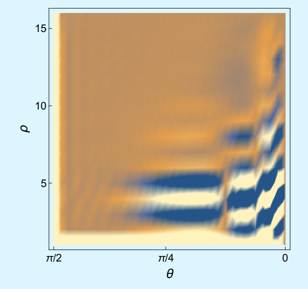
- Recipe:
  - Compute the wavefunction $\phi(\mathrm{r}) \equiv \phi(\rho, \theta)$
  - Pick a number of pairs $(\mathrm{r}_1, \mathrm{r}_2)$ with different $\Delta\mathrm{r} \equiv \mathrm{r}_1 - \mathrm{r}_2$
  - Compare to the universal correlator $C_k(\Delta r)$

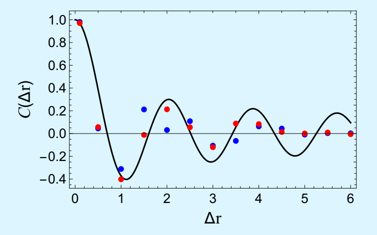
- Intensity distribution follows the Porter-Thomas law:

$$P(I) = \frac{e^{-I/2}}{\sqrt{2\pi I}}, \quad I \equiv \frac{|\psi|^2}{\langle |\psi|^2 \rangle}$$

- Note for later: stable periodic orbits lead to caustics, i.e. rare but strong spikes in the tails of Porter-Thomas!

Weak wave chaos for LLM geometries





Left: Scalar wavefunction in the (ρ, θ) plane in black-and-white. Right: normalized correlation function $C(\Delta r)$ for black-and-white (blue) and grayscale (red), compared to the Berry random wave prediction (black).

The trend: gray geometries show stronger chaos

D1-D5 supertubes

- Lunin-Mathur solution – take a string with some profile and let it source the geometry:

$$ds^2 = \frac{1}{\sqrt{H_1 H_5}}\Big[-\left(dt - A_i dx^i\right)^2 + \left(dy + B_i dx^i\right)^2$$
$$+\sqrt{H_1 H_5}\left(dr^2 + r^2 d\Omega_3^2\right) + \frac{r_0^2}{r^2}\left(\cosh\Sigma dt + \sinh\Sigma dy\right)^2\Big].$$

- Two charges (Q_1 and Q_5) hence 1/4 BPS

- Circular string – integrable, but any other profile nonintegrable

D1-D5-P supertubes

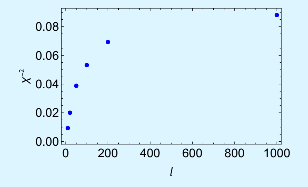
- Many ways to construct the third charge ("bubbling supertubes")
- The interesting way for us: deform the deep IR, i.e. the cap, for example Giusto et al 1211.0306
- Conundrum – this naively corresponds to a fractional spectral flow in the CFT ($\alpha = 2s/k$). But the P-momentum along the circle is still integer:

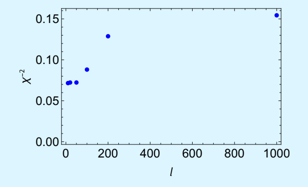
$$h - \bar{h} = \frac{n}{(N_1, N_5)} = \frac{s(s+1)}{k}.$$

Decoupling limit

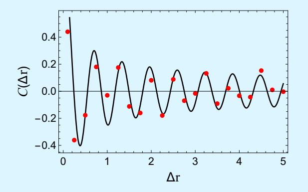
- Domain-wall structure: $\mathrm{AdS}_3 \times \mathbb{S}^3 \mapsto \mathrm{BTZ} \times \mathbb{S}^3 \mapsto \mathrm{flat}$
- Always take the decoupling limit (get rid of the flat region)
- Reasons: (1) can do AdS/CFT (2) flat region irrelevant for wave chaos (3) interesting black-hole physics is likely in IR
- AdS throat length $\ell \sim \sqrt{Q_1 Q_5}/a$

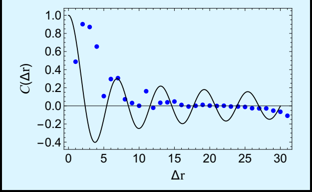
Wave chaos in D1-D5-P supertubes





Proximity to the Berry random wave correlator (left) and to Thomas-Porter distribution (right) for supertubes with different throat lengths $\mathfrak{l}$



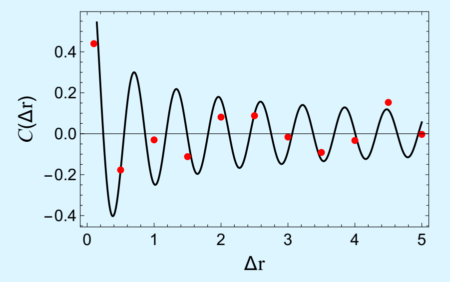


Two-point correlator in D1-D5-P supertube for $\mathfrak{l} = 1000$ (left) vs. $\mathfrak{l} = 10$ (right).

Superstrata

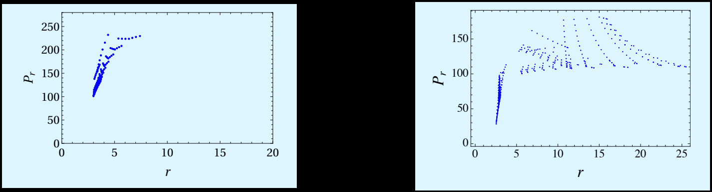
- The essence: take a supertube as "seed" and add oscillations to fluxes
- Geodesic chaos studied in 1709.01107 – from our viewpoint one should rather look for wave chaos
- Regions ($AdS_3 \mapsto \mathrm{BTZ} \mapsto \mathrm{flat}$): $r \sim \sqrt{n}a$, $r \sim \sqrt{Q_{1,5}}$, $r \gg \sqrt{Q_{1,5}}$

Wave chaos in superstrata

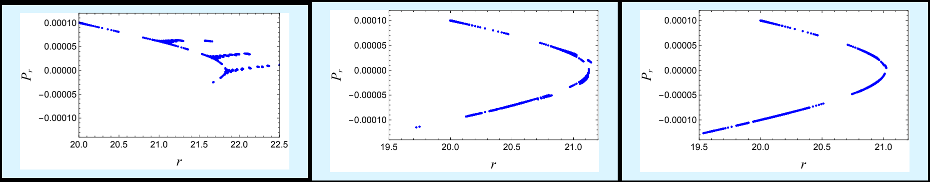


- Correlation functions are very close to the Berry random wave (Bessel) form
- The strongest wave chaos we have found so far

Superstrata and 3-charge supertube – nearly regular orbits

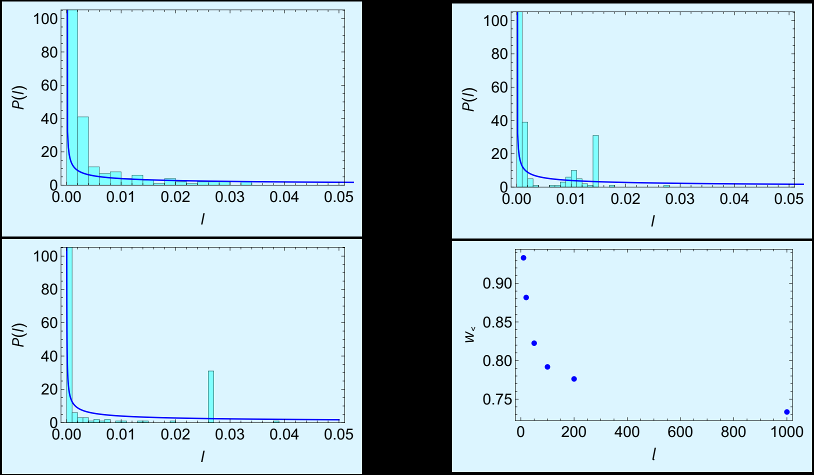


Poincare sections for a 3-charge supertube, with $\mathfrak{l} = 1/2$ (left) and $\mathfrak{l} = 1/200$ (right)



Sequence of Poincare sections for increasing black-holish throat length $(0.010, 0.025, 0.070)$, left to right)

Caustics from periodic orbits in AdS_2



Porter-Thomas histograms for $l = 10, 100, 1000$ and the cumulative weight up to some $I = I_c \gg 1$ (bottom right). Long AdS throats have exponentially many periodic orbits leading to caustics and near-integrable dynamics.

Resolving the Paradox

In a mixed system the spectrum splits into a regular subsequence supported on islands and irregular supported on the chaotic sea

- **Sea** states become **random** waves → correlation **converges** to $\mathbf{J_0}$
- The **regular** ones grow and contribute localized, **caustic**-dominated intensity to $\mathbf{P(I)}$ → tail **spikes**
- Classical **trajectories** see the growing **regular** measure directly → geodesics look **more** integrable.

2 Berry wave chaos – CFT picture?

Towards the CFT picture

- CFT under control at weak coupling/orbifold point vs. gravity under control in classical SUGRA regime $\Rightarrow$ hard to match
- Remember: no RMT, no OTOC... but also no (semi)classical waves

- Proxy for chaos: Renyi entropies $S^{(\alpha)}$
- Coefficients c_i of the state ψ in some basis: $\psi = \sum_i c_i|i\rangle$
- Shannon entropy $S^{(\alpha\to 1)}$: $S_S = -\sum_i |c_i|^2 \log |c_i|^2$
- Participation entropy $S^{(2)}$: $S_p = -\log \sum_{i=1}^N |c_i|^4$

- Not SUSY protected
- But: can likely only grow, not decay as g grows

Example: CFT states for superstrata

- The idea: construct a coherent state out of microstates that have correct charges; relate the coefficients to the SUGRA metric functions
- Elementary states $s \in \{++, +-, -+, --, 00\}$
- $(k, m, n) = (2, 1, n)$ superstratum microstate:

$$\psi\left(N_1, N_{21n}\right) = \left(|++\rangle_1\right)^{N_1} \prod_k \left[J^+_{-1} \frac{\left(L_{-1} - J^0_{-1}\right)^k}{k!} |00\rangle_2 \right]^{N_{21n}}$$

- Superstratum coherent (SUGRA) state:

$$\psi\left(A_1, B_{21n}\right) = \sum_{N_1, N_{21n}} \left(A_1 |++\rangle_1\right)^{N_1} \prod_k \left[B_{21n} J^+_{-1} \frac{\left(L_{-1} - J^0_{-1}\right)^k}{k!} |00\rangle_2 \right]^{N_{21n}}$$

$$\text{constraint:}\ N_1 + \sum_n N_{21n} = N$$

Superstratum complexity

- The CFT state of general (k,m,n) superstrata in supergravity regime is coherent state:

$$\psi(A_1, B_{k,m,n,0}) = \sum_{p=1}^{N/k} (A_1 | + + \rangle_1)^{N_1} (B_{k,m,n,0} | k, m, n, 0\rangle)^p, \quad N_1 + kp = N$$

- Variance analysis $\sigma/\bar{p} \sim 1/\sqrt{N}$ shows that distribution is sharply peaked and Gaussian in the large-N limit
- **Approach:** Analysing the problem for $k = 1$ using the binomial theorem, and then generalizing it for arbitrary *k* via partition function analysis and saddle-point approximation
- **Entropy Scaling**: $S_S \simeq \log \sqrt{N} = \frac{1}{2} \log N$
- States occupy the effective number of modes $N_{\text{eff}} \simeq \sqrt{N} \ll N$, forming a subspace of vanishing size the total Hilbert space for large N
- Participation entropy and Shannon entropy have the same scaling
- Since the two entropies provide compatible behavior, further we consider only Shannon entropy

3-Charge Supertube complexity

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- Supergravity state again is a coherent state:

$$\psi(\{A_k^{(s)}, B_{k,m_k}\}) = \sum_{\{N_{k,m_k}^{(s)}\}} c\left(\{N_{k,m_k}\}, \{N_k^{(s)}\}\right) \left[\prod_s \prod_k (A_k^{(s)} |s\rangle_k)^{N_k^{(s)}} \prod_{k,m_k} \left(B_{k,m_k} \frac{(J_{-1}^+)_k^{m_k}}{m_k!} |00\rangle_k\right)^{N_{k,m_k}}\right]$$

- **Approach:** As in the superstratum case, generating function Z_N and probability distribution P are evaluated using Stirling's and saddle-point approximation, yielding a Gaussian profile once again

$$P(N) \approx \prod_{s,k} \frac{1}{\sqrt{2\pi\sigma_{s,k}^2}} \exp\left(-\frac{(N_k^{(s)} - \bar{N}_k^{(s)})^2}{2\sigma_{s,k}^2}\right) \prod_{k,m_k} \frac{1}{\sqrt{2\pi\sigma_{k,m_k}^2}} \exp\left(-\frac{(N_{k,m_k}^{(0,0)} - \bar{N}_{k,m_k}^{(0,0)})^2}{2\sigma_{k,m_k}^2}\right)$$

$$S_{\mathrm{S}} \approx \sum_s \sum_k \left(\frac{1}{2}\log(2\pi\sigma_{s,k}^2) + \frac{1}{2}\right) + \sum_{k,m_k} \left(\frac{1}{2}\log(2\pi\sigma_{k,m_k}^2) + \frac{1}{2}\right) \quad N = \sum_s \sum_k \sigma_{s,k}^2 + \sum_{k,m_k} \sigma_{k,m_k}^2$$

- **Entropy Scaling**: $S_{\mathrm{S}} \simeq \log\left(2\pi N\sqrt{\lambda(1-\lambda)}\right) \implies S_S \sim \log N \implies N_{\mathrm{eff}} = N$

- States occupy an effective number of modes $N_{\mathrm{eff}} = N$ reproducing the exact scaling expected for black holes, which implies they are more black-holish in the sense of complexity than superstrata

2-Charge Supertube and 1/2-BPS LLM states

- In the case of **2-charge supertube** we also have a coherent state:

$$\psi(\{A_k^{(s)}\}) = \sum_{\{N_k^{(s)}\}} c_k^{(s)2} \prod_{k,s} \left(A_k^{(s)}|s\rangle_k\right)^{N_k^{(s)}0}$$

- We are using the same approach as in the 3-charge case
- **Entropy Scaling:** $S_S \simeq \frac{1}{2}\log\left(2\pi N\right) \sim \log\sqrt{N} \equiv \log N_{\text{eff}}$

(for fixed parameters $M_s = 1, M = 2$ to ensure a meaningful comparison)

- States occupy a smaller effective subspace, as in the case of superstrata
- **No Complexity in 1/2-BPS LLM States:**
  - Disk+ring configuration completely fixed by radii $R_{1,2,3}$
  - The state is a unique Slater determinant at the weakly coupled level
  - Radial localization driven by Schur polynomial symmetry $\rightarrow$ nothing to count
  - General coherent states introduce only gauge overcounting ($\log \mathrm{Vol}\, U(N)$)

CFT complexity – no hierarchy

- Bottom line – BPS hierarchy not visible in N_{eff} of the CFT:

$$\frac{1}{2}\mathrm{BPS} \to \frac{1}{4}\mathrm{BPS} \to \frac{1}{8}\mathrm{BPS} \to \frac{1}{8}\mathrm{BPS} \Leftrightarrow 0 \to N^{1/2} \to N \to N^{1/2} \quad \mathrm{WTF?!}$$

- Why the disagreement with Berry wave chaos?
- Answer: complexity does not necessarily imply chaos!

Fractionated superstrata – fractionation $\leftrightarrow$ fortuity?

Fractionated superstratum states

- Idea: construct a genuine fractional spectral flow, that *has* to be written in terms of $J^+_{-1/q}$
- Fractionated microstate:

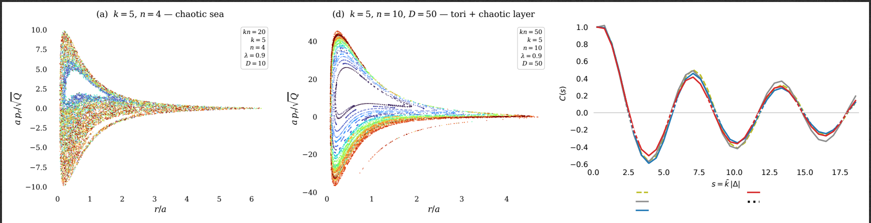
$$\psi\,(N_1, N_{211}, N_{212}, N_{213}) = (|++\rangle_q)^{N_1} \prod_k \left[\left(J^+_{-1/q}\right)^{qk} |00\rangle_{q^2 p}\right]^{N_{2q}} \times$$
$$\times \prod_l \left[\left(J^+_{-1/q}\right)^{ql} |++\rangle_{q(ql+1)}\right]^{N_{3q}} \prod_p \left[\left(J^+_{-1/q}\right)^{qp} |--\rangle_{q(qp-1)}\right]^{N_{3q}}$$

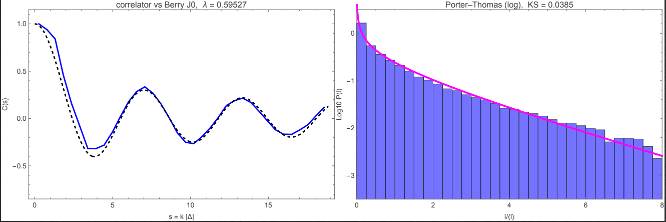
- Roughly: create momentum in units of $1/k$ and thus act with k times more operators
- The sourcing string k-wound instead of sinly-wound
- Metrics can be written explicitly: cohomogeneity-2 ($Q_1 = Q_5$) and cohomogeneity-3 ($Q_1 \neq Q_5$, generic) – SUGRA solutions are functions of $(r, theta, \zeta)$
- 4 quantum numbers (instead of 3 quantum numbers)

Fractionation = fortuity?

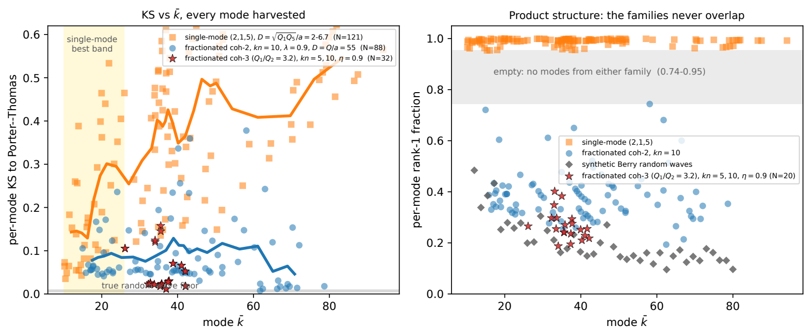
- Acting by $J^+_{-1/k}$ imposes an additional consistency condition: k must divide $N_1 N_5$
- Consequence: changing $N_1 \to N_1 \pm 1$ or $N_5 \to N_5 \pm 1$ in general destroys the divisibility by k and the state thus ceases to exist
- SUGRA consequence: 1 additional parameter in the metric, reduced symmetry
- Unlike fortuity, momentum fractionation is well-defined also at $N = \infty$

Genuine Wave Randomness





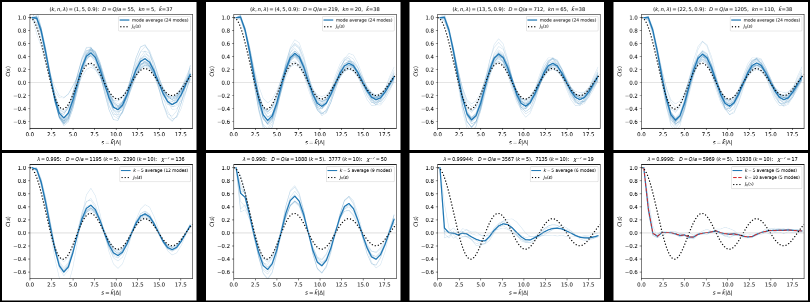
Fractionated vs. "naive" superstrata



Comparison of Berry wave chaos $(2, 1, n)$ superstrata (orange) and for two families of fractionated superstrata (blue squares, red stars)
Left: Kolmogorov-Smirnov test of the 2-pt correlator. Right: Separability of the equations of motion (rank of the PDE matrix).

Bottom line: clear "gap" between non-fractionated and fractionated microstates

Wave chaos in fractionated superstrata: from weak to strong to weak



Blue: Wave chaos first strengthens, then weakens, and finally disappears as $\mathfrak{l} = Q/a$ grows. Black dots: Berry random wave field.

Remember: near-extremal black holes have $\mathfrak{l} \sim 1/T$

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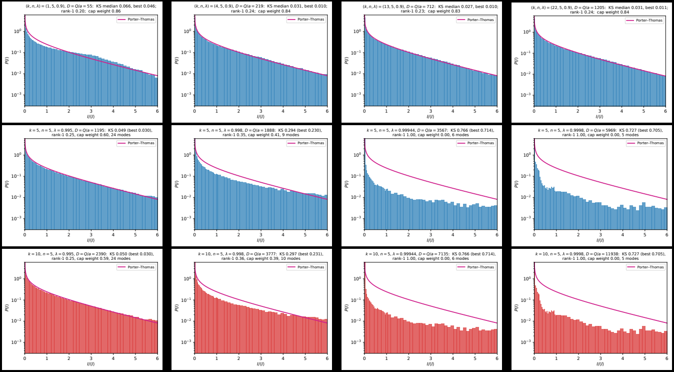

Blue: Cumulative weight histograms vs. Porter-Thomas – again the same weak-strong-weak trend.
Red: Same thing, same throat lengths as the second line in blue, but different (k, n) quantum numbers – *exactly* the same wave chaos behavior

Nonseparability for fractionated superstrata

- Analytic non-separability of coh-2 fractionated superstrata from HJ equation:

$$
\begin{aligned}
&\left(r^2+a^2\right)\left(\partial_r S_r\right)^2+\left(\partial_\theta S_\theta\right)^2=\Sigma\mathcal{W}\\
&\Sigma\mathcal{W}=-\sqrt{\mathcal{P}}g^{AB}p_A p_B, \quad A,B\in\{u,v,\varphi_1,\varphi_2\}\\
&\Sigma\mathcal{W}=\Sigma\left[\mathcal{P}\left(2p_u p_v-\mathcal{F}p_u^2\right)-\frac{M_1^2}{\left(r^2+a^2\right)\sin^2\theta}-\frac{M_2^2}{r^2\cos^2\theta}\right]\equiv\\
&\qquad\equiv\Sigma\mathcal{P}\left(2p_u p_v-\mathcal{F}p_u^2\right)+\widetilde{W}\\
&\Sigma\mathcal{W}=\Sigma\mathcal{W}\Big|_{b=0}-\frac{1}{2}\mathcal{C}\mathcal{F}, \quad \mathcal{C}\equiv 8kRp_u\left(m_1+m_2+2kRp_v\right)
\end{aligned}
$$

- $\Sigma\mathcal{W}\Big|_{b=0}$ is the round supertube potential – separable
- For general $\mathcal{F}$ need to tune $\mathcal{C}=0$ for separability – but that makes m_1 dependent on m_2 $\quad\Rightarrow\quad$ no fractionation
- Cohomogeneity-3 cases reduce to the same form of $\Sigma\mathcal{W}$ as cohomogeneity-2

Separability for fractionated superstrata *with long throat*

- Throat length $\mathfrak{l} = \sqrt{Q_1 Q_5}/a$ with a determined by quantum numbers k, n, λ
- In the infinite throat limit $\mathfrak{l} \to \infty$ with *any* limits for $Q_{1,5}$ and a separately we get:

$$\mathcal{F} = -\frac{|b_4|^2}{k^2 R^2 r^2}\left[1 - \frac{a^2 \sin^2\theta}{r^2} - \left(\frac{a^2 \cos^2\theta}{r^2 + a^2}\right)^{kn} + \mathcal{O}\left(\frac{a^4}{r^4}\right)\right].$$

- Outcome: $\mathcal{F}$ separates itself, no need to tune $\mathcal{C}$

- Remember PUZZLE 1 – bulk probes in black hole geometries have integrable dynamics
- Fractionated superstrata reproduce this but not the vanilla $(2, 1, n)$ superstrata

Conclusions

- SUGRA regime – chaos hierarchy for monotone states:
  1/2 BPS → 1/4 BPS → 1/8 BPS
- Fortuitous states are exceptional: *stronger* wave chaos in the chaotic regime *but near-integrable* regime for long throats
- What happens when we finally form a black hole? Where is the fast scrambling in gravity?
- Is there a red line?


That's all Folks!