Gravitational instantons with weakened AdS boundary conditions

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In collaboration with Felipe Díaz, Borja Diez, Gastón Giribet, Rodrigo Olea, Eleftherios Papantonopoulos and Per Sundell

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AdS/CFT correspondence

- The fundamental dictionary of AdS/CFT is the GKPW relation[1]

$$\mathcal{Z}_{\text{grav}}[\phi]\big|_{\phi\to\phi_{(0)}} = \mathcal{Z}_{\text{CFT}}[\phi_{(0)}]\,,$$

where $\phi_{(0)}$ is the boundary value of a bulk field.

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- The gravitational action plays the role of the generating functional of connected correlators in the dual CFT, e.g. the one-point function

$$\langle \mathcal{O}(x)\rangle = \frac{\delta I_{\text{on-shell}}}{\delta\phi_{(0)}(x)}\Bigg|_{\phi_{(0)}=0}$$

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- To have a meaningful result, the gravitational action must have a well-posed variational principle for the holographic sources.

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Renormalization in AdS

- In $D = d+1$ dimensions, AlAdS spacetimes admit the expansion[2]

$$\mathrm{d}s^2 = \frac{\ell^2}{z^2}\left[\mathrm{d}z^2 + \mathrm{d}x^i\mathrm{d}x^j\left(g_{ij}^{(0)}(x) + \frac{z}{\ell}g_{ij}^{(1)}(x) + \frac{z^2}{\ell^2}g_{ij}^{(2)}(x) + \frac{z^3}{\ell^3}g_{ij}^{(3)}(x) + \dots\right)\right].$$

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- In the case of Einstein-AdS gravity

$$I[g_{\mu\nu}] = \frac{1}{16\pi G}\int_{\mathcal{M}}\mathrm{d}^{d+1}x\sqrt{|g|}\left(R + \frac{d(d-1)}{\ell^2}\right) + \frac{1}{8\pi G}\int_{\partial\mathcal{M}}\mathrm{d}^d x\sqrt{|h|}\,K\,,$$

the coefficients are obtained by solving the EOM order by order in z, i.e.

$$g_{ij}^{(1)} = 0\,, \qquad g_{ij}^{(2)} = -\frac{\ell^2}{d-2}\left(\mathcal{R}_{ij}^{(0)} - \frac{1}{2(d-1)}\mathcal{R}^{(0)}g_{ij}^{(0)}\right), \qquad \dots$$

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- The on-shell action and variations thereof are divergent for AlAdS spaces.

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- The action can be rendered finite by adding boundary counterterms[3]

$$I_{\rm ct} = \frac{1}{8\pi G}\int_{\partial\mathcal{M}} \mathrm{d}^d x\sqrt{|h|}\Bigg[\frac{d-1}{\ell} + \frac{\ell}{2(d-2)}\mathcal{R} + \frac{\ell^3}{2(d-4)(d-2)^2}\left(\mathcal{R}_{ij}\mathcal{R}^{ij} - \frac{d}{4(d-1)}\mathcal{R}^2\right) + \dots\Bigg],$$

where $\mathcal{R}_{ij} = \mathcal{R}_{ij}(h)$ and $\mathcal{R} = h^{ij}\mathcal{R}_{ij}$ (Kostas Skenderis' talk).

[3] Henningson, Skenderis *hep-th/9806087* ; Balasubramanian, Kraus *hep-th/9902121*; Emparan, Johnson, Myers, *hep-th/9903238*; de Haro, Solodukhin, Skenderis, *hep-th/0002230*; Bianchi, Freedman, Skenderis, *hep-th/0112119*; Skenderis, *hep-th/0209067*.

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- The variational principle is then

$$\delta(I+I_{\rm ct})\big|_{\rm EOM}=\int_{\partial\mathcal{M}}\mathrm{d}^dx\sqrt{|h|}\,T^{ij}\delta h_{ij}\,,$$

is well-posed for $\delta h_{ij}|_{\partial\mathcal{M}}=0$, where

$$T^{ij}=\frac{1}{8\pi G}(K^{ij}-h^{ij}K)+\frac{\delta I_{\rm ct}}{\delta h_{ij}}\,.$$

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- Invariance under boundary diffeomorphisms implies $D^iT_{ij}=0$. Then,

$$Q[\xi]=\int_\infty \mathrm{d}^{d-1}x\sqrt{|\sigma|}\,v_iT^i_j\xi^i\,,$$

where v^i is a time-like unit normal, and ξ^i an asymptotic Killing vector.

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- The holographic responses can be obtained by taking $z \to 0$, giving

$$\lim_{z\to 0}\, \delta(I + I_{\mathrm{ct}})\big|_{\mathrm{EOM}} = \int_{\partial\mathcal{M}} \mathrm{d}^d x \sqrt{|g_{(0)}|}\, \tau^{ij}\, \delta g^{(0)}_{ij} \,,$$

where, for d odd, the holographic stress tensor is

$$\tau_{ij} = \lim_{z\to 0} \left(\frac{\ell^{d-2}}{z^{d-2}} T_{ij} \right) = \frac{d}{16\pi G \ell}\, g^{(d)}_{ij} \,.$$

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- The same correlators can be obtained in an alternative way.

Conformal Gravity

- Four-dimensional gravity theory with Weyl symmetry $g_{\mu\nu} \to g'_{\mu\nu} = e^{2\Omega(x)} g_{\mu\nu}$

$$I_{\mathrm{CG}}[g_{\mu\nu}] = \alpha \int_{\mathcal{M}} \mathrm{d}^4 x \sqrt{|g|}\, W^{\mu\nu}_{\lambda\rho} W^{\lambda\rho}_{\mu\nu}\,,$$

where the Weyl and Schouten tensors are respectively defined by

$$W^{\mu\nu}_{\lambda\rho} = R^{\mu\nu}_{\lambda\rho} + 4\,\delta^{[\mu}_{[\lambda} S^{\nu]}_{\rho]} \qquad \text{and} \qquad S^{\mu}_{\nu} = \frac{1}{2}\left(R^{\mu}_{\nu} - \frac{1}{6}\delta^{\mu}_{\nu} R\right).$$

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- Dirichlet boundary conditions for the holographic sources.[4]

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- Dirichlet boundary conditions for the holographic sources.[4]
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- On-shell equivalent to renormalized Einstein-AdS gravity.[6]

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- Arbitrary variations

$$\delta I_{\mathrm{CG}} = 4\alpha \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{|g|}\,\delta g^{\mu\nu}B_{\mu\nu} + \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{|g|}\,\nabla_\mu\Theta^\mu\,,$$

where, defining the Cotton tensor $C_{\mu\nu\lambda} = 2\nabla_{[\lambda}S_{\nu]\mu}$, we have

$$\begin{aligned} B^\mu_\nu &= \nabla^\lambda C^\mu{}_{\nu\lambda} + S^\rho_\lambda W^{\mu\lambda}_{\nu\rho}\,,\\ \Theta^\mu &= -4\alpha\left(\delta g_{\nu\sigma}C^{\sigma\mu\nu} + W^{\rho\sigma\mu\nu}\nabla_\rho\delta g_{\nu\sigma}\right)\,. \end{aligned}$$

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- The on-shell variation is finite for weakened AdS boundary conditions[7]

$$\lim_{z\to 0} \delta I_{\mathrm{CG}} \big|_{\mathrm{EOM}} = \int_{\partial\mathcal{M}} \mathrm{d}^3x \sqrt{|g_{(0)}|} \left(\tau^{ij} \delta g^{(0)}_{ij} + P^{ij} \delta g^{(1)}_{ij} \right) \,.$$

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- Weyl and diffeomorphism invariance imply ($\psi^{(n)}_{ij} := g^{(n)}_{ij} - \frac{1}{3}g^{(0)}_{ij}g^{(n)}$)

$$g^{(0)}_{ij}\tau^{ij} + \frac{1}{2}\psi^{(1)}_{ij}P^{ij} = 0\,,\quad g^{(0)}_{ij}P^{ij} = 0\,,\quad \mathcal{D}^{(0)}_i\left(\tau^i_j + P^{ik}g^{(1)}_{jk}\right) = \tfrac{1}{2}P^{ik}\mathcal{D}^j_{(0)}g^{(1)}_{ik}\,.$$

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- The holographic responses are[8]

$$
\begin{aligned}
\tau_{ij} = &-\Bigg[\frac{2}{\ell}\left(E^{(3)}_{ij} + \frac{1}{3}E^{(2)}_{ij}\,\mathrm{tr}\,g^{(1)}\right) - \frac{4}{\ell}E^{(2)}_{ij}\psi^{(1)k}_{j} + \frac{1}{\ell}g^{(0)}_{ij}E^{(2)}_{kl}\psi^{kl}_{(1)} \\
&+ \frac{1}{2\ell^3}\psi^{(1)}_{ij}\psi^{(1)}_{kl}\psi^{kl}_{(1)} - \frac{1}{\ell^3}\psi^{(1)}_{kl}\left(\psi^{(1)k}_{j}\psi^{(1)l}_{j} - \frac{1}{3}g^{(0)}_{ij}\psi^{(1)k}_{m}\psi^{lm}_{(1)}\right)\Bigg] \\
&- 4\mathcal{D}^{k}_{(0)}B^{(1)}_{ijk} + i \leftrightarrow j\,, \\
P_{ij} = &\,\frac{4}{\ell}E^{(2)}_{ij}\,,
\end{aligned}
$$

where the electric and magnetic parts of the Weyl tensor are defined as

$$
\begin{aligned}
E_{ij} &= n_\alpha n^\beta W^{\alpha}{}_{i\beta j} = E^{(2)}_{ij} + \frac{z}{\ell}E^{(3)}_{ij} + \cdots\,, \\
B_{ijk} &= n_\alpha W^{\alpha}{}_{ijk} = \frac{\ell}{z}B^{(1)}_{ijk} + B^{(2)}_{ijk} + \cdots\,,
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whose specific form in terms of the FG coefficients are known.[8]

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- Einstein-AdS spaces ($S^\mu_\nu = -\frac{1}{2\ell^2}\delta^\mu_\nu$) are Bach-flat, since

$$B^\mu_\nu = \nabla^\lambda C^\mu{}_{\nu\lambda} + S^\rho_\lambda W^{\mu\lambda}_{\nu\rho}$$

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- The Weyl tensor evaluated on Einstein spaces is

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- Choosing the coupling $\alpha = \frac{\ell^2}{64\pi G}$ and using the identity

$$W^{\mu\nu}_{\lambda\rho}W^{\lambda\rho}_{\mu\nu} = \frac{1}{4}\delta^{\mu_1\ldots\mu_4}_{\nu_1\ldots\nu_4}W^{\nu_1\nu_2}_{\mu_1\mu_2}W^{\nu_3\nu_4}_{\mu_3\mu_4}\,,$$

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the Conformal Gravity action evaluated on Einstein spaces becomes[9]

$$I_{\mathrm{CG}}\big|_{\mathrm{E}} = \frac{1}{16\pi G}\int_{\mathcal{M}}\mathrm{d}^4x\sqrt{|g|}\left[R + \frac{6}{\ell^2} + \frac{\ell^2}{4}\left(R^2 - 4R^\mu_\nu R^\nu_\mu + R^{\mu\nu}_{\lambda\rho}R^{\lambda\rho}_{\mu\nu}\right)\right]\,.$$

- It is finite for AlAdS spaces (Nicolas Boulanger's talk).[10]

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- Using the Chern-Gauss-Bonnet theorem

$$\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{|g|}\,\mathcal{G} = 32\pi^2\chi(\mathcal{M}) + \int_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{|h|}\,\mathcal{B}\,,$$

where $\chi(\mathcal{M})$ is the Euler characteristic and $\mathcal{B}$ is the Chern form

$$\mathcal{B} = 4\,\delta^{i_1 i_2 i_3}_{j_1 j_2 j_3} K^{j_1}_{i_1}\left(\frac{1}{2}\mathcal{R}^{j_2 j_3}_{i_2 i_3}(h) - \frac{1}{3}K^{j_2}_{i_2}K^{j_3}_{i_3}\right)\,,$$

one can show that, after a FG expansion, the holographic stress tensor is[11]

$$\lim_{z\to 0}\delta I^{(\mathrm{ren})}_{\mathrm{EH}}\Big|_{\mathrm{EOM}} = \frac{1}{2}\int \mathrm{d}^3x\sqrt{|g_{(0)}|}\,\tau_{ij}\delta g^{ij}_{(0)}\,, \quad \text{where} \quad \tau_{ij} = \frac{3}{16\pi G\ell}g^{(3)}_{ij}\,.$$

[11] Mišković and Olea, *0902.2082*; Anastasiou, Mišković, Olea, Papadimitriou *2003.06425*.

[12] Anastasiou, Araya, Olea *2010.15146*; Anastasiou, Araya, Corral, Olea *2308.09140*; Anastasiou, Araya, Boulanger, Olea, Rovere *2607.03679*.

- Using the Chern-Gauss-Bonnet theorem

$$\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{|g|}\,\mathcal{G} = 32\pi^2\chi(\mathcal{M}) + \int_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{|h|}\,\mathcal{B}\,,$$

where $\chi(\mathcal{M})$ is the Euler characteristic and $\mathcal{B}$ is the Chern form

$$\mathcal{B} = 4\,\delta^{i_1i_2i_3}_{j_1j_2j_3}K^{j_1}_{i_1}\left(\frac{1}{2}\mathcal{R}^{j_2j_3}_{i_2i_3}(h) - \frac{1}{3}K^{j_2}_{i_2}K^{j_3}_{i_3}\right)\,,$$

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Embedding Einstein-AdS gravity in Conformal Gravity provides the suitable counterterms to renormalize the action and variations thereof.

- It has been shown up to 8D (Nicolas Boulanger's talk).[12]

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- A conformally-Einstein space satisfies $g_{\mu\nu} = e^{2\Omega(x)}\bar{g}_{\mu\nu}$, where $\bar{R}_{\mu\nu} = -\frac{3}{\ell^2}\bar{g}_{\mu\nu}$.

[13] Kozameh, Newman, Tod, *Gen.Rel.Grav. 17 (1985) 343-352.*

[14] Riegert, *Phys.Rev.Lett. 53 (1984) 315-318.*

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$$B_{\mu\nu} = 0 \quad \text{and} \quad C_{\lambda\mu\nu} + V^\rho W_{\mu\nu\lambda\rho} = 0\,,$$

where $\nabla_\mu\Omega := V_\mu = 4W^{\lambda\rho}_{\sigma\mu}C^\sigma{}_{\lambda\rho}/W^2$ and $W^2 = W_{\mu\nu\lambda\rho}W^{\mu\nu\lambda\rho} \neq 0$.

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- For instance, the Riegert black hole[14]

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega^2\,, \quad f(r) = 1 - 3\beta\gamma - \frac{(2-3\beta\gamma)\beta}{r} + \gamma r - kr^2\,,$$

with β, γ, k integration constants, is conformally related to Schwarzschild BH.[15]

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- Although they are locally related, their global properties may differ.[16]

- Non-conformally Einstein spaces have been found.[17]

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Self duality in Conformal Gravity

- The Euclidean Conformal Gravity action can be rewritten as[18]

$$I_{\rm CG} = \alpha \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{|g|}\, W^{\mu\nu}_{\lambda\rho} W^{\lambda\rho}_{\mu\nu}$$

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&= \frac{\alpha}{2} \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{|g|} \left(W^{\mu\nu}_{\lambda\rho} \pm \tilde{W}^{\mu\nu}_{\lambda\rho} \right)^2 \mp 16\pi^2 \alpha P_1[\mathcal{M}]\,,
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where $\tilde{W}_{\mu\nu\lambda\rho} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}W^{\alpha\beta}_{\lambda\rho}$ and the Chern-Pontryagin index is defined as

$$
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$$

- Therefore, (anti-)self-dual configurations saturate the bound

$$
I_{\rm CG} \pm 16\pi^2\alpha P_1[\mathcal{M}] = \frac{\alpha}{2}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{|g|}\left(W^{\mu\nu}_{\lambda\rho} \pm \tilde{W}^{\mu\nu}_{\lambda\rho}\right)^2 \geq 0\,.
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- **Theorem:**[19] Let g be a Riemannian metric with (anti-)self-dual Weyl tensor. Then, the conformal class of g contains a Ricci-flat metric iff

$$\mathcal{P} := 4C_{\mu\nu\lambda}C^{\mu\nu\lambda} - V_\mu V^\mu W_{\alpha\beta\gamma\delta}W^{\alpha\beta\gamma\delta} = 0\,,$$

$$\mathcal{T}_{\mu\nu} := S_{\mu\nu} + \nabla_\mu V_\nu + V_\mu V_\nu - \frac{1}{2}g_{\mu\nu}V_\lambda V^\lambda = 0.$$

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[20] LeBrun, *J. Diff. Geom. 34 (1991) 223.*

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- In the following, an example beyond the LeBrun metric is presented.[20]

[19] M. Dunajski, P. Tod, *Commun. Math. Phys. 331 (2014) 351.*

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Taub-NUT with weakened AdS asymptotics

- The Euclidean Taub-NUT in Conformal Gravity

$$\mathrm{d}s^2 = f(r)\left(\mathrm{d}\tau + 2n\cos\theta \mathrm{d}\phi\right)^2 + \frac{\mathrm{d}r^2}{f(r)} + \left(r^2 - n^2\right)\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right),$$

where the metric function determined by $B_{\mu\nu} = 0$ is[21]

$$f(r) = \frac{r^2 + n^2}{r^2 - n^2} - \frac{2mr}{r^2 - n^2} + \frac{r^4 - 6n^2 r^2 - 3n^4}{\ell^2 (r^2 - n^2)} + \frac{a\left(r^2 + \frac{n^2}{3}\right)}{r^2 - n^2} + \frac{br^3}{r^2 - n^2}.$$

[21] Liu, Lu, *1212.6264*; CC, Giribet, Olea, *2105.10574*.

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$$6b\ell^2 m + \ell^2 a^2 + 2\ell^2 a - 8an^2 = 0\,.$$

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- The asymptotic behavior as $r \to \infty$ is

$$f(r) = \frac{r^2}{\ell^2} + br + 1 - \frac{5n^2}{\ell^2} + a + \frac{bn^2 - 2m}{r} + \mathcal{O}\left(r^{-2}\right)\,.$$

[21] Liu, Lu, *1212.6264*; CC, Giribet, Olea, *2105.10574*.

- There is a curve in parameter space where $W^{\mu\nu}_{\lambda\rho} = \pm\tilde{W}^{\mu\nu}_{\lambda\rho}$, i.e.

$$m = \pm\frac{n}{G}\left(1 - \frac{4n^2}{\ell^2} + \frac{a}{2}\right) \quad \text{and} \quad a = \mp 3bn\,.$$

[22] CC, Giribet, Olea, *2105.10574*.

[23] CC, Diez, Papantonopoulos *2510.11854*.

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$$f(r) = \frac{r-n}{r+n} + \frac{b(r-n)^2}{r+n} + \frac{(3n+r)(r-n)^2}{(r+n)\ell^2}\,,$$

which is regular at $r = n$ as long as $\tau \sim \tau + 8\pi n$.

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- The Euclidean on-shell action is $I = \pm 16\pi^2\alpha P_1[\mathcal{M}]$, where

$$P_1[\mathcal{M}] = 2 - \frac{16n^2}{\ell^2}\left(1 - \frac{2n^2}{\ell^2}\right) + 4b^2n^2\,.$$

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- The metric is non-conformally Ricci-flat, since $\mathcal{P} = 0$ and[23]

$$\mathcal{T}_{\mu\nu} = -\frac{[(bn-1)\ell^2 + 4n^2][(b\ell^2 + 4n^2)^2 - 4\ell^2]}{2\ell^2[8n^2 + 3b\ell^2 n - \ell^2(br+2)]^2}\, g_{\mu\nu}\,.$$

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[23] CC, Diez, Papantonopoulos *2510.11854*.

Holographic vorticity of Bach-flat Taub-NUT

- Consider the Lorentzian Bach-flat Taub-NUT metric

$$\mathrm{d}s^2 = -f(r)(\mathrm{d}t + 2n(\cos\vartheta + C)\mathrm{d}\varphi)^2 + \frac{\mathrm{d}r^2}{f(r)} + (r^2+n^2)\mathrm{d}\Omega^2_{(2)}\,,$$

which can be obtained from the Euclidean metric via $\tau \to it$ and $n \to in$,

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- We map this solution into the FG gauge via a radial diffeomorphism

$$r = \frac{\ell}{z} + \frac{b\ell^2}{2} + \left[\frac{5}{4}n^2 + \frac{\ell}{16}\left(b^2\ell^2 - 4(a+1)\right)\right]z + \frac{1}{6}\left(2m + bn^2\ell^2\right)z^2 + \ell\left[n^4 - \frac{bm}{8} - \frac{n^2}{48}\left(8a + 3b^2\ell^2 + 12\right)\right]z^3 + \dots$$

- The FG coefficients, for $n \geq 1$, can be written as[24]

$$g_{ij}^{(n)} = A_{(n)}\, u_i\, u_j + B_{(n)}\, h_{ij}\,,$$

where $u = u^i \partial_i = \partial_t$ is the 3-velocity, $h_{ij} = \gamma_{ij}^{(0)} + u_i u_j$ is the transverse projector, and, up to the holographic order, the coefficients are

[24] CC, Diaz, Diez, Papantonopoulos, Sundell, *26XX.XXXXX*.

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$$\begin{aligned}
A_{(1)} &= 0\,, & B_{(1)} &= -\ell\, b\\
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A_{(3)} &= \frac{4m}{3\ell} + \frac{2n^2 b}{3\ell}\,, & B_{(3)} &= \frac{2m}{3\ell} + \frac{ab\ell}{4} - \frac{b^3\ell^3}{16} + \frac{(3\ell^2 + 19n^2)b}{12\ell}\,.
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- Notice that $g_{ij}^{(1)}$ is nonvanishing as long as $b \neq 0$.

[24] CC, Diaz, Diez, Papantonopoulos, Sundell, *26XX.XXXXX*.

- The boundary metric (Nikolay Bobev's talk)

$$
\mathrm{d}s^2_{(0)} = g^{(0)}_{ij}\mathrm{d}x^i\mathrm{d}x^j = -\left(\mathrm{d}t + 2n(C+\cos\vartheta)\mathrm{d}\varphi\right)^2 + \ell^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right) ,
$$

is not conformally flat, since its Cotton tensor is

$$
C^{(0)}_{ij} := \frac{1}{2}\varepsilon^{(0)}_{ikl}\, C^{(0)kl}_{j} = \frac{n}{\ell^4}\left(1+\frac{4n^2}{\ell^2}\right)(h_{ij}+2u_iu_j) \,.
$$

[25] Leigh and Petkou, *0704.0531*; Mansi, Petkou, Tagliabue, *0808.1212*, *0808.1213*; Leigh, Petkou, Petropoulos, *1108.1393* ; Bakas, *0809.4852*, *0812.0152*; Miskovic and Olea, *0902.2082*; Caldarelli, Leigh, Petkou, Petropoulos, Pozzoli, Siampos, *1206.4351*; Mukhopadhyay, Petkou, Petropoulos, Pozzoli, Siampos, *1309.2310*.

- The boundary metric (Nikolay Bobev's talk)

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\mathrm{d}s^2_{(0)} = g^{(0)}_{ij}\mathrm{d}x^i\mathrm{d}x^j = -\left(\mathrm{d}t + 2n(C+\cos\vartheta)\mathrm{d}\varphi\right)^2 + \ell^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right) ,
$$

is not conformally flat, since its Cotton tensor is

$$
C^{(0)}_{ij} := \frac{1}{2}\varepsilon^{(0)}_{ikl}\, C^{(0)kl}_{j} = \frac{n}{\ell^4}\left(1+\frac{4n^2}{\ell^2}\right)(h_{ij}+2u_iu_j) \,.
$$

- The Cotton tensor represents a conformal perfect fluid with vorticity,[25]

$$
\omega_{ij} := \mathcal{D}^{(0)}_{[i}u_{j]} = \frac{n}{\ell}\,\varepsilon^{(0)}_{ijk}\,u^k \neq 0\,,
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where $\varepsilon^{(0)}_{ijk}$ is the Levi-Civita tensor of $g^{(0)}_{ij}$.

[25] Leigh and Petkou, *0704.0531*; Mansi, Petkou, Tagliabue, *0808.1212*, *0808.1213*; Leigh, Petkou, Petropoulos, *1108.1393* ; Bakas, *0809.4852*, *0812.0152*; Miskovic and Olea, *0902.2082*; Caldarelli, Leigh, Petkou, Petropoulos, Pozzoli, Siampos, *1206.4351*; Mukhopadhyay, Petkou, Petropoulos, Pozzoli, Siampos, *1309.2310*.

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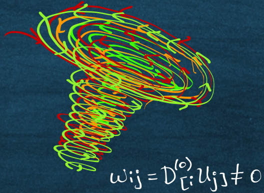

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- The holographic response to $g_{ij}^{(0)}$ is a non-ideal fluid[26]

$$\tau_{ij} = p\left(h_{ij} + 2u_i u_j\right) - \frac{ab\ell^2}{6n^2}\Pi_{ij}\,,$$

where the pressure and dissipative terms are given by

$$p = \frac{4}{\ell^4}\left[m + \frac{b}{12}\left(a\ell^2 + 18n^2\right)\right] \quad \text{and} \quad \Pi_{ij} = h_i^m h_j^n \left(\omega_m{}^k \omega_{kn} - \frac{1}{2}h_{mn}\omega_{kl}\omega^{kl}\right).$$

[26] CC, Diaz, Diez, Papantonopoulos, Sundell, *26XX.XXXXX*.

[27] Anastasiou, Araya, Olea, *2010.15146*.

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- The partially massless response can be expressed as[27]

$$P_j^i = -\frac{2}{\ell^3}\left[\psi^i_{(2)j} - \frac{1}{4}g_{(1)}\psi^i_{(1)j} + \ell^2\left(\mathcal{R}^i_{(0)j} - \frac{1}{3}\mathcal{R}_{(0)}\delta^i_j\right)\right].$$

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- For the Bach-flat Taub-NUT metric, is

$$P^i_j = \frac{2a}{3\ell^3}\begin{pmatrix} -2 & 0 & -6n(C+\cos\vartheta) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

[26] CC, Diaz, Diez, Papantonopoulos, Sundell, *26XX.XXXXX*.

[27] Anastasiou, Araya, Olea, *2010.15146*.

- Diffeomorphism invariance implies $\delta_\xi g^{(0)}_{ij} = \mathcal{L}_\xi g^{(0)}_{ij}$ and $\delta_\xi g^{(1)}_{ij} = \mathcal{L}_\xi g^{(1)}_{ij}$, then

$$Q[\xi] = \int_\infty \mathrm{d}^2x\sqrt{|\sigma|}\, v_i T^i_j\, \xi^j\,, \quad \text{where} \quad T^i_j = 2\tau^i_j + 2P^{ik} g^{(1)}_{jk}\,,$$

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- For the Bach-flat Taub-NUT metric, conserved charges are[28]

$$Q[\partial_t] := M = 16\pi\alpha\ell^2 p\,,$$
$$Q[\partial_\varphi] := J = 32\pi\alpha\ell^2 Cnp = 2MCn\,,$$

with p being the pressure of the holographic fluid, that is,

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Take-home messages

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- Gravitational instantons in Conformal Gravity provide an interesting scenario for studying holographic vorticity of non-ideal fluids as a consequence of their relaxed AdS boundary conditions.
- Conformal Gravity is finite, and it has a well-posed variational principle with Dirichlet boundary conditions for the two holographic sources, that is, $g_{(0)ij}$ and $g_{(1)ij}$.
- The linear mode of non-Einstein Bach-flat metrics is captured by the conserved charges, and it induces a dissipative term in the holographic stress tensor.

Thank you!