A (COVARIANT) NOETHER THEOREM FOR POISSON-LIE SYMMETRIES

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WHAT'S MISSING IN NOETHER'S THEOREMS?

Idea: Discretization + Quantization → Classical symmetries/reps are **"deformed"** (Lie groups → quantum groups)

Example: "quantum group spin networks" $SU(2) \mapsto \mathcal{U}_q(\mathfrak{su}(2))$

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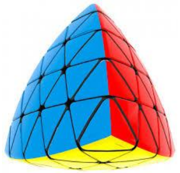

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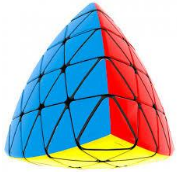

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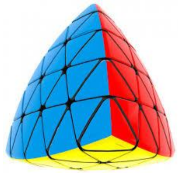

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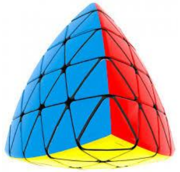

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Poisson-Lie Groups [Lu, Weinstein ,1990], [Chari, Pressely, 1995]

1. Lie group $\mathcal{G}$ with "compatible" Poisson structure $\pi_{\mathcal{G}}$:

$$\mathcal{G} = (\mathbb{R}^3, +) \quad \text{with } \{x_i, x_j\} = \epsilon_{ijk} x_k$$

$$\Longrightarrow \{x_i + a_i, x_j + a_j\} = \epsilon_{ijk}(x_k + a_k)$$

2. Canonical non-Abelian notion of $\mathfrak{g}^*$ and $\mathcal{G}^*$:

$$(\mathcal{G}, \pi_{\mathcal{G}}) \mapsto (\mathfrak{g}, \delta) \quad \alpha \in \mathfrak{g},\ X_i \in \mathfrak{g}^*$$

$$\Longrightarrow \langle \delta(\alpha), X_1 \otimes X_2 \rangle =: \langle \alpha, [X_1, X_2]_* \rangle$$

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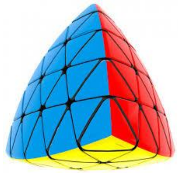

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<u>Recall:</u> Noether **always says $\mathfrak{g}^*$ is Abelian** ($[\cdot,\cdot]_* = 0 \iff \pi_{\mathcal{G}} = 0$)

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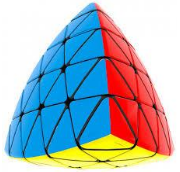

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Extend Noether to PL groups $\Longrightarrow$ new charges & new symmetries!

BEYOND NOETHER: POISSON-LIE SYMMETRIES

[Blohmann, 2023] [Wald, Zoupas, 2000]

Input

$\varphi \in \mathcal{F}$

$\mathcal{L}(\varphi, \partial\varphi, \ldots)$

$\mathbb{e} : \mathfrak{g} \mapsto \mathfrak{X}(\mathcal{F})$

[Takens, 1979]

$\mathbb{d}\mathcal{L} = \mathcal{E}_{\mathrm{EL}}\mathbb{d}\varphi + d\theta$

$\longrightarrow$

$(\mathbb{d}, \mathbb{i}_\bullet, \mathbb{L}_\bullet)$

Cov. Phase Space

$$\Omega = \int_\Sigma n_\mu \mathbb{d}\theta^\mu$$

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$$\mathbb{Q} = Q^{-1}\mathbb{d}Q \implies \boxed{Q(\varphi) = P\exp\left(\int_{\varphi_0}^{\varphi} \mathbb{Q}\right)} \quad ②$$

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[Girelli, **C.P.**, Riello, 2025]

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Punchline: Poisson-Lie symmetries genuinely extend Noether's notions!

$$\mathbb{L}_{\mathbb{e}(\alpha)}\mathcal{L} \neq \nabla_\mu \langle R^\mu, \alpha \rangle$$

2D DILATON GRAVITY PT. 1

[Girelli, **C.P.,** Riello, 2025]
[Klimčík, Ševera, 1995]
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[Many more…]

- Many well known field theories have PL symmetries! (Particles, strings, 3D gravity, spin chains, etc…) (curved/non-cotangent phase space)

- Focus on **2D Dilaton Gravity**: Topological, holographic, describes near horizon physics of near extremal BH's in higher dimensions, (dual to Schwarzian/conformal particle)

$$S_{\mathrm{DG}}[g,\Phi;U(\Phi)] = -\frac{1}{16\pi G_N}\int_M \Big(\Phi R + U(\Phi)\Big) + \text{bdry terms}$$

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[Mertens, Turiaci, 2023], [Ikeda, 1993]

- Change to first order formalism:

$$S_{\mathrm{DG}}[g,\Phi;U(\Phi)] \mapsto S_{1\mathrm{st}}[X,A;\pi(X)] \qquad (X^I) = (X^0, X^1, \Phi) \quad (A_I) = (e_0, e_1, \omega)$$

where $\pi^{IJ}(X)$ **is a Poisson bracket** encoding $U(\Phi)$:

$$\boxed{\pi^{01} = \{X^0, X^1\} = \frac{U(\Phi)}{2}, \quad \pi^{a2} = \{X^a, \Phi\} = \epsilon^a{}_b X^b}$$

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Remark: **Choices of $U(\Phi)$, equivalently $\pi^{IJ}(X)$, define the model (e.g. JT gravity, Liouville, etc)**

So which of them have Poisson-Lie symmetries?

2D DILATON GRAVITY PT. 2

- $S_{1\text{st}}[X, A; \pi(X)]$ has (non-linear) SL(2,R) **Noether** gauge symmetry: [Ikeda, 1993], [Grumiller et al, 2008]

$$\boxed{q_{\mathcal{Y}} = X^I \mathcal{Y}_I, \quad \{X^I, X^J\} = \pi^{IJ}(X), \quad d\mathcal{C}(X) \approx 0}$$

(note special case $\pi^{IJ}(X) = \tilde{c}^{IJ}{}_K X^K$)

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  - **Punchline 1:** Only 2 non-trivial classes of compatible models:*

And a Jordanian "twist" case

$$\boxed{U(\Phi) = -\Lambda\Phi}$$

(Flat $\Lambda = 0$, JT $\Lambda = -2$, or De-Sitter $\Lambda = 2$ Gravity)

$$\boxed{U(\Phi) = \pm \frac{\sinh\left(2\pi\tilde{b}^2\Phi\right)}{\pi\tilde{b}^2}}$$

(Liouville gravity, q-schwarzian)
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  - **Punchline 2:** Demand $\circledast$ is also the defining property of PL groups, so these Dilaton gravity models will have Poisson-Lie symmetries!

POISSON-LIE SYMMETRIES IN JT GRAVITY PT. 1

Example (JT Gravity): $U(\Phi) = 2\Phi$ [Mertens, Turiaci, 2023], [Girelli, **C.P.,** Riello, Zwikel, 20XX]

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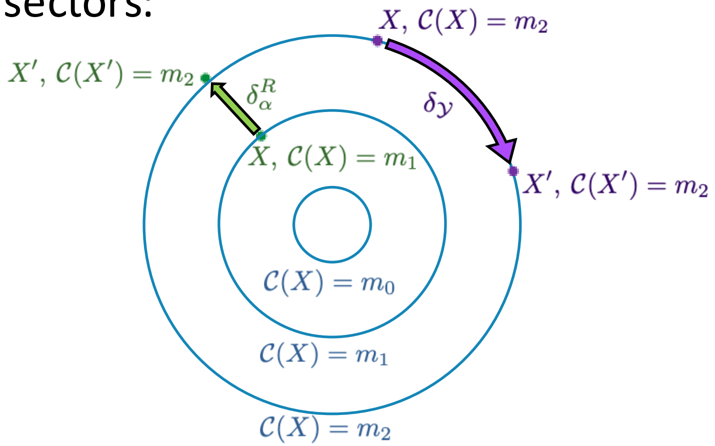
Local geometry is unchanged!



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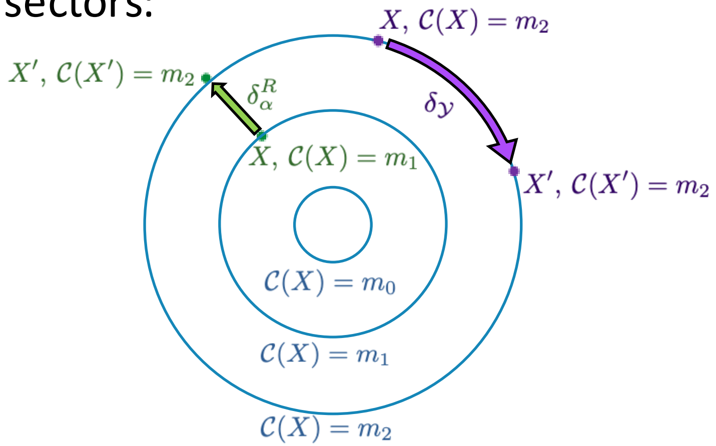

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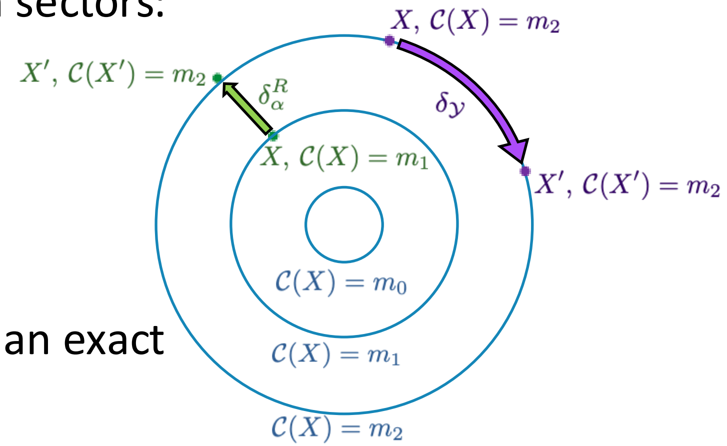

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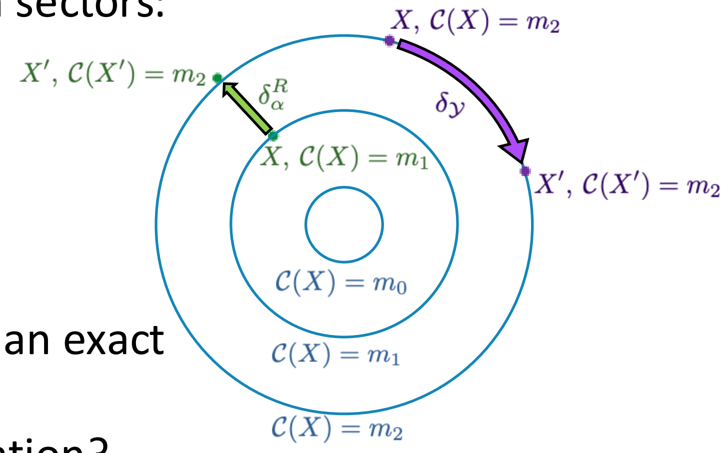

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  - **No?** Because what is the physical meaning of such a transformation?

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1. $\mathbb{L}_{\mathbb{e}(\alpha)} \mathcal{E}_{\mathrm{EL}} \approx 0$
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- So, for operators supported on the boundary, we find

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- Conjecture,

$$\langle \delta^R_\alpha \mathcal{O}[X,A]\rangle \propto \chi(M)\langle \mathcal{O}[X,A]\rangle$$ + corrections?

[Mertens, Turiaci, 2023]

from saddle-approximation evidence, the relation between X and the mass, and classical results.

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  - **Q:** What does this mean for the "structure" of the theory, classically? Quantum? Stability? Implications for no hair and no global symmetry conjectures?

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  - **Q:** What does this mean for the "structure" of the theory, classically? Quantum? Stability? Implications for no hair and no global symmetry conjectures?

- **Outlook:** Can these methods be used to derive from the action, instead of taking as an ansatz, the right non-commutative structure of GR? (1+1d and 2+1d "well" understood, look forward to 3+1d).

[Girelli, **C.P.**, Riello, Zwikel, 20XX] [Dupuis, Girelli, Freidel, Osumanu, Rennert, 2020]

Based on [2505.14942] and work in progress.

THANKS!

ΕΥΧΑΡΙΣΤΏ!

Christopher Pollack

PhD Candidate, UWaterloo, Applied Math

Resident PhD, Perimeter Institute

Supervised by **Florian Girelli & Aldo Riello**

[Lu, Weinstein ,1990],
[Chari, Pressely, 1995],
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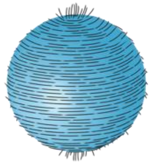
POISSON-LIE GROUPS



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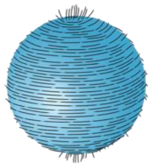

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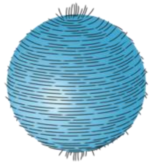

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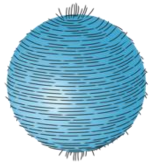

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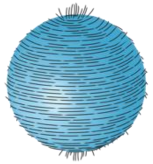

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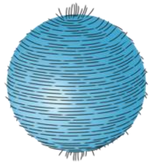

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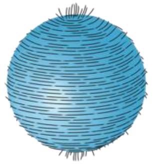

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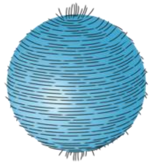

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Note: PL groups are extremely common, e.g. when configuration space/phase space has a group structure.