Double Scaling Limit of the BMN Matrix Model

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Based on Y.A., Goro Ishiki, Shinji Shimasaki [arXiv: 2607.25116]

Introduction

Matrix Models can be
non-perturbative formulation of string/M theory

c=1 matrix model, BFSS model, IKKT model,…

BMN matrix model (a.k.a. plane-wave matrix model)

$$S = \frac{1}{g^2}\int dt\,\mathrm{tr}\left(\frac{1}{2}(D_tX^i)^2 - \frac{1}{4}\left(-\frac{\mu}{3}\varepsilon^{abc}X_c + i[X^a, X^b]\right)^2 + \frac{1}{2}[X^a, X^m]^2 + \frac{1}{4}[X^m, X^n]^2 - \frac{1}{2}\left(\frac{\mu}{6}\right)^2 X_m X^m + \text{fermions}\right)$$

$$\begin{pmatrix} i=1,\dots,8,9, \\ a=1,2,3, \\ m=4,\dots,8,9 \end{pmatrix}$$

… the mass-deformation of the BFSS
w/ maximal supersymmetry

[Berenstein, Maldacena, Nastase '02]

The double scaling limit
is an important concept in various matrix models in many contexts,
e.g. emergent geometry from a matrix model

Introduction

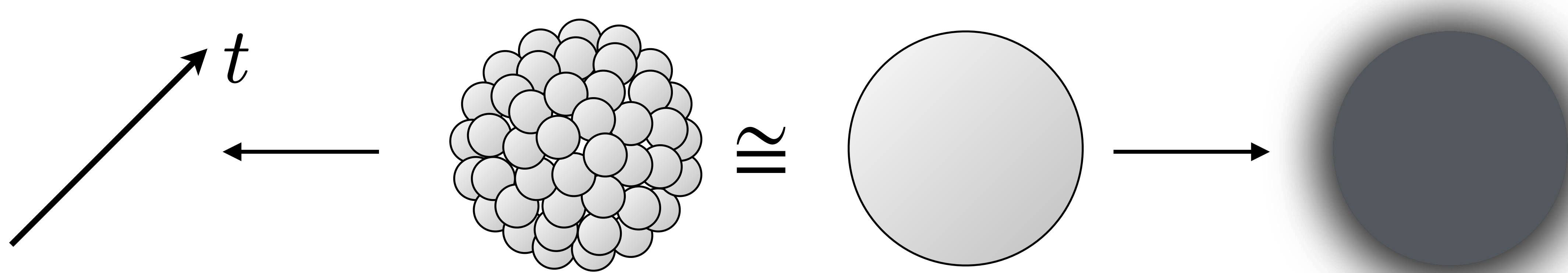
<u>Gauge/gravity duality</u>

[Lin, Lunin, Maldacena ’04; Lin, Maldacena ’05]

Vacuum
BMN matrix model

$\Longleftrightarrow$

Gravity solution
IIA/11D SUGRA



matrix model

$$X^a = \frac{\mu}{3} L_a$$

L_a: $SU(2)$ generators
$(a = 1,2,3)$

blown-up
D0s/gravitons

p-branes

gravitational field

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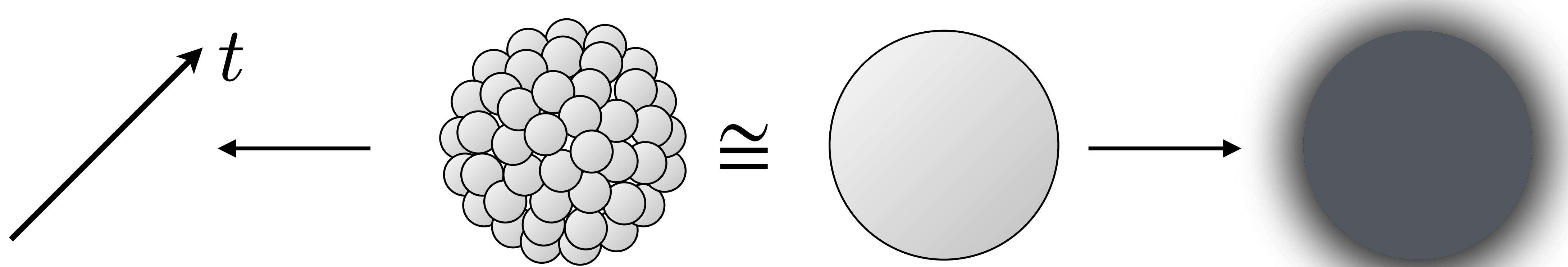

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as brane configs

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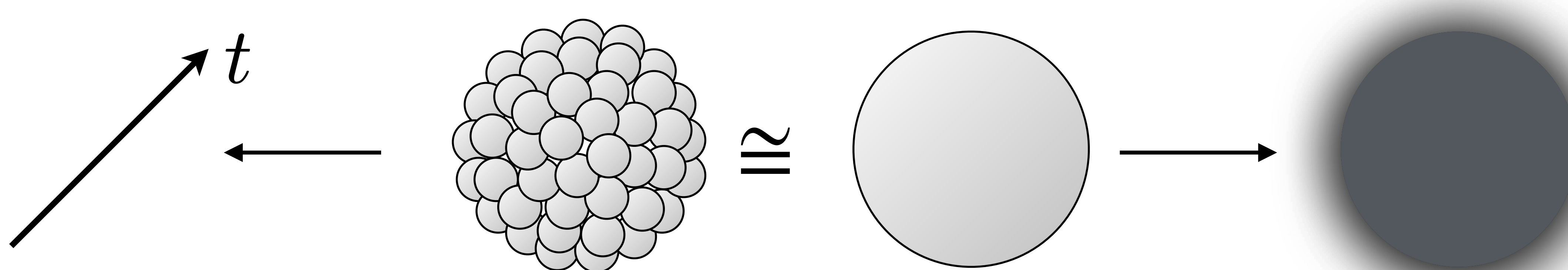

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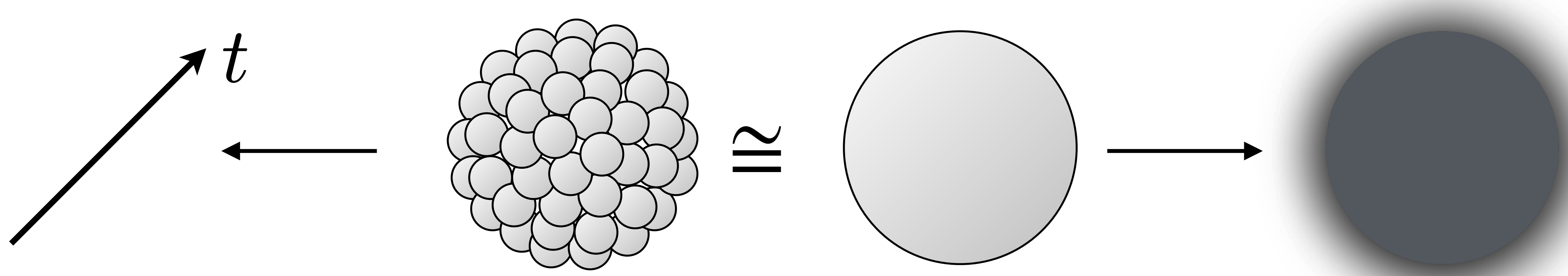

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Many degenerate vacua

Each matrix-model vacuum should describe the corresponding brane config. and thus its gravitational field.

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The vacua of the BMN model

The BMN model has many degenerate vacua, protected by susy.

[Dasgupta, Sheikh-Jabbari, Raamsdonk '02]

$$X^a = \frac{\mu}{3}\begin{pmatrix} \mathbf{1}_{N_1} \otimes L_a^{[n_1]} & & \\ & \mathbf{1}_{N_2} \otimes L_a^{[n_2]} & \\ & & \ddots \end{pmatrix}$$

$(a = 1,2,3)$

$L_a^{[n_s]}$: $SU(2)$ generator in the n_s dimensional irrep.

N_s: multiplicity of this irrep.

$$X^m = 0$$

$(m = 4,\ldots,9)$

The vacuum is parameterised by $\{N_s, n_s\}_{s=1,2,\cdots}$

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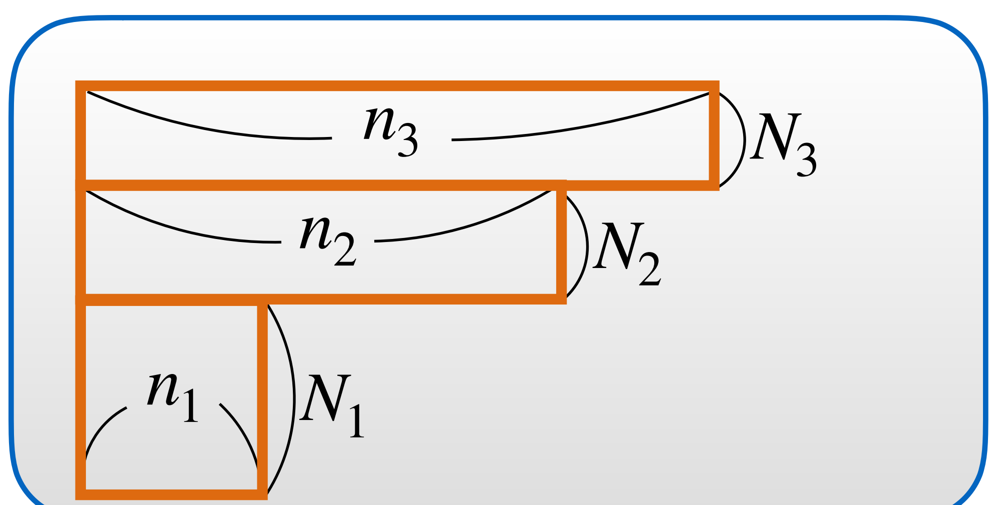
Partition of N

$$N = \sum_s N_s n_s$$



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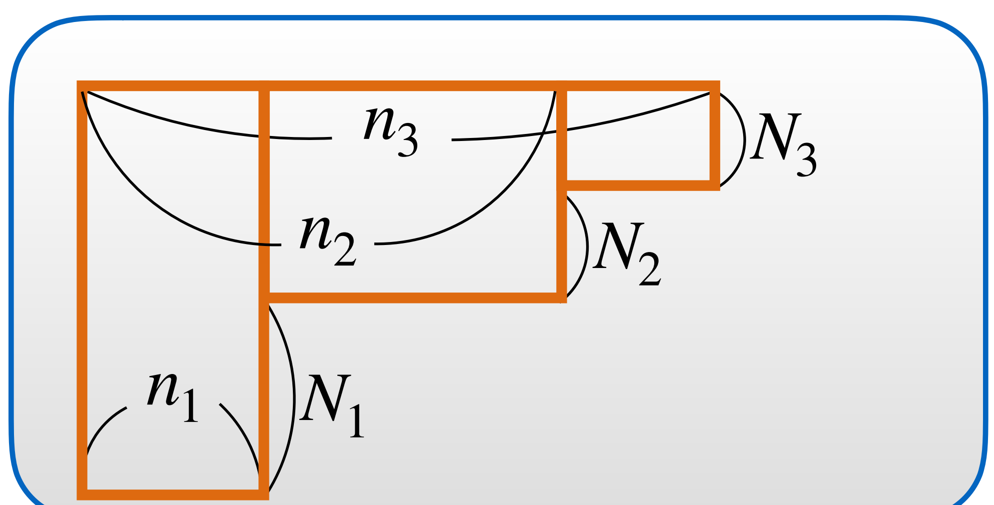

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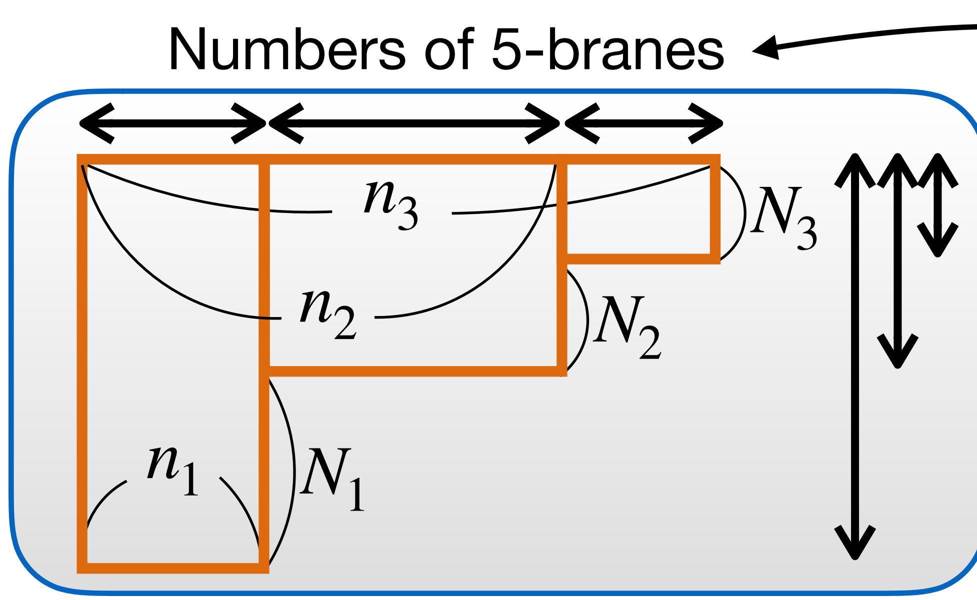

finite

5-brane config.

infinite

p^+ of 5-branes

~ numbers of 2-branes

[Maldacena, Sheikh-Jabbari, Raamsdonk '02]

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Double scaling limit for 5-branes

BMN vacuum

- On the gravity side, the plane-wave solution (with branes) and the 5-brane solution were constructed.

  [Lin, Lunin, Maldacena ’04; Lin, Maldacena ’05]

- On the gravity side, a double scaling limit to reproduce the simplest 5-brane solution from a plane-wave solution was identified.

  [Ling, Mohazab, Shieh, Anders, Raamsdonk ’06]

- The double scaling limit was generalised to the case for general 5-brane solutions on both sides, and the existence of the double scaling limit in the gauge theory was numerically checked.

  [Y.A., Ishiki, Okada, Shimasaki ’14; Y.A., Ishiki, Matsumoto, Shimasaki, Watanabe ’22]

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The double scaling limit was analytically derived solely on the gauge-theory side without referring to any info on the gravity side.

[Y.A., Ishiki, Shimasaki, ’26]

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Overview of my talk

- Focus on a 1/4-BPS sector of the BMN matrix model

We consider the BMN model around a vacuum for 5-branes.
The path integral is reduced to a simpler matrix integral
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- Separate the matrix eigenvalues into two parts
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  - The part that reproduces the 5-brane solution
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- Find the limit that keeps the probe part non-trivial

  … the double scaling limit that describes the 5-brane system

Focus on a 1/4-BPS sector

Vacuum for 5-branes

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$(a = 1,2,3)$

※ $\mu = 6$ for simplicity

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<u>Vacuum for 5-branes</u>

forming 5-branes

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※ We take $n_2 < n_1$

A 5-brane limit: n_1: fixed, $N_1 \to \infty$ and $\lambda = g^2 N_1 \to \infty$

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How fast do we take λ to infinity?

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Quarter-BPS sector (4 supercharges)

[Y.A., Ishiki, Okada, Shimasaki '12]

The BPS sector in which

$$\phi = X^3 + X^8 \sinh\tau - iX^9 \cosh\tau$$

is invariant

$$= \bar{\epsilon}\gamma^\mu \epsilon \, X_\mu \qquad \begin{cases} \gamma^{4567}\epsilon(\tau) = \epsilon(\tau) \\ \gamma^{03}\epsilon(\tau) = \epsilon(-\tau) \end{cases}$$

(after a double Wick rotation $t = -i\tau, \; X^9 = iX^0$)

Focus on a 1/4-BPS sector

Localising locus in the localisation computation

[Y.A., Ishiki, Okada, Shimasaki '12]

same as the vacuum "moduli"

$$\phi \to 2\begin{pmatrix} \mathbf{1}_{N_1}\otimes L_3^{[n_1]} & \\ & \mathbf{1}_{N_2}\otimes L_3^{[n_2]} \end{pmatrix} + i\begin{pmatrix} \mathscr{X}\otimes \mathbf{1}_{n_1} & \\ & \mathscr{Y}\otimes \mathbf{1}_{n_2} \end{pmatrix}$$

x_i : eigenvalue of $\mathscr{X}$

y_i : eigenvalue of $\mathscr{Y}$

Partition fn. in the 1/4-BPS sector

$$Z = \int DX\, D\Psi\, e^{-S} = \int [dx][dy]\, e^{-S_{\text{eff}}}$$

Exact result
(up to instanton effect)

$$\begin{aligned} S_{\text{eff}} = &\frac{2}{g^2}\sum_{i=1}^{N_1} n_1 x_i^2 - \sum_{J=0}^{n_1-1}\sum_{i,j\neq i}^{N_1}\ln\left[\frac{\{(2J+2)^2+(x_i-x_j)^2\}\{(2J)^2+(x_i-x_j)^2\}}{\{(2J+1)^2+(x_i-x_j)^2\}^2}\right]^{\frac{1}{2}} \\ &+\frac{2}{g^2}\sum_{i=1}^{N_2} n_2 y_i^2 - \sum_{J=0}^{n_2-1}\sum_{i,j\neq i}^{N_2}\ln\left[\frac{\{(2J+2)^2+(y_i-y_j)^2\}\{(2J)^2+(y_i-y_j)^2\}}{\{(2J+1)^2+(y_i-y_j)^2\}^2}\right]^{\frac{1}{2}} \\ &-\sum_{J=\frac{|n_1-n_2|}{2}}^{\frac{n_1+n_2}{2}-1}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\ln\left[\frac{\{(2J+2)^2+(x_i-y_j)^2\}\{(2J)^2+(x_i-y_j)^2\}}{\{(2J+1)^2+(x_i-y_j)^2\}^2}\right] \end{aligned}$$

Separate the matrix eigenvalues

$$Z = \int [dx][dy]\, e^{-S_{\text{eff}}}$$
$$= \int [dy]\, e^{-s_2(y)} \int [dx]\, e^{-s_1(x)} \prod_{J=\frac{|n_1-n_2|}{2}}^{\frac{n_1+n_2}{2}-1} \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} \left[\frac{\{(2J+2)^2+(x_i-y_j)^2\}\{(2J)^2+(x_i-y_j)^2\}}{\{(2J+1)^2+(x_i-y_j)^2\}^2} \right]$$

$$s_s(x) = \frac{2n_s}{g^2} \sum_{i=1}^{N_s} x_i^2 - \sum_{J=0}^{n_s-1} \sum_{i,j\neq i}^{N_s} \ln \left[\frac{\{(2J+2)^2+(x_i-x_j)^2\}\{(2J)^2+(x_i-x_j)^2\}}{\{(2J+1)^2+(x_i-x_j)^2\}^2} \right]^{\frac{1}{2}}$$

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For the time being, we focus on $Z_1(y)$ from now on while leaving y_i unintegrated:

$$\langle \mathcal{O}(x) \rangle_1 = \frac{1}{Z_1(y)} \int [dx]\, \mathcal{O}(x)\, e^{-s_1(x)} \prod_{J=\frac{|n_1-n_2|}{2}}^{\frac{n_1+n_2}{2}-1} \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} \left[\frac{\{(2J+2)^2+(x_i - y_j)^2\}\{(2J)^2+(x_i - y_j)^2\}}{\{(2J+1)^2+(x_i - y_j)^2\}^2} \right]$$

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Eigenval. density: $\rho^{(1)}(x) = \sum_{i=1}^{N_1} \delta(x - x_i)$

Resolvents:
$$\begin{cases} R^{(1)}(z) = \sum_{i=1}^{N_1} \dfrac{1}{z - x_i} \\ R^{(2)}(z) = \sum_{i=1}^{N_2} \dfrac{1}{z - y_i} \end{cases}$$

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(i) Loop equation for x_i

$$0 = \int [dx] \sum_{k=1}^{N_1} \frac{\partial}{\partial x_k} \left(\frac{1}{z - x_k} e^{-s_1(x)} \prod_{J=\frac{|n_1-n_2|}{2}}^{\frac{n_1+n_2}{2}-1} \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} \left[\frac{\{(2J+2)^2 + (x_i - y_j)^2\}\{(2J)^2 + (x_i - y_j)^2\}}{\{(2J+1)^2 + (x_i - y_j)^2\}^2} \right] \right)$$

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$(\lambda = g^2 N_1)$

$$\begin{aligned} 0 = \left\langle R^{(1)}(z)^2 \right\rangle &+ \int dx \frac{1}{z - x} \Big\langle \left(R^{(1)}(x + 2i) - 2R^{(1)}(x + i) + \mathrm{c.c.} \right) \rho^{(1)}(x) \\ &+ \sum_{J=1}^{n_1 - 1} \left(R^{(1)}(x + (2J+2)i) + R^{(1)}(x + (2J)i) - 2R^{(1)}(x + (2J+1)i) + \mathrm{c.c.} \right) \rho^{(1)}(x) \Big\rangle_1 \\ &+ \int dx \frac{1}{z - x} \sum_{J=\frac{|n_1 - n_2|}{2}}^{\frac{n_1+n_2}{2}-1} \left(R^{(2)}(x + (2J+2)i) + R^{(2)}(x + (2J)i) - 2R^{(2)}(x + (2J+1)i) + \mathrm{c.c.} \right) \left\langle \rho^{(1)}(x) \right\rangle_1 \\ &- \frac{4 n_1 N_1}{\lambda} \int dx \frac{x}{z - x} \langle \rho^{(1)}(x) \rangle_1 \end{aligned}$$

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(ii) Derivative of free energy

$$\ln Z_1(y) = \ln \int [dx] \sum_{k=1}^{N_1} e^{-s_1(x)} \prod_{J=\frac{|n_1-n_2|}{2}}^{\frac{n_1+n_2}{2}-1} \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} \left[\frac{\{(2J+2)^2 + (x_i - y_j)^2\}\{(2J)^2 + (x_i - y_j)^2\}}{\{(2J+1)^2 + (x_i - y_j)^2\}^2} \right]$$

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$$\ln Z_1(y) = \ln \int [dx] \sum_{k=1}^{N_1} e^{-s_1(x)} \prod_{J=\frac{|n_1-n_2|}{2}}^{\frac{n_1+n_2}{2}-1} \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} \left[\frac{\{(2J+2)^2 + (x_i - y_j)^2\}\{(2J)^2 + (x_i - y_j)^2\}}{\{(2J+1)^2 + (x_i - y_j)^2\}^2} \right]$$

$$\frac{\partial}{\partial y_j} \ln Z_1(y) = \left\langle \sum_{J=\frac{|n_1-n_2|}{2}}^{\frac{n_1+n_2}{2}-1} \left(R^{(1)}(y_j + (2J+2)i) + R^{(1)}(y_j + (2J)i) - 2R^{(1)}(y_j + (2J+1)i) + \mathrm{c.c.} \right) \right\rangle_1$$

Separate the matrix eigenvalues

Eigenval. density: $\rho^{(1)}(x) = \sum_{i=1}^{N_1} \delta(x - x_i)$

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A similar structure appears in the loop eq.

Find the double scaling limit

1/N$_1$ expansion

$$\ln Z_1(y) = \sum_{g=0}^{\infty} \sum_{h=0}^{\infty} N_1^{2-2g-h} N_2^h \, f_{g,h}(y) \qquad \langle R^{(1)}(z) \rangle_1 = \sum_{g=0}^{\infty} \sum_{h=0}^{\infty} N_1^{1-2g-h} N_2^h \, r_{g,h}^{(1)}(y)$$

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Loop eq. to leading order

$$0 = \sum_{J=0}^{n_1-1}\left(r_{00}^{(1)}(x+(2J+2)i) + r_{00}^{(1)}(x+(2J)i) - 2r_{00}^{(1)}(x+(2J+1)i) + \mathrm{c.c.}\right) - \frac{4n_1}{\lambda}x$$

when x is inside the support $[-x_m, x_m]$ of $\rho^{(1)}(x)$

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Loop eq. to leading order

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when x is inside the support $[-x_m, x_m]$ of $\rho^{(1)}(x)$

One can explicitly solve this equation for large x_m/n_1.

$$r^{(1)}_{00}(z) = \frac{8}{x_m^4}\left(-\frac{1}{3}z^3 + \frac{x_m^2}{2}z + \frac{1}{3}(z^2 - x_m^2)^{\frac{3}{2}} + O\left(\frac{n_1}{x_m}\right)\right)$$

$$x_m = (8\lambda)^{1/4} + O(\lambda^0)$$

[Y.A., Ishiki, Shimasaki, ’26]

Find the double scaling limit

Derivative of free energy to non-trivial leading order

Free energy $$\ln Z_1(y) = N_1^2\left(\underbrace{f_{00}}_{\text{constant in } y} + \frac{N_2}{N_1}\underbrace{f_{01}(y)}_{\text{non-trivial leading order}} + \frac{N_2^2}{N_1^2} f_{02}(y) + \cdots\right) + \cdots$$

Find the double scaling limit

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The relation of the free energy gives

$$\frac{\partial}{\partial y_j}f_{01}(y) = \frac{1}{N_2}\sum_{J=\frac{|n_1-n_2|}{2}}^{\frac{n_1+n_2}{2}-1}\left(r_{00}^{(1)}(y_j+(2J+2)i) + r_{00}^{(1)}(y_j+(2J)i) - 2r_{00}^{(1)}(y_j+(2J+1)i) + \mathrm{c.c.}\right)$$

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constant in y

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The relation of the free energy gives

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Plugging the solution $r_{00}^{(1)}$ to it, we finally obtain

$$f_{01}(y) = \frac{2n_2}{N_2 \lambda} \sum_{i=1}^{N_2} y_i^2 - \frac{n_1 A(n_1) e^{-\frac{\pi}{n_1}(8\lambda)^{1/4}}}{\pi N_2 \lambda^{5/8}} \sin \frac{\pi n_2}{n_1} \sum_{i=1}^{N_2} \cosh \frac{\pi y_i}{n_1}$$

$$+ O\left(e^{-\frac{2\pi}{n_1}(8\lambda)^{1/4}} \right)$$

Find the double scaling limit

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Find the double scaling limit

Loop eq. to next-to-leading order

$$0 = \sum_{J=0}^{n_1-1}\left(r_{01}^{(1)}(x+(2J+2)i) + r_{01}^{(1)}(x+(2J)i) - 2r_{01}^{(1)}(x+(2J+1)i) + \mathrm{c.c.}\right)$$
$$+\frac{1}{N_2}\sum_{J=\frac{|n_1-n_2|}{2}}^{\frac{n_1+n_2}{2}-1}\left(R^{(2)}(x+(2J+2)i) + R^{(2)}(x+(2J)i) - 2R^{(2)}(x+(2J+1)i) + \mathrm{c.c.}\right)$$

when x is inside the support $[-x_m, x_m]$ of $\rho^{(1)}(x)$

Derivative of free energy to non-trivial next-to-leading order

Using the relation of the free energy, we obtain

$$f_{02}(y) = \frac{1}{2N_2^2}\sum_{i,j\neq i}\sum_{k=1}^{\infty}\left(\sum_{J=kn_1}^{kn_1+n_2-1} - \sum_{J=kn_1-n_2}^{kn_1-1}\right)\ln\frac{\left\{(2J+2)^2+(y_i-y_j)^2\right\}\left\{(2J)^2+(y_i-y_j)^2\right\}}{\left\{(2J+1)^2+(y_i-y_j)^2\right\}^2}$$
$$+O\left(e^{-\frac{\pi}{n_1}(8\lambda)^{1/4}}\right)$$

Find the double scaling limit

Loop eq. to next-to-leading order

$$0=\sum_{J=0}^{n_1-1}\left(r_{01}^{(1)}(x+(2J+2)i)+r_{01}^{(1)}(x+(2J)i)-2r_{01}^{(1)}(x+(2J+1)i)+\mathrm{c.c.}\right)$$
$$+\frac{1}{N_2}\sum_{J=\frac{|n_1-n_2|}{2}}^{\frac{n_1+n_2}{2}-1}\left(R^{(2)}(x+(2J+2)i)+R^{(2)}(x+(2J)i)-2R^{(2)}(x+(2J+1)i)+\mathrm{c.c.}\right)$$

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$$f_{02}(y)=\frac{1}{2N_2^2}\sum_{i,j\neq i}\sum_{k=1}^{\infty}\left(\sum_{J=kn_1}^{kn_1+n_2-1}-\sum_{J=kn_1-n_2}^{kn_1-1}\right)\ln\frac{\left\{(2J+2)^2+(y_i-y_j)^2\right\}\left\{(2J)^2+(y_i-y_j)^2\right\}}{\left\{(2J+1)^2+(y_i-y_j)^2\right\}^2}$$
$$+O\left(e^{-\frac{\pi}{n_1}(8\lambda)^{1/4}}\right)$$

➡ We now have obtained the free energy as

$$\ln Z_1(y)=N_1^2\left(f_{00}+\frac{N_2}{N_1}f_{01}(y)+\frac{N_2^2}{N_1^2}f_{02}(y)+\cdots\right)+\cdots$$

Find the double scaling limit

Let's come back to the probe part y_i

$$Z = \int [dy]\, e^{-s_2(y)} Z_1(y)$$

$$s_2(y) = \frac{2n_2}{g^2}\sum_{i=1}^{N_2} y_i^2 - \frac{1}{2}\sum_{J=0}^{n_2-1}\sum_{i,j\neq i}^{N_2} \ln \frac{\{(2J+2)^2+(y_i-y_j)^2\}\{(2J)^2+(y_i-y_j)^2\}}{\{(2J+1)^2+(y_i-y_j)^2\}^2}$$

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$$-\ln Z_1(y) = -N_1^2 f_{00} - \frac{2n_2}{g^2}\sum_{i=1}^{N_2} y_i^2 + \frac{n_1 N_1 A(n_1) e^{-\frac{\pi}{n_1}(8\lambda)^{1/4}}}{\pi\lambda^{5/8}} \sin\frac{\pi n_2}{n_1}\sum_{i=1}^{N_2}\cosh\frac{\pi y_i}{n_1}$$
$$-\frac{1}{2}\sum_{k=1}^{\infty}\left(\sum_{J=kn_1}^{kn_1+n_2-1} - \sum_{J=kn_1-n_2}^{kn_1-1}\right)\sum_{i,j\neq i}^{N_2} \ln\frac{\left\{(2J+2)^2+(y_i-y_j)^2\right\}\left\{(2J)^2+(y_i-y_j)^2\right\}}{\left\{(2J+1)^2+(y_i-y_j)^2\right\}^2} + \cdots$$

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cancel

$$-\ln Z_1(y) = -N_1^2 f_{00} - \cancel{\frac{2n_2}{g^2}\sum_{i=1}^{N_2} y_i^2} + \frac{n_1 N_1 A(n_1) e^{-\frac{\pi}{n_1}(8\lambda)^{1/4}}}{\pi \lambda^{5/8}} \sin\frac{\pi n_2}{n_1} \sum_{i=1}^{N_2} \cosh\frac{\pi y_i}{n_1}$$

$$-\frac{1}{2}\sum_{k=1}^{\infty}\left(\sum_{J=kn_1}^{kn_1+n_2-1} - \sum_{J=kn_1-n_2}^{kn_1-1}\right)\sum_{i,j\neq i}^{N_2} \ln \frac{\left\{(2J+2)^2+(y_i-y_j)^2\right\}\left\{(2J)^2+(y_i-y_j)^2\right\}}{\left\{(2J+1)^2+(y_i-y_j)^2\right\}^2} + \cdots$$

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cancel

combined

$$-\ln Z_1(y) = -N_1^2 f_{00} - \cancel{\frac{2n_2}{g^2}\sum_{i=1}^{N_2} y_i^2} + \frac{n_1 N_1 A(n_1) e^{-\frac{\pi}{n_1}(8\lambda)^{1/4}}}{\pi\lambda^{5/8}} \sin\frac{\pi n_2}{n_1}\sum_{i=1}^{N_2}\cosh\frac{\pi y_i}{n_1}$$

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cancel

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$$-\ln Z_1(y) = -N_1^2 f_{00} - \cancel{\frac{2n_2}{g^2}\sum_{i=1}^{N_2} y_i^2} + \boxed{\frac{n_1 N_1 A(n_1) e^{-\frac{\pi}{n_1}(8\lambda)^{1/4}}}{\pi\lambda^{5/8}} \sin\frac{\pi n_2}{n_1}\sum_{i=1}^{N_2}\cosh\frac{\pi y_i}{n_1}}$$

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cosh potential term

Find the double scaling limit

The large-N_2 saddle point eq. for y_i

$$\rho_0^{(2)}(y) = N_2 \lim_{N_2\to\infty} \sum_{i=1}^{N_2} \frac{1}{N_2} \langle \delta(y - y_i) \rangle$$

$$0 = \text{const.} + \frac{n_1 N_1 A(n_1) e^{-\frac{\pi}{n_1}(8\lambda)^{1/4}}}{\pi^2 \lambda^{5/8}} \sin\frac{\pi n_2}{n_1} \sum_{i=1}^{N_2} \cosh\frac{\pi y_i}{n_1} + \rho_0^{(2)}(y)$$

$$- \sum_{k=-\infty}^{\infty} \frac{1}{\pi} \int_{-y_m}^{y_m} dy' \left(\frac{|2kn_1 + 2n_2|}{(y-y')^2 + |2kn_1 + 2n_2|^2} - \frac{|2kn_1|}{(y-y')^2 + |2kn_1|^2} \right) \rho_0^{(2)}(y') + \cdots$$

[Y.A., Ishiki, Shimasaki, ’26]

Find the double scaling limit

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The double scaling limit

[Y.A., Ishiki, Shimasaki, '26]

n_1: fixed, $N_1 \to \infty$ and $\lambda = g^2 N_1 \to \infty$

Find the double scaling limit

The large-N_2 saddle point eq. for y_i

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$$\frac{4}{\pi^2 g_0}$$

The double scaling limit

[Y.A., Ishiki, Shimasaki, '26]

n_1: fixed, $N_1 \to \infty$ and $\lambda = g^2 N_1 \to \infty$

with $\boxed{g_0 = \frac{4\lambda^{5/8}}{n_1 N_1 A(n_1)} e^{\frac{\pi}{n_1}(8\lambda)^{1/4}} \text{ fixed}}$

Find the double scaling limit

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$$\rho_0^{(2)}(y) = N_2 \lim_{N_2\to\infty} \sum_{i=1}^{N_2} \frac{1}{N_2}\langle \delta(y-y_i)\rangle$$

$$0 = \underset{\text{chem. pot.}}{\text{const.}} + \underbrace{\boxed{\frac{n_1 N_1 A(n_1) e^{-\frac{\pi}{n_1}(8\lambda)^{1/4}}}{\pi^2 \lambda^{5/8}}}}_{\frac{4}{\pi^2 g_0}} \sin\frac{\pi n_2}{n_1} \sum_{i=1}^{N_2} \cosh\frac{\pi y_i}{n_1} + \rho_0^{(2)}(y)$$

$$- \sum_{k=-\infty}^{\infty} \frac{1}{\pi} \int_{-y_m}^{y_m} dy' \left(\frac{|2kn_1 + 2n_2|}{(y-y')^2 + |2kn_1+2n_2|^2} - \frac{|2kn_1|}{(y-y')^2 + |2kn_1|^2} \right) \rho_0^{(2)}(y') \underbrace{+ \cdots}_{\to 0}$$

The double scaling limit

[Y.A., Ishiki, Shimasaki, '26]

n_1: fixed, $N_1 \to \infty$ and $\lambda = g^2 N_1 \to \infty$

with $\boxed{g_0 = \frac{4\lambda^{5/8}}{n_1 N_1 A(n_1)} e^{\frac{\pi}{n_1}(8\lambda)^{1/4}} \text{ fixed}}$

This saddle point eq. is equivalent to the equation that determines the NS5-brane solution in the gravity dual.

[Lin, Maldacena '05; Ling, Mohazab, Shieh, Anders, Raamsdonk '06; Y.A., Ishiki, Matsumoto, Shimasaki, Watanabe '22]

Summary and discussion

- We have derived the double scaling limit by keeping the probe part finite and non-trivial solely on the gauge theory side.
  The saddle pt. eq. and the double scaling limit are exactly the ones obtained on the gravity side.

- It is straightforward to generalise the discussion of this double scaling limit to general 5-brane solutions.

  [Y.A., Ishiki, Shimasaki, '26]

- The obtained double-scaled matrix integral is considered to be the 1/4-BPS sector of type IIA Little String Theory.
  g_0 should correspond to the effective string coupling.

  [Lin, Maldacena '05; Y.A., Ishiki, Matsumoto, Shimasaki, Watanabe '22]

- The procedure should be applicable to other matrix models.

Backups

Find the double scaling limit

$$\sum_{J=\frac{|n_1-n_2|}{2}}^{\frac{n_1+n_2}{2}-1}\left(r_{00}^{(1)}(y+(2J+2)i)+r_{00}^{(1)}(y+(2J)i)-2r_{00}^{(1)}(y+(2J+1)i)+\mathrm{c.c.}\right)$$

$$=\int_{-\infty}^{\infty}dx\, S_{n_1,n_2}(y-x)\sum_{J=0}^{n_1-1}\left(r_{00}^{(1)}(x+(2J+2)i)+r_{00}^{(1)}(x+(2J)i)-2r_{00}^{(1)}(x+(2J+1)i)+\mathrm{c.c.}\right)$$

[Y.A., Ishiki, Shimasaki, '26]

$$S_{n_1,n_2}(y-x)=\frac{1}{2\pi i}\sum_{k=-\infty}^{\infty}\left(\frac{1}{y+i(n_1+n_2)-x+2in_1k}-\frac{1}{y+i(n_1-n_2)-x+2in_1k}\right)$$

$$=\frac{1}{2n_1}\frac{\sin\frac{\pi n_2}{n_1}}{\cosh\frac{\pi(y-x)}{n_1}+\cos\frac{\pi n_2}{n_1}}$$

$$=\frac{1}{n_1}\sum_{k=1}^{\infty}(-1)^{k-1}e^{-\frac{k\pi|y-x|}{n_1}}\sin\frac{k\pi n_2}{n_1}$$

Find the double scaling limit

Derivative of free energy to non-trivial leading order

※ f_{00} is constant in y

The relation of the free energy gives

$$\frac{\partial}{\partial y_j} f_{01}(y) = \frac{1}{N_2} \sum_{J=\frac{|n_1-n_2|}{2}}^{\frac{n_1+n_2}{2}-1} \left(r_{00}^{(1)}(y_j + (2J+2)i) + r_{00}^{(1)}(y_j + (2J)i) - 2r_{00}^{(1)}(y_j + (2J+1)i) + \mathrm{c.c.} \right)$$

The r.h.s. can be converted to the one appearing in the loop eq.

$$\sum_{J=\frac{|n_1-n_2|}{2}}^{\frac{n_1+n_2}{2}-1} \left(r_{00}^{(1)}(y + (2J+2)i) + r_{00}^{(1)}(y + (2J)i) - 2r_{00}^{(1)}(y + (2J+1)i) + \mathrm{c.c.} \right)$$

$$= \int_{-\infty}^{\infty} dx\, \underline{\overset{\exists}{S}_{n_1,n_2}(y-x)} \sum_{J=0}^{n_1-1} \left(r_{00}^{(1)}(x + (2J+2)i) + r_{00}^{(1)}(x + (2J)i) - 2r_{00}^{(1)}(x + (2J+1)i) + \mathrm{c.c.} \right)$$

[Y.A., Ishiki, Shimasaki, '26]

Plugging the solution $r_{00}^{(1)}$ to it, we finally obtain

$$f_{01}(y) = \frac{2n_2}{N_2\lambda} \sum_{i=1}^{N_2} y_i^2 - \frac{n_1 A(n_1) e^{-\frac{\pi}{n_1}(8\lambda)^{1/4}}}{\pi N_2 \lambda^{5/8}} \sin\frac{\pi n_2}{n_1} \sum_{i=1}^{N_2} \cosh\frac{\pi y_i}{n_1} + O\left(e^{-\frac{2\pi}{n_1}(8\lambda)^{1/4}}\right)$$

DSL for the gauge/gravity

More precisely for the gauge/gravity duality, the double scaling limit is

$$n_1 > n_2 \to \infty,\ g_0 N_2 \to \infty \text{ with } \frac{n_1}{y_m} \text{ and } \frac{n_2}{y_m} \text{ fixed}$$

$$g_0 = \frac{4\lambda^{5/8}}{n_1 N_1 A(n_1)} e^{\frac{\pi}{n_1}(8\lambda)^{1/4}} : \text{fixed}$$

Introduction

5-brane geometry in the Matrix Model

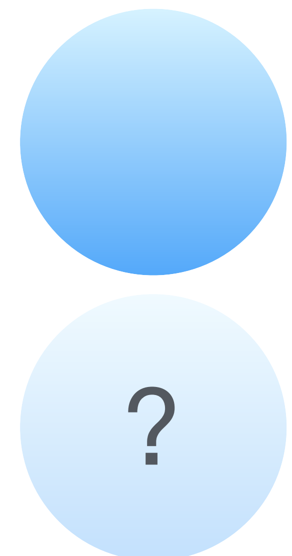
Vacuum:

$$X^a = \frac{\mu}{3} L_a$$

(a=1,2,3)

$$X^m = 0$$

(m=4,...,9)



fuzzy 2-sphere ··· 2-brane!

should be (fuzzy) 5-sphere…

5-branes have been a long-standing problem

To solve this problem, we ought to notice that geometrical features should be seen at strong coupling

Localisation

How the localisation works

[Pestun '07;...]

Let us consider a system with a fermionic trf. δ

$$Z_{\mathcal{O}} = \int DX\, \mathcal{O} e^{-S[X]} \qquad \delta S = 0 \qquad \delta \mathcal{O} = 0$$

Introduce a deformation δV whose bosonic part is positive-definite.

$$Z_{\mathcal{O}}(t) := \int DX\, \mathcal{O} e^{-S[X] - t\delta V[X]} \qquad \delta^2 V = 0$$

Then $$\frac{dZ_{\mathcal{O}}(t)}{dt} := -\int DX\, \delta(\mathcal{O} V e^{-S[X] - t\delta V[X]}) = 0$$

if DX is SUSY inv. & the surface term vanishes

The original path integral can be computed by config. localised around $\delta V = 0$:

$$Z_{\mathcal{O}} = Z_{\mathcal{O}}(0) = \lim_{t \to +\infty} Z_{\mathcal{O}}(t)$$

→ 1-loop computation becomes exact

Localisation

<u>What will this 1/4-BPS mean?</u> [Y.A., Ishiki, Okada, Shimasaki '12]

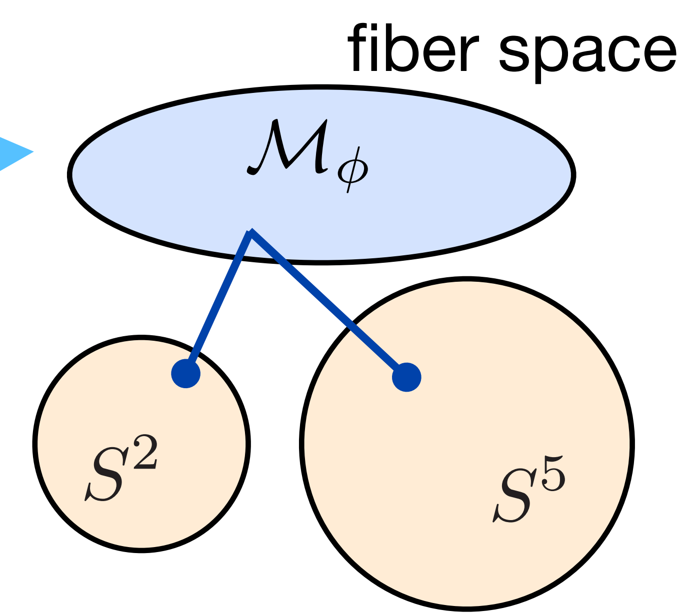
In the original BMN model,

$$\phi = X^3 + i\left(\sin\frac{\mu t}{6}X^8 - \cos\frac{\mu t}{6}X^9\right)$$

This would describe the radial directions of $S^2 \times S^5$

insertion of ϕ
$\langle f(\phi(\tau))\rangle$

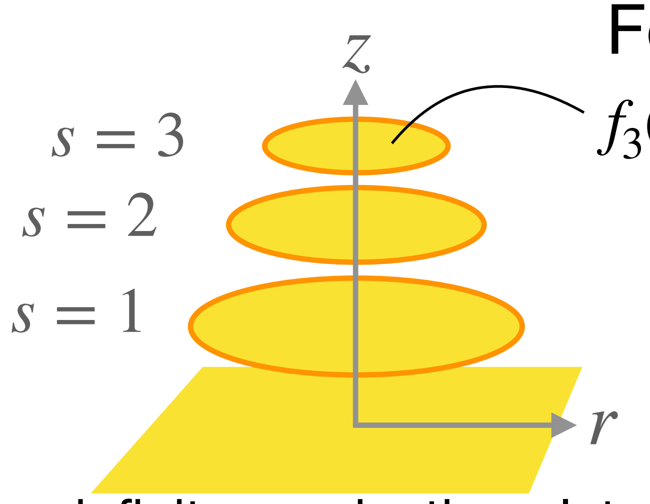
$R\times SO(3)\times SO(6)$
$\rightarrow SO(2)\times SO(5)$
a point on $S^2 \times S^5$



We expect it describes low-energy d.o.f. In fact, ϕ turns out to be time-independent, meaning that it only has a "0-energy mode."

Theory on 5-branes

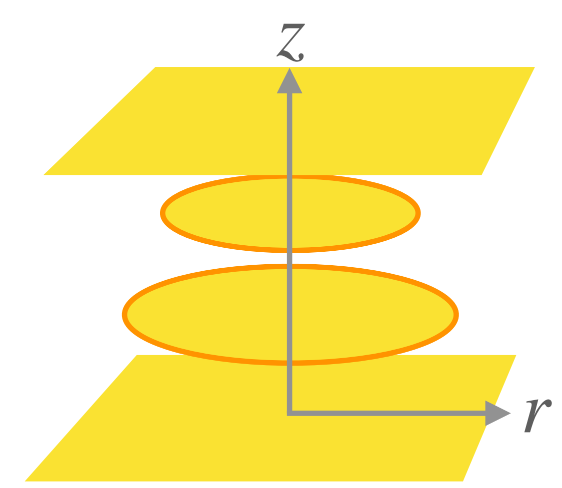
There's a double scaling limit to obtain the gravity solution for NS5-branes on the gravity side:



For general Λ,

$f_3(u) = f(u) + g(u)$

sol. for NS5-branes



For $\Lambda = 1$,

$$V = V_{\text{b.g.}} + \phi_1[f(u); N_5, r, z] \longrightarrow V = \frac{1}{g_0} \sin\frac{2z}{N_5} I_0\left(\frac{2r}{N_5}\right) = V_{\text{NS5 b.g.}}$$

$$f(u) - \frac{1}{\pi}\int_{-R}^{R} du' \frac{\frac{\pi}{2}(2N_5)}{\frac{\pi^2}{4}(2N_5)^2 + (u-u')^2} f(u') = C - V_0 N_5 u^2$$

DSL: $R \to \infty,\ V_0 \to \infty$ with $g_0 \propto \dfrac{\exp[\frac{2R}{N_5}]}{V_0(RN_5)^{3/2}}$: fixed

[Ling, Mohazab, Shieh, Anders, Raamsdonk '06]

Theory on 5-branes

The DSL is expected to keep all orders of $1/N$ expansion.

The expectation value of a function of ϕ is

$$\langle \mathcal{O}(\phi) \rangle = \sum_{h=0}^{\infty} d_h(\lambda) N_1^{-h}$$

If g_0 is the coupling of the theory on NS5-branes (LST),

$$\langle \mathcal{O}(\phi) \rangle = C(\lambda) \sum_{h=0}^{\infty} c_h \, g_0^h \quad \Rightarrow \quad \frac{d_{h+1}}{d_h} = \frac{c_{n+1}}{c_n} \lambda^{5/8} \exp\left[\pi \frac{(8\lambda)^{1/4}}{n_1}\right]$$

Finiteness of c_1/c_0 was checked numerically

[Y.A., Ishiki, Matsumoto, Shimasaki, Watanabe '22]

DSL beyond the planar sector