# Vafa-Witten Theory and Mock Modularity

Aradhita Chattopadhyaya

Ramanujan Fellow

Chennai Mathematical Institute

CHENNAI MATHEMATICAL INSTITUTE

Anusandhan National Research Foundation

# Guess who?

Hint: The first person from the west to write about India!

# A STRONG COUPLING TEST OF $S$-DUALITY

Cumrun Vafa

*Lyman Laboratory of Physics, Harvard University*
*Cambridge, MA 02138 USA*

and

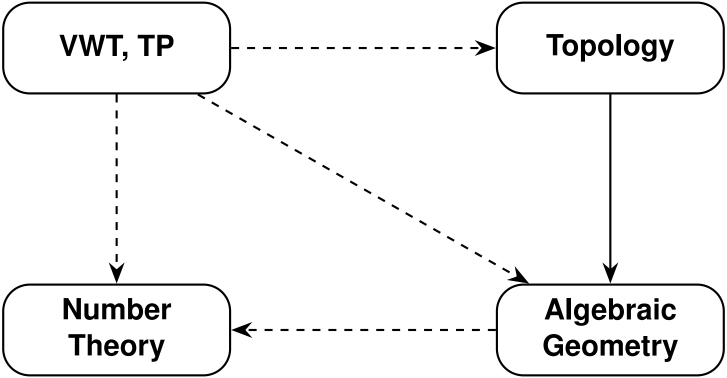
*School of Natural Sciences, Institute for Advanced Study*
*Olden Lane, Princeton, NJ 08540, USA*

and

Edward Witten

*School of Natural Sciences, Institute for Advanced Study*
*Olden Lane, Princeton, NJ 08540, USA*

Vafa, Witten (1994)


VWT, TP
Topology
Number Theory
Algebraic Geometry

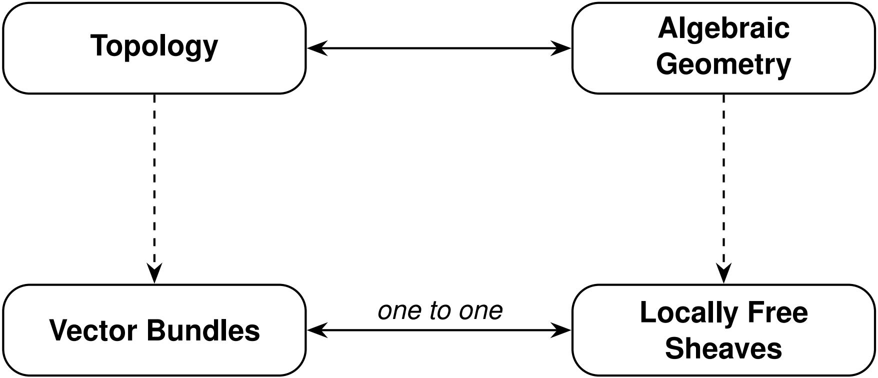
# VWT: Main Focus

- They computed the instanton partition function for $\mathcal{N}$=4 twisted Yang Mills theory on a 4 manifold $M$, with a gauge group $SU(N)$.
- $M = \underbrace{K3, \mathbb{CP}^2}_{VW:1994,\ N=2}, \underbrace{\text{Hizerbruch, del} - \text{Pezzo}}_{\text{Alexandrov, Manschot, Pioline}}$.

# Symmetries

- $\mathcal{N}$=4 super Yang Mills have a global symmetry $SU(4)$.
- The fundamental representation **4** is identified as $(\mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{2})$ of the local spin(4).
- There is an $SU(2)$ which interchanges these two $(\mathbf{1}, \mathbf{2})$ which is the global symmetry for the twisted theories.
- This allows us to write all equations with space-time and gauge group indices alone.

- $Z_M$ computes the Euler characteristic of moduli space of instantons $n \sim \int_M \mathrm{Tr} F \wedge F$ on $M$.
- Expect to obtain an invariant under the change of couplings in the theory. $g \to 1/g$. (Test of S-duality)
- Use methods from Algebraic geometry. (ADHM construction)


Topology
Algebraic Geometry
Vector Bundles
one to one
Locally Free Sheaves

Vafa-Witten Theory and Algebraic Geometry

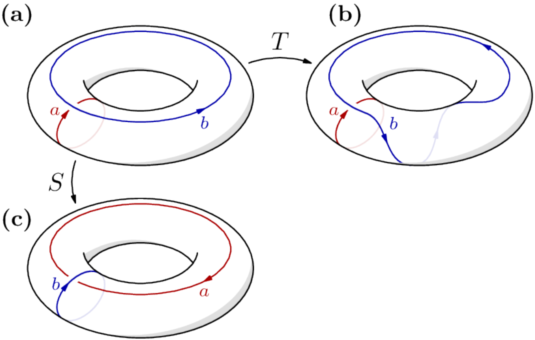
# VWT context

- Write the instanton equations as zeros of a section $F^+(A) = s(A) = 0$.
- Donaldson Uhlenbeck Yau theorem ensures that counting a $\underbrace{\text{certain class}}_{semi-stable,\ coherent}$ of sheaves and YM instantons in VWT are the same.
- Direct counting was done by Klyachko-Yoshioka (1991) for rank 2 sheaves.

  Zagier (1975) computed mock modularity of the resultant function.

Introduction to modular, mock modular forms

# Modular forms a functions on a torus



# Modular forms

If $\tau$ is in upper half plane, $f(\tau)$ is a modular form of weight $k$ if under an $SL(2\mathbb{Z})$ transformation:

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

$$ad - bc = 1, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}.$$

$k$ is the weight of the modular form.

Generators for the full modular group are given by,

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Due to the $T$ generator we can write them as

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_n q^n, \quad q = e^{2\pi i \tau}$$

These definitions may be extended to subgroups of $SL_2(\mathbb{Z})$.

If $a_0 = 0$, we have a cusp form.

In general we have:

$$f\left(\frac{a\tau+b}{c\tau+d}\right)=\epsilon(a,b,c,d)(c\tau+d)^k f(\tau)$$

In these cases we can write $f(\tau)$ in terms of Fourier coefficients

$$f(\tau)=\sum_{n\in\mathbb{Z}+\lambda} a_n q^n, \quad q=e^{2\pi i\tau}$$

for $\lambda\in\mathbb{Q}$.

Further generalization can occur with $f$ being vector valued of dimension $d$, and $\epsilon$ being a $d\times d$ matrix.

# Example: Dedekind eta function

Partition function of a closed bosonic string on a torus with parameter $\tau$ is given by,

$$\begin{aligned}
Z_{\text{boson}}(\tau) &= \frac{1}{\eta(\tau)} \\
\eta(\tau+1) &= e^{\frac{\pi i}{12}}\,\eta(\tau) \\
\eta(-\frac{1}{\tau}) &= \underbrace{\sqrt{-i\tau}}_{\text{weight } 1/2}\;\eta(\tau)
\end{aligned}$$

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \qquad q = e^{2\pi i\tau}$$

# $\Theta$ functions

These are modular forms of weight $1/2$ in $\Gamma_0(4)$.

$$\Theta_0(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}, \quad \Theta_{1/2}(\tau) = \sum_{n\in\mathbb{Z}+1/2} q^{n^2}$$

*S* transformations:

$$\begin{aligned} \Theta_0(-\frac{1}{\tau}) &= \frac{(-i\tau)^{1/2}}{\sqrt{2}} \left(\Theta_0(\tau) + \Theta_{1/2}(\tau)\right) \\ \Theta_{1/2}(-\frac{1}{\tau}) &= \frac{(-i\tau)^{1/2}}{\sqrt{2}} \left(\Theta_0(\tau) - \Theta_{1/2}(\tau)\right) \end{aligned}$$

# Mock modular forms

The function $f$ is modular for $f\colon \mathbb{H} \to \mathbb{C}$ such that under $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ transformation

$$f(\gamma\tau) = (c\tau + d)^k f(\tau)$$

Completion of Mock modular form $\hat{g}(\tau, \bar{\tau})$:

$$\partial_{\bar{\tau}} \hat{g}(\tau, \bar{\tau}) \sim f(\tau).$$

So essentially, $\hat{g}(\tau, \bar{\tau}) \sim \underbrace{g(\tau)}_{\text{mock modular}} + g^*(\tau, \bar{\tau})$.

Mockness is a trade off between holomorphicity and modularity.

# Mock Jacobi Form

The function $f$ is a Jacobi form of index $m$ and weight $k$ for $f$: $\mathbb{H} \times \mathbb{C} \to \mathbb{C}$ such that under $\gamma \in SL_2(\mathbb{Z})$ transformation property is

$$f(\gamma\tau, \gamma z) = (c\tau + d)^k e^{2\pi i m z/(c\tau+d)^2} f(\tau, z)$$

Also, under $z \to z + \ell\tau + n$ with $(\ell, n) \in \mathbb{Z}^2$

$$f(\tau, z + \ell\tau + n) = e^{-2\pi i m(\ell^2\tau + 2nz)} f(\tau, z)$$

Mockness comes as a trade off as before.

For mock Jacobi forms there is a notion of depth, which corresponds to the rank of the gauge group in VWT.

# Depth of a Mock modular form

The depth of a mock modular form is $n-1$ when its completion is given by $R(\tau,\bar{\tau})$, ie, the number of iterations of the integral

$$R_{n-1}(\tau,\bar{\tau}) = \# \int_{-\bar{\tau}}^{i\infty} d\omega_1 \int_{\omega_1}^{i\infty} d\omega_2 \ldots g(\omega_1,\omega_2,\ldots\omega_{n-1}) + \ldots$$

In the context of VWT, depth$= N-1$ for gauge group $SU(N)$.

# VWT and Number theory

# Results for $K3$

If $M = K3$ and the gauge group is $U(1)$ the partition function is given by $Z_{K3} = \frac{1}{\eta^{24}} = \frac{1}{\eta^{\chi(K3)}}$

For gauge group $SU(2)$ some Hecke operation on the original result for $K3$,

$$Z_{K3,SU(2)}(\tau) = \frac{1}{8} Z_{K3}(2\tau) + \frac{1}{4} Z_{K3}(\frac{\tau}{2}) + \frac{1}{4} Z_{K3}(\frac{\tau+1}{2})$$

For $SU(N)$ with prime $N$ the result is

$$Z_N^{K3} = \frac{1}{2N^2} Z_{K3}(N\tau) + \frac{1}{2N} \sum_{b=0}^{N-1} Z_{K3}(\frac{\tau+b}{N})$$

# $SU(2)$, $M = \mathbb{CP}^2$

For the gauge group $SU(2)$ the holomorphic part of the partition function reads:

$$
\begin{aligned}
Z_{2,0}(\tau) &= \frac{3}{\eta^6(\tau)} \sum_{n=0}^{\infty} H(4n) q^n = 3\frac{f_{2,0}(\tau)}{\eta^6(\tau)}, \\
Z_{2,1}(\tau) &= \frac{3}{\eta^6(\tau)} \sum_{n=1}^{\infty} H(4n-1) q^{n-1/4} = 3\frac{f_{2,1}(\tau)}{\eta^6(\tau)},
\end{aligned}
$$

where the subscripts $Z_{2,\mu}(\tau)$ correspond to the magnetic flux $\mu = 0, 1$.

Klyachko(1991), Yoshioka(1993,1994)

- $H(4n-\mu)$ is the Hurwitz class number
- The full partition function is a modular form of weight $-3/2$ and the holomorphic part is given by a mock modular form of same weight.
- $H(n)$ counts the number of solutions $b^2-4ac=-n$, with $|b|\leq a\leq c$, if $|b|=a$, or, $a=c$ then $b>0$, with some automorphic factor $g_n$.

  If $ax^2+bxy+cy^2$ represents a quadratic form, for $x^2+xy+y^2$, $n=3$, $g_3=1/3$ and $x^2+y^2$, $n=4$, $g_4=1/2$

$x^2 + xy + y^2$ remains invariant under the $SL_2(\mathbb{Z})$ transformation

$$x \to -y, \quad y \to x + y$$

The corresponding matrix $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^3 = -\mathbb{I}, \quad g_3 = 1/3.$

Similarly, $x^2 + y^2$ remains invariant under the $SL_2(\mathbb{Z})$ transformation

$$x \to -y, \quad y \to x$$

The corresponding matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = -\mathbb{I}, \quad g_4 = 1/2.$

# $SU(3)$

The growth pattern for the instanton numbers when these nos. are prime is interesting.

Moreover there is a modular form which transform with the same multiplicity as those of $f_{3,\mu}$, so we can produce a (mock) cusp form $S_\mu$.

Growth of these numbers is very close to $\sim p^{3/2}$.

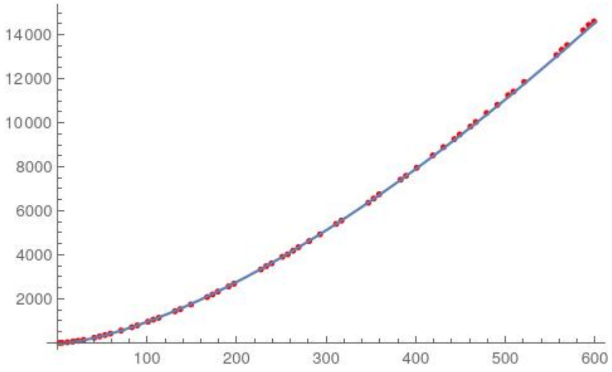



Figure: The dots represent $s_1(n + 2/3)$, with x axis going as $3n + 2$, and the line is given by $0.993519x^{3/2}$ and $n$ is on integers.

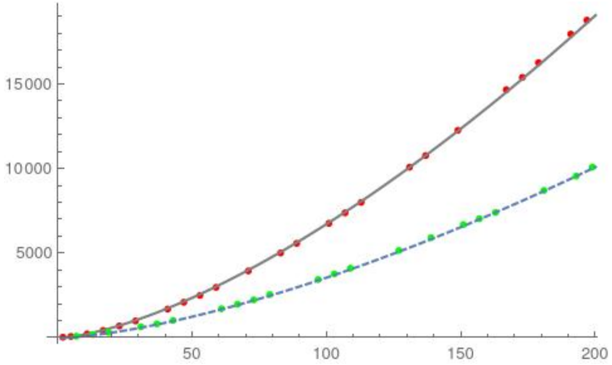



Figure: The red dots represent the coefficient $s_0(p)$ for prime $p = 3n - 1$ and the grey line is given by $6.75467x^{3/2}$, the green dots represent $s_0(p)$ for prime $p = 3n + 1$ and the dashed line is given by $3.57843x^{3/2}$

# Modularity check

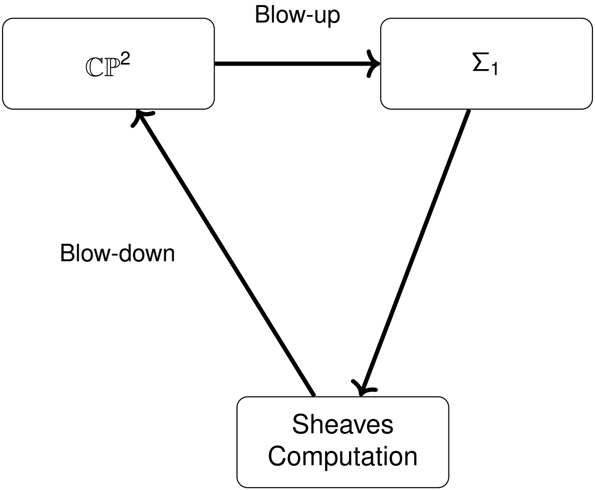
- $f_{2,\mu}$ is a mock modular form of depth 1 and its shadow is given by theta functions, for $SU(3)$ a depth 2 object whose shadow is $\hat{f}_{2,\mu}$.
- For $SU(2)$ the result was computed in 1975 by Don Zagier.
- However, it is not easy to check the modularity in this method.

# A short detour/Remark

The class number generating function $H(4n-\mu)$ also appears in the computation of black hole microstates in $\mathcal{N}=4$ type II theories on $K3 \times T^2$ and CHL orbifolds.

# Modularity check continued

- Series of work by Dominic Joyce (2002-04)- techniques of stack invariants.
- Results for the gauge group $SU(N), N \geq 2$ was computed using the above method by Manschot (2011-14).
- The results may be generalized to Hizerburch and del Pezzo surfaces. Alexandrov, Manschot, Pioline (2019).


Blow-up
$\mathbb{CP}^2$
$\Sigma_1$
Blow-down
Sheaves Computation

# Results

- For the gauge group $G = SU(N)$ in VWT corresponding to rank $N$ sheaves the results are given in terms of Appell functions:

$$g_{N,\mu} \sim \sum_{k\in\Lambda+\mu} \frac{q^{Q(k)/2} e^{2\pi i B(k,v)}}{\prod_{j=1}^{N}(1 - e^{2\pi i B(u,d_j)} q^{B(d_j,k)})}$$

Manschot (2011-14)

- $\Lambda$ is given in terms of $A_N$ root lattice.
- $g_{N,\mu}$ encodes the instanton partition function $Z_N$.

# Main ideas

This method allows one to obtain the modular completion

Works of Vigneras (1977)

Schematically we need to replace the sign functions in the geometric sums by generalization of error functions.

These generalized error functions have a structure of iterated integrals which describe the depth of the mock modular forms.

# Ongoing work 2609.XXXX

$$M_n(\tau) = \# \int_{-\bar{\tau}}^{i\infty} d\omega_1 \int_{\omega_1}^{i\infty} d\omega_2 \cdots \prod_{j=1}^{n} \frac{e^{2\pi i \omega_j^2 u_j}}{\sqrt{-i(\omega_j + \tau)}}$$

# BPS states

These Appell functions capture the refined BPS invariants of 1/2 BPS states.

This refinement allows us to access the Betti numbers of these moduli spaces.

Type IIA with a non-compact CY=$\mathrm{Tot}(K_S)$.

Information on the sheaves can be thought of as that obtained from $D4 - D2 - D0$ branes system.

Rank correspond to coincident $D4$ branes.

# Holomorphic anomaly

Vafa-Witten's prediction on the holomorphic anomaly:

$$D_N Z_N = \sum_{k=1}^{N-1} k(N-k) Z_k Z_{N-k}$$

To achieve this we require

$$\hat{g}_{N,\mu}(\tau,z) = \underbrace{g_{N,\mu}(\tau,z)}_{\text{Mock Jacobi}} + \sum_{k,\nu} \overline{\Theta_k(\tau,z)}\, g_{N-k,\nu}(\tau,z)$$

# Physics perspective

For the gauge group $SU(2)$ this anomaly has been recently understood boundary contributions of anti-instantons. Dabholkar, Putrov, Witten (2020).

For higher rank gauge groups path integral computation in $\mathcal{N} = 2^*$ theory (which under the massless limit goes to $\mathcal{N} = 4$ VWT) was done to obtain non-holomorphic completion by Manschot, Moore (2021).

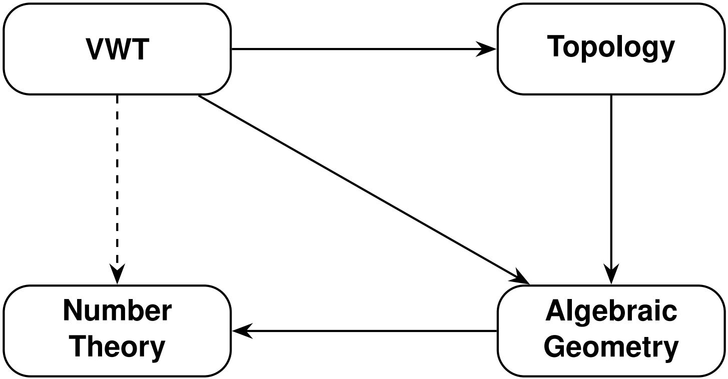
Modular completion of $g_{N,\mu}$:

$$\hat{g}_{N,\mu}(\tau, z, \bar{\tau}, \bar{z}) = g_{N,\mu}(\tau, z) + \sum_{m,\nu} \overline{\Theta_{m,\nu}(\tau)} g_{N-m,\nu}(\tau, z)$$

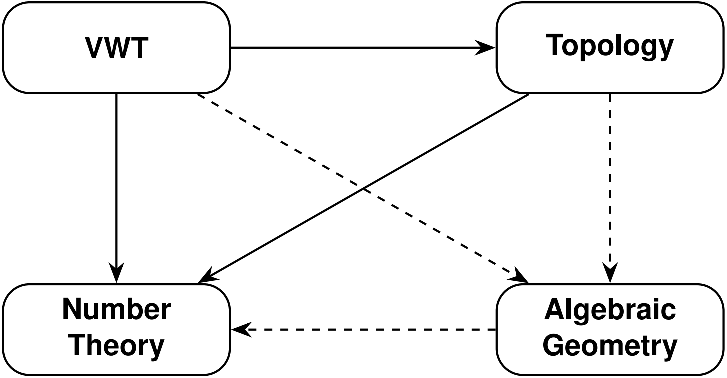
We constructed a generic class of Appell functions and showed the above holds.

Glue vectors are given by $\nu$.

$\overline{\Theta}$ is given with the kernels $M_k$.


VWT
Topology
Number Theory
Algebraic Geometry

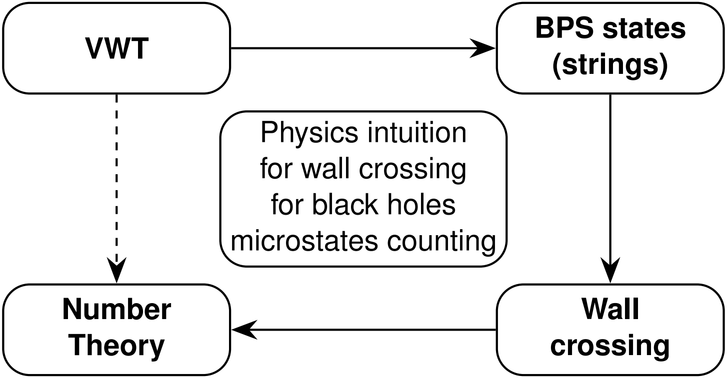
# Open Problem



# Recent work (2026 SA,AC)

For collinear charges $D4$ brane charges are given by positive integer partions of $N$

$$\hat{h}_S^{\mathrm{ref}} \sim \hat{g}.$$




THANK YOU