Bicovariant Coderivations, Dualities, Vector Fields

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Motivations and Plan

Quantum Groups(QG) are a special sort of Hopf algebras that posses a classical limit. In fact, there are two kinds of QG: quantize enveloping algebras (Drinfeld) and matrix (Woronowicz, FRT), which are "mutually dual" each other (FRT).
Differential calculus is a basic tool in NCG. They are natural object for matrix QGs, since in the classical limit one can reconstruct Cartan calculus on Lie group manifolds. Here we argue that for quantized enveloping algebras more natural are codifferential calculi.

PLAN:

- First-Order Codifferential Caluli (FOCC)
- Hopf algebra bicocovariant FOCC
- Duality FOCC/ FODC
- quantum Lie algebra structure
- quantum tangent space and quantum universal Killing vectors
- Some examples
- Summary and Conclusions

Coderivations

Differential calculus on associative unital -algebra $\mathcal{A} = (\mathcal{A}, \mu = \cdot, 1 = 1_{\mathcal{A}})$ is a derivation

$$d : \mathcal{A} \to \Omega \in {}_{\mathcal{A}}\mathfrak{M}_{\mathcal{A}}, \quad \text{such that} \quad d(f \cdot g) = df.g + f.dg$$

We also assume that $\Omega = \mathrm{Span}\{f.dg\} = \mathrm{Span}\{df.g\}$ (FODC). Dually, let $\mathcal{C} = (\mathcal{C}, \Delta, \varepsilon = \varepsilon_{\mathcal{C}})$ be a coalgebra. Coderivation (Y. Doi 1981)

$${}^{\mathcal{C}}\mathfrak{M}^{\mathcal{C}} \ni \Upsilon \xrightarrow{\delta} \mathcal{C}, \quad \text{such that} \quad \Delta \circ \delta = (\mathrm{id} \otimes \delta) \circ \Delta_L + (\delta \otimes \mathrm{id}) \circ \Delta_R\,,$$

$$\text{or} \quad \delta(m)_{(1)} \otimes \delta(m)_{(2)} = m_{(-1)} \otimes \delta(m_{<0>}) + \delta(m_{<0>}) \otimes m_{(1)}\,.$$

Restriction of δ to any subbicomodule $\Upsilon_1 \subset \Upsilon$ becomes a coderivation on Υ_1.

The image of any coderivation $\mathrm{Im}\delta$ is a coideal in $\mathcal{C}$

$$\epsilon \circ \delta = 0\,, \quad \text{i.e.} \quad \mathrm{Im}\delta \subset \mathrm{Ker}\varepsilon\,.$$

Universal coderivations

Similarly to the case of algebras, for each coalgebra there exists a universal coderivation [Doi81]. Firstly, for a coalgebra $(\mathcal{C}, \Delta, \varepsilon)$ one defines a universal bicomodule $\Upsilon_{\mathcal{C}}^{U}$ as a quotient

$$\Upsilon_{\mathcal{C}}^{U} = \mathcal{C} \otimes \mathcal{C} \,/\, \mathrm{Im}\Delta \equiv \mathrm{Coker}\Delta$$

together with $([\Delta(a)] \equiv [a_{(1)} \otimes a_{(2)}] = 0)$

$$\begin{aligned} \Delta_L^U[a \otimes b] &:= a_{(1)} \otimes [a_{(2)} \otimes b], \\ \Delta_R^U[a \otimes b] &:= [a \otimes b_{(1)}] \otimes b_{(2)}\,, \end{aligned}$$

where $[a \otimes b]$ denotes the corresponding equivalence class of a simple tensor $a \otimes b \in \mathcal{C} \otimes \mathcal{C}$ $([\mathrm{Im}\Delta] = 0)$. Then one gets the exact sequence of bicomodules

$$0 \longrightarrow \mathcal{C} \xrightarrow{\Delta} \mathcal{C} \otimes \mathcal{C} \xrightarrow{\pi} \Upsilon_{\mathcal{C}}^{U} \longrightarrow 0\,,$$

where $\pi(a \otimes b) = [a \otimes b]$ is a canonical projection. The universal coderivation is defined by

$$\delta^U[a \otimes b] := a\varepsilon(b) - b\varepsilon(a)$$

Proposition.(Y. Doi) There is one-to-one correspondence between

$$\mathrm{Com}^{(\mathcal{C},\mathcal{C})}(\Upsilon, \Upsilon_{\mathcal{C}}^{U}) \longleftrightarrow \mathrm{Coder}(\Upsilon, \mathcal{C}) \ni \delta$$

which is an isomorphism of $\mathbb{K}$-spaces.

$$\hat{\delta}(m) \doteq [\delta(m_{<0>}) \otimes m_{(1)}] = -[m_{(-1)} \otimes \delta(m_{<0>})] .$$

The last two expressions differ by $[\Delta(\delta(m))] = 0$. Therefore, $\hat{\delta}(\Upsilon) \subset \Upsilon_{\mathcal{C}}^{U}$ is a subbicomodule. Moreover,

$$\delta = \delta^{U} \circ \hat{\delta} .$$

Definition First order codifferential calculus (FOCC) over a coalgebra $\mathcal{C}$ is a pair (Υ, δ), where $\Upsilon \in {}^{\mathcal{C}}\mathfrak{M}^{\mathcal{C}}$ and $\delta : \Upsilon \to \mathcal{C}$ is a coderivation, such that the corresponding bicomodule homomorphism $\hat{\delta}$ is injective, i.e. $\mathrm{Ker}\hat{\delta} = 0$.

Corollary. This allows to classify FOCC classifying subbicomodules of $\Upsilon_{\mathcal{C}}^{U}$ (in fact, up to a coalgebra automorphism $\phi : \mathcal{C} \to \mathcal{C}$).

Bicovariant bimodules and Yetter-Drinfeld (YD) modules

Let $H = (H, \Delta, \mu = \cdot, \varepsilon, 1, S)$ be a Hopf algebra.

Definition. Bicovariant bimodule (a.k.a. Hopf bimodule) over Hopf algebra H is an object $(M, \triangleright, \triangleleft, \Delta_L, \Delta_R) \in {}^H_H\mathfrak{M}^H_H$, such that:

- $(M, \triangleright, \triangleleft) \in {}_H\mathfrak{M}_H$ is a bimodule, $(M, \Delta_L, \Delta_R) \in {}^H\mathfrak{M}^H$ is a bicomodule, and the following compatibility conditions are satisfied

$$\Delta_L(a \triangleright m \triangleleft b) = (a_{(1)} \cdot m_{(-1)} \cdot b_{(1)}) \otimes (a_{(2)} \triangleright m_{<o>} \triangleleft b_{(2)}),$$

  and $(1 \triangleright m = m \triangleleft 1 = m)$

$$\Delta_R(a \triangleright m \triangleleft b) = (a_{(1)} \triangleright m_{<o>} \triangleleft b_{(1)}) \otimes (a_{(2)} \cdot m_{(1)} \cdot b_{(2)}).$$

The structure theorem for such objects indicates (e.g. Schauenburg '93) that $V_L \otimes H \cong M \cong H \otimes V_R$, where $(V_L, \Xi_L, \blacktriangleright) \in {}^H_H\mathfrak{YD}$,

$$\Xi_L(a \blacktriangleright u) = a_{(1)} v_{(-1)} S(a_{(3)}) \otimes a_{(2)} \blacktriangleright v_{<o>}.$$

Thus $M \simeq V_L \otimes H$ has the following Hopf bimodule structure

$$\Delta_L(v \otimes a) = v_{(-1)} a_{(1)} \otimes v_{<0>} \otimes a_{(2)}, \quad \Delta_R(v \otimes a) = v \otimes a_{(1)} \otimes a_{(2)},$$

and $\qquad a \triangleright (v \otimes b) \triangleleft c = (a_{(1)} \blacktriangleright v) \otimes a_{(2)} bc.$

The case of universal bicomodule Υ_H^U

Theorem. For any Hopf algebra the universal bicomodule Υ_H^U is bicovariant if we define the bimodule structure

$$x \triangleright [a \otimes b] \triangleleft y \doteq [x_{(1)} a y_{(2)} \otimes x_{(2)} b y_{(1)}]$$

Further, $\Upsilon_H^U \cong \bar{H}_L \otimes H \cong H \otimes \bar{H}_R$, where $\bar{H}_L = (\bar{H}, \Xi_L, \blacktriangleright) \in {}^H_H\mathfrak{YD}$ and $\bar{H} \doteq H/\{1\mathbb{K}\} \ni \bar{c} = \overline{c + \lambda 1}\,, a \in H$. Here

$$\Xi_L(\bar{c}) = c_{(1)} \otimes \overline{c_{(2)}}, \quad x \blacktriangleright \bar{c} = \overline{x_{(1)} c S(x_{(2)})}\,.$$

bears a canonical Yetter-Drinfeld structure inherited from H.

Notice that $\bar{H}_L \subset \Upsilon_H^U$ by the identification $[c \otimes 1] \mapsto \bar{c}$ provides a left-left Y-D embedding $\bar{H}_L \subset \Upsilon_H^U$, hence $\delta(\bar{c}) = c - \varepsilon(c)1$.

Bicovariant coderivation

Definition Let $\Upsilon \in {}^H_H\mathfrak{M}^H_H$ then a coderivation (Υ, δ) over H is called bicovariant if

$$\delta(a \triangleright m \triangleleft b) = a\,\delta(m)\,b$$

Proposition 1. The universal FOCC $\delta^U : \Upsilon^U_H \to H$ is bicovariant.

2. Let $\Upsilon \in {}^H_H\mathfrak{M}^H_H$ then a coderivation (Υ, δ) over H is bicovariant iff $\hat{\delta}(\Upsilon)$ is bicovariant subbicomodule in Υ^U_H.

3. Classification of bicovariabt FOCC can be reduced to the classification of Hopf subbicomodules in Υ^U_H (up to a Hopf algebra automorphism) or equivalently, left-left YD subcomodules in $\bar{H}_L$.
More exactly, we are seeking non-isomorphic Y-D subbimodules $U \subset \bar{H}$, i.e., subspaces satisfying the following conditions

$$\mathrm{Ad}^L_H U \subset U\,, \quad \Xi_L(U) \subset H \otimes U\,.$$

Canonical (left) quantum Lie algebra structure on $\bar{H}_L$

Definition (E. Beggs, S. Majid) A vector space $\mathcal{L}$ equipped with a braiding $\tau_q : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L} \otimes \mathcal{L}$ and a bracket $[,]_q : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ satisfying the following compatibility condition: braided anticommutativity: $[\,,]_q$ vanishes on $\ker(\mathrm{id} - \tau_q)$; three braided Jacobi identities on $\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L}$

$$\begin{aligned}
[\,,]_q \circ ([\,,]_q \otimes \mathrm{id}) &= [\,,]_q \circ (\mathrm{id} \otimes [\,,]_q) \circ ((\mathrm{id} - \tau_q) \otimes \mathrm{id})\,, \\
\tau_q \circ (\mathrm{id} \otimes [\,,]_q) &= ([\,,]_q \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau_q) \circ (\tau_q \otimes \mathrm{id})\,, \\
\tau_q \circ ([\,,]_q \otimes \mathrm{id}) &- (\mathrm{id} \otimes [\,,]_q) \circ (\tau_q \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau_q) \\
&= ([\,,]_q \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau_q) \circ ((\mathrm{id} - \tau_q^2) \otimes \mathrm{id})\,.
\end{aligned}$$

Theorem For any Hopf algebra H the Y-D module $\overline{H}_L$, equipped with the canonical Y-D braiding $\tau_q : \bar{H} \otimes \bar{H} \to \bar{H} \otimes \bar{H}$

$$\tau_q(\overline{X} \otimes \overline{Y}) \doteq \mathrm{ad}^L_{X_{(1)}} \overline{Y} \otimes \overline{X_{(2)}} = \overline{X_{(1)} Y S(X_{(2)})} \otimes \overline{X_{(3)}}$$

and the (quantum) bracket

$$[\overline{X}, \overline{Y}]_q \doteq \mathrm{ad}^L_{\delta(\overline{X})} \overline{Y} = \overline{X_{(1)} Y S(X_{(2)})} - \varepsilon(X) \overline{Y}$$

bears a (left) quantum quantum Lie algebra structure, where $\delta(\overline{X}) \doteq \delta^U([X \otimes 1]) = X - \varepsilon(X) 1 \in \ker \varepsilon$.

Canonical duality between Hopf algebras

For every Hopf algebra $(H, \mu, \Delta, S, 1, \varepsilon)$ there exists a canonical dual Hopf algebra

$$(H^\circ, \mu^\circ = \Delta^* \mid_{H^\circ \otimes H^\circ} = \star, \Delta^\circ = \mu^* \mid_{H^\circ}, S^\circ = S^* \mid_{H^\circ}, 1^\circ = \varepsilon \mid_{H^\circ}, \epsilon^\circ = 1)\,,$$

where H° is the biggest subalgebra of $(H^*, \Delta^* = \star, 1 \equiv \varepsilon)$ which has well defined coproduct $\Delta^\circ = \mu^* \mid_{H^\circ} : H^\circ \to H^\circ \otimes H^\circ \subset H^* \otimes H^* \subset (H \otimes H)^*$ (a.k.a restricted dual). In the finite-dimensional case $H^\circ = H^*$. There is a canonical duality between Hopf algebras H, H° given by the evaluation map

$$< X|\alpha > \doteq \alpha(X)\,, \quad \text{such that}$$

$$< XY|\alpha > = \alpha_{(1)}(X)\alpha_{(2)}(Y)\,, \qquad < X|\alpha \star \beta > = \alpha(X_{(1)})\beta(X_{(2)})\,,$$

where $\Delta^\circ(\alpha) = \alpha_{(1)} \otimes \alpha_{(2)}$ and $\star$ denotes the convolution product. Of course, we also have $\epsilon^\circ(\alpha) =< 1|\alpha >$, $\varepsilon(X) =< X|\varepsilon >$ and $< S(X)|\alpha > =< X|S^\circ(\alpha) >$.

Consider a universal FODC $d^U : H^\circ \to \Omega^U_1(H^\circ) \equiv \ker \mu^\circ \subset \ker \Delta^*$, which is now a bicovariant FODC; $d^U\alpha = \alpha \otimes \varepsilon - \varepsilon \otimes \alpha$.

Remarks on Woronowicz bicovariant calculi

The vector space $H \otimes H$ enjoys two nonisomorphic Hopf bimodule structures which turns out to be dual each other

i) with the actions $a \triangleright (b \otimes c) \triangleleft d = a_{(1)} b d_{(1)} \otimes a_{(2)} c d_{(2)}$ and the coactions $\Delta_L(a \otimes b) = a_{(1)} \otimes (a_{(2)} \otimes b)\,, \quad \Delta_R(a \otimes b) = (a \otimes b_{(1)}) \otimes b_{(2)}$. The universal bicovariant bimodule Υ_H^U becomes its factor bimodule.

ii) with the actions $a \triangleright (b \otimes c) \triangleleft d = ab \otimes cd$ and the coactions $\Delta_L(a \otimes b) = a_{(1)} b_{(1)} \otimes (a_{(2)} \otimes b_{(2)})\,, \quad \Delta_R(a \otimes b) = (a_{(1)} \otimes b_{(1)}) \otimes a_{(2)} b_{(2)}$ (c.f. Woronowicz). In this case $\ker \mu \cong H \otimes (\ker \varepsilon)_R$ is a bicovariant subbimodule of $H \otimes H$.

$$d^U : H \to \ker \mu \cong H \otimes (\ker \varepsilon)_R\,, \quad d^U a \doteq a \otimes 1 - 1 \otimes a$$

where $(\ker \varepsilon)_R \in \mathfrak{YD}_H^H$. The universal FODC is bicovariant.

According to Woronowicz prescription, firstly one chooses a Y-D submodule $R \subset \ker \varepsilon$ and then we set $N = r^{-1}(H \otimes R) \subset \ker \mu$. Thus $\ker \mu / N \simeq H \otimes \ker \varepsilon / R$. The isomorphism is realized by Woronowicz map $r(a \otimes b) = a b_{(1)} \otimes b_{(2)}$. Moreover, any bicovariant FODC on H can be obtained in this way. Obviously, there is also left-handed version of this provided by the Woronowicz map $s'(a \otimes b) = a_{(1)} \otimes a_{(2)} b$; $N = s'^{-1}(L \otimes H) \subset \ker \mu$.

Duality FOCC – FODC, the bicovariant (universal) case

Assume a bicovariant bimodule structure of type i) on $H \otimes H$ and the type ii) (Woronowicz type) on $H^\circ \otimes H^\circ$.
Theorem The pairing extends naturally to the pairing between $H \otimes H$ and $H^\circ \otimes H^\circ$ as well as to the pairing between Υ_H^U and $\ker \mu^\circ = \{\omega = \sum_i \alpha^i \otimes \beta^i \in H^\circ \otimes H^\circ | \sum_i (\alpha^i \star \beta^i)(X) = 0 \text{ for } X \in H\}$, where

$$< [X \otimes Y] | \omega >= \sum_i \alpha^i(X) \beta^i(Y) .$$

The following, preserving the bicovariance properties are thus satisfied

$$< Z \triangleright [X \otimes Y] | \omega >=< Z \otimes [X \otimes Y] | \Delta_L^\circ(\omega) >$$

where $\Delta_L^\circ(\omega) = \alpha_{(1)} \star \beta_{(1)} \otimes (\alpha_{(2)} \otimes \beta_{(2)})$

$$< [X \otimes Y] \triangleleft Z | \omega >=< [X \otimes Y] \otimes Z | \Delta_R^\circ(\omega) >$$

where $\Delta_R^\circ(\omega) = (\alpha_{(1)} \otimes \beta_{(1)}) \otimes \alpha_{(2)} \star \beta_{(2)}$.

$$< \Delta_L^U([X \otimes Y])|\gamma \otimes \omega >=< [X \otimes Y]|\gamma \triangleright \omega >$$

where $\gamma \triangleright \omega = \sum_i \gamma \star \alpha^i \otimes \beta^i$.

$$< \Delta_R^U([X \otimes Y])|\omega \otimes \gamma >=< [X \otimes Y]|\omega \triangleleft \gamma >$$

where $\omega \triangleleft \gamma = \sum_i \alpha^i \otimes \beta^i \star \gamma$. Moreover

$$< \delta^U[X \otimes Y]|\alpha >=< [X \otimes Y]|d^U \alpha >= \alpha(X)\varepsilon(Y) - \alpha(Y)\varepsilon(X) .$$

Finally,

$$< [X \otimes 1] \triangleleft Y|\omega >=< r(\overline{X} \otimes Y)|\omega >=< \overline{X} \otimes Y|s'(\omega) >=< [X \otimes 1] \otimes Y|\Delta^\circ(\omega) >$$

where $r(\overline{X} \otimes Y) = [XY_{(1)} \otimes Y_{(2)}]$ and $s'(\omega) = \sum_i$

Using isomorpisms of bicovariant bimoduules $\Upsilon^U_H \cong \overline{H}_L \otimes H$ and $\ker \mu^\circ \cong (\ker \epsilon^\circ)_L \otimes H^\circ$ one gets

Lemma The pairing reduces to the pairing between left-left Y-D modules $\overline{H}_L$ and $(\ker \epsilon^\circ)_L = \{\alpha \in H^\circ | \alpha(1) = 0\}$

$$< \overline{X} | \alpha > \doteq \alpha(X) \,, \quad \text{for} \quad \alpha \in \ker \epsilon^\circ \,.$$

In particular, there is a duality between left-left Y-D module structures on $\overline{H}_L \in {}^H_H\mathfrak{YD}$ and $(\ker \epsilon^\circ)_L \in {}^{H^\circ}_{H^\circ}\mathfrak{YD}$

$$\langle \mathrm{ad}^L_Y \overline{X} | \alpha \rangle = \langle Y \otimes \overline{X} | \widetilde{\Xi}_L(\alpha) \rangle \,,$$

where $\widetilde{\Xi}_L(\alpha) = \alpha_{(1)} \star S^\circ(\alpha_{(3)}) \otimes \alpha_{(2)}$.

$$\langle \overline{X} | \beta \succ \alpha \rangle = \langle \Xi_L(\overline{X}) | \beta \otimes \alpha \rangle \,,$$

where $\beta \succ \alpha = \beta \star \alpha$, $\beta \in H^\circ$ and $\Xi_L(\overline{X}) = X_{(1)} \otimes \bar{X}_{(2)}$.
Of course, the similar results are valid if we set a universal bicovariant FODC on H and a universal FOCC on H°.

Universal FODC and universal vector fields

Consider a universal FODC over H°, $d^U : H^\circ \to \ker \mu^\circ \cong \ker \varepsilon^\circ \otimes H^\circ$. A right (module) dual $(\ker \mu^\circ)^\dagger \doteq \mathrm{Hom}_{(-,H^\circ)}(\ker \mu^\circ, H^\circ)$, the space of right module morphisms, i.e. H°-linear maps which admits a bimodule structure defined on it

$$(\alpha.X.\beta)(\omega \star \gamma) \doteq \alpha \star X(\beta \star \omega) \star \gamma$$

where $X \in (\ker \mu^\circ)^\dagger, \alpha, \beta, \gamma \in H^\circ, \omega \in \ker \mu^\circ$. Further, with any element X one can associate the endomorphism $X^\triangleright \in \mathrm{End}^0_{\mathbb{K}}(H^\circ) \doteq \{A \in \mathrm{End}_{\mathbb{K}}(H^\circ) | A(1^\circ) = 0\}$

$$X^\triangleright(\alpha) \doteq X(d^U \alpha) = X(\alpha \otimes \varepsilon - \varepsilon \otimes \alpha)$$

via the quantum contraction. In fact $(\ker \mu^\circ)^\dagger \cong \mathrm{Hom}_{\mathbb{K}}(\overline{H^\circ}, H^\circ) \cong \mathrm{End}^0_{\mathbb{K}}(H^\circ)$. Thus, the following Leibniz rule is satisfied

$$X^\triangleright(\alpha \star \beta) = X^\triangleright(\alpha) \star \beta + (X.\alpha)^\triangleright(\beta)\,.$$

In this way, the pair $((\ker \mu^\circ)^\dagger, \triangleright)$ satisfies the requirements of the abstract definition of a (right) Cartan pair from [AB'96].

Since $\ker\mu^\circ \cong (\ker\varepsilon^\circ)_L \otimes H^\circ$ is a right free comodule, then [1]

$$(\ker\mu^\circ)^* \cong \mathrm{Com}^{(-,H^\circ)}(\ker\mu^\circ, H^\circ) \subset \mathrm{Hom}_{\mathbb{K}}(\ker\mu^\circ, H^\circ)\,,$$

by $\tilde{v}(d\alpha \star \beta) \doteq (v \otimes \mathrm{id})\Delta^\circ_R(d\alpha \star \beta) = v(d\alpha_{(1)} \star \beta_{(1)})\alpha_{(2)} \star \beta_{(2)}$ is a right comodule map and $v = \epsilon^\circ \circ \tilde{v} \in (\ker\mu^\circ)^*$.
Similarly, we can also associate a right module map $\hat{v} \in (\ker\mu^\circ)^\dagger \doteq \mathrm{Mod}_{(-,H^\circ)}(\ker\mu^\circ, H^\circ) \subset \mathrm{Hom}_{\mathbb{K}}(\ker\mu^\circ, H^\circ)$ by

$$\hat{v}(d^U\alpha \star \beta) \doteq \tilde{v}(d^U\alpha) \star \beta = v(d^U\alpha_{(1)})\alpha_{(2)} \star \beta$$

$(\ker\mu^\circ)^* \ni v \to \hat{v} \in (\ker\mu^\circ)^\dagger \subset \mathrm{Hom}_{\mathbb{K}}(\ker\mu^\circ, H^\circ)$, Both maps agrees on differential $d^U\alpha$. Finally, one also has the vector field action

$$v^\triangleright(\alpha) \doteq \hat{v}(d^U\alpha) = \tilde{v}(d^U\alpha) = v(d^U\alpha_{(1)})\alpha_{(2)}\,,$$

i.e., $v^\triangleright = (v \circ d^U \otimes \mathrm{id}) \circ \Delta^\circ \in \mathrm{End}^0_{\mathbb{K}}(H^\circ)\,.$

[1] Vector space isomorphism!

Woronowicz quantum tangent space

Definition [Woronowicz] A vector space (notice that $\alpha(1) = \epsilon^\circ(\alpha)$ and $X(1^\circ) = \varepsilon(X)$.)

$$\mathcal{T}_\circ = \{v \in (\ker \mu^\circ)^* \mid v(d^U\alpha \star \beta) = v(d^U\alpha)\beta(1)\}$$

will be called a universal quantum tangent space for H°.
Thus restricting all three maps to the subdomain $\mathcal{T}_\circ$ give rise to the following:
Lemma $v \in \mathcal{T}_\circ \iff \hat{v} = \tilde{v} \in \mathrm{Hom}_{\mathbb{K}}(\ker \mu^\circ, H^\circ)$, i.e.

$$\tilde{v}(d^U\alpha \star \beta) = \hat{v}(d^U\alpha \star \beta) = \tilde{v}(d^U\alpha) \star \beta = v(d^U\alpha_{(1)})\alpha_{(2)} \star \beta\,.$$

Reduced (special) quantum tangent space

$\bar{H} \ni \overline{X} \mapsto \eth(\bar{X}) \in \mathcal{T}_\circ$ one can associate

$$\eth(\overline{X})(d\alpha \star \beta) \doteq \alpha(\delta(X))\beta(1) = (\alpha(X) - \varepsilon(X)\alpha(1))\beta(1)$$

an element of the universal quantum tangent space. Therefore, the image $\eth(\overline{H}_L) \subset \mathcal{T}_\circ$ one can call a special universal quantum tangent space. This image can be further extend to

$$\widetilde{\eth(\overline{H}_L)} = \widehat{\eth(\overline{H}_L)} \subset \mathrm{Com}^{(-,H^\circ)}(\ker \mu^\circ, H^\circ) \cap \mathrm{Mod}_{(-,H^\circ)}(\ker \mu^\circ, H^\circ) \subset (\ker \mu^\circ)^\dagger\,,$$

or $\eth(\overline{H}_L)^\triangleright \subset \mathrm{End}^0_{\mathbb{K}}(H^\circ)$, which we would like to call a special (or covariant) universal quantum vector fields. We find

$$\widehat{\eth(\overline{X})}(d\alpha) \equiv \eth(\overline{X})^\triangleright(\alpha) = \alpha_{(1)}(X)\alpha_{(2)} - \varepsilon(X)\alpha$$

However, it should be mentioned that the first map $\overline{X} \mapsto \eth(\overline{X})$ does not need, in general, to be injective. This is because a canonical map $H \to (H^\circ)^*$ is not always injective. The assumption that an algebra H is proper, guarantees injectivity.

Quantum universal Killing vector fields

Proposition Assume that H is a proper Hopf algebra. The subspace of covariant universal vector fields $\widehat{\eth(\overline{H}_L)} \subset (\ker \mu^\circ)^\dagger$ (or $\eth(\overline{X})^\triangleright \subset \mathrm{End}^0_{\mathbb{K}}(H^\circ)$), bears induced quantum Lie algebra structure if we define

$$\underline{\tau}_q(\eth(\overline{X})^\triangleright \otimes \eth(\overline{Y})^\triangleright) \doteq (\eth()^\triangleright \otimes \eth()^\triangleright)(\tau_q(\bar{X} \otimes \bar{Y}))\,,$$

and

$$\underline{[\eth(\overline{X})^\triangleright, \eth(\overline{Y})^\triangleright]}_q \doteq \eth([\bar{X}, \bar{Y}]_q)^\triangleright\,.$$

The inherited left Y-D structure over H is defined by the left coaction

$$\underline{\Xi}_L \circ \eth()^\triangleright \doteq (\mathrm{id} \otimes \eth()^\triangleright) \circ \Xi_L\,,$$

and the left action

$$\underline{\blacktriangleright} \circ \eth()^\triangleright \doteq \eth()^\triangleright) \circ \blacktriangleright\ .$$

Or the other way around, one can define an action $\bar{H}_L \ni \bar{X} \mapsto \bar{X}^{\underline{\triangleright}} \in \mathrm{End}^0_{\mathbb{K}}(H^\circ)$ by

$$\bar{X}^{\underline{\triangleright}}(\alpha) \doteq \alpha_{(1)}(X)\alpha_{(2)} - \varepsilon(X)\alpha \quad \text{for} \quad \alpha \in H^\circ\,.$$

Quantize enveloping Hopf algebra $U_q(\mathfrak{sl}(2))$

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1, & KE &= q^2 EK, \\ [E,F] &= \frac{K - K^{-1}}{q - q^{-1}}, & KF &= q^{-2} FK. \end{aligned}$$

$$\begin{array}{ccc} \Delta(K) = K \otimes K & \varepsilon(K) = 1 & S(K) = K^{-1} \\ \Delta(E) = E \otimes K + 1 \otimes E & \varepsilon(E) = 0 & S(E) = -EK^{-1} \\ \Delta(F) = F \otimes 1 + K^{-1} \otimes F & \varepsilon(F) = 0 & S(F) = -KF \end{array}$$

There is a family of left Y-D modules generated by singletons

$$\mathcal{L} < \overline{K^n} > \qquad \text{for} \quad n \in \mathbb{Z} \setminus \{0\}$$

For $n > 0$ the dimension is finite.
For $n < 0$ and the dimension is infinite.

Lowest dimensional bicovariant $U_q(\mathfrak{sl}(2))$ FOCC

The lowest-dimensional bicovariant FOCC, $n = 1$, have four elements:

$$\begin{aligned} \upsilon_{00} &= \overline{K}, & \upsilon_{10} &= \overline{E}, \\ \upsilon_{01} &= \overline{FK}, & \upsilon_{11} &= \overline{EF - q^2 FE}. \end{aligned}$$

the second lowest-dimensional bicovariant FOCC, $n = 2$, is nine dimensional.

$$\begin{aligned} \upsilon_{00} &= \overline{K^2}, & \upsilon_{10} &= \overline{EK}, & \upsilon_{20} &= \overline{E^2}, \\ \upsilon_{01} &= \overline{FK^2}, & \upsilon_{11} &= \overline{(EF - q^4 FE)K}, & \upsilon_{12} &= \overline{E^2F - q^4 FE^2}, \\ \upsilon_{02} &= \overline{F^2K^2}, & \upsilon_{21} &= \overline{(EF^2 - q^4 F^2E)K}, & \upsilon_{22} &= \ . \end{aligned}$$

$$\upsilon_{22} = \overline{E^2F^2 - (q^2 + q^4)EF^2E + q^6 F^2E^2}$$

Quantum Lie algebra structure for $n = 1$. Braiding.

$$\tau_q(v_{00} \otimes v_{00}) = v_{00} \otimes v_{00} \qquad \tau_q(v_{00} \otimes v_{10}) = q^2 v_{10} \otimes v_{00}$$
$$\tau_q(v_{00} \otimes v_{11}) = v_{11} \otimes v_{00} \qquad \tau_q(v_{00} \otimes v_{01}) = q^{-2} v_{01} \otimes v_{00}$$

$$\tau_q(v_{10} \otimes v_{00}) = v_{00} \otimes v_{10} + (1 - q^2) v_{10} \otimes v_{00} \qquad \tau_q(v_{10} \otimes v_{10}) = v_{10} \otimes v_{10}$$
$$\tau_q(v_{10} \otimes v_{11}) = v_{11} \otimes v_{10} - (q^3 + q) v_{10} \otimes v_{00} \qquad \tau_q(v_{10} \otimes v_{01}) = v_{01} \otimes v_{10} + v_{11} \otimes v_{00}$$

$$\tau_q(v_{01} \otimes v_{00}) = v_{00} \otimes v_{01} + (1 - q^{-2}) v_{01} \otimes v_{00} \qquad \tau_q(v_{01} \otimes v_{10}) = v_{10} \otimes v_{01} - v_{11} \otimes v_{00}$$
$$\tau_q(v_{01} \otimes v_{11}) = v_{11} \otimes v_{01} + (q + q^{-1}) v_{01} \otimes v_{00} \qquad \tau_q(v_{01} \otimes v_{01}) = v_{01} \otimes v_{01}$$

$$\begin{aligned}
\tau_q(v_{11} \otimes v_{00}) &= v_{00} \otimes v_{11} + (2 - q^2 - q^{-2})(v_{11} \otimes v_{00} + v_{01} \otimes v_{10} - v_{10} \otimes v_{01}) \\
\tau_q(v_{11} \otimes v_{11}) &= v_{11} \otimes v_{11} + (q^{-1} - q^3)(v_{11} \otimes v_{00} + v_{01} \otimes v_{10} - v_{10} \otimes v_{01}) \\
\tau_q(v_{11} \otimes v_{10}) &= q^{-2} v_{10} \otimes v_{11} + (1 - q^{-2}) v_{11} \otimes v_{10} + (q + q^{-1}) v_{10} \otimes v_{00} \\
\tau_q(v_{11} \otimes v_{01}) &= q^2 v_{01} \otimes v_{11} + (1 - q^2) v_{11} \otimes v_{01} - (q^3 + q) v_{01} \otimes v_{00}
\end{aligned}$$

Quantum Lie algebra structure for $n = 1$. Brackets.

$$[\upsilon_{00}, \upsilon_{00}]_q = 0 \qquad [\upsilon_{00}, \upsilon_{11}]_q = 0 \qquad [\upsilon_{00}, \upsilon_{10}]_q = (q^2 - 1)\upsilon_{10}$$
$$[\upsilon_{00}, \upsilon_{01}]_q = (q^{-2} - 1)\upsilon_{01}$$
$$[\upsilon_{10}, \upsilon_{00}]_q = (1 - q^2)\upsilon_{10} \qquad [\upsilon_{10}, \upsilon_{11}]_q = -(q^3 - q)\upsilon_{10} \qquad [\upsilon_{10}, \upsilon_{10}]_q = 0$$
$$[\upsilon_{10}, \upsilon_{01}]_q = \upsilon_{11}$$
$$[\upsilon_{01}, \upsilon_{00}]_q = (1 - q^{-2})\upsilon_{01} \qquad [\upsilon_{01}, \upsilon_{11}]_q = (q - q^{-1})\upsilon_{10} \qquad [\upsilon_{01}, \upsilon_{10}]_q = -\upsilon_{11}$$
$$[\upsilon_{01}, \upsilon_{01}]_q = 0$$
$$[\upsilon_{11}, \upsilon_{00}]_q = (2 - q^2 - q^{-2})\upsilon_{11} \qquad [\upsilon_{11}, \upsilon_{11}]_q = (q - q^3)\upsilon_{11}$$
$$[\upsilon_{11}, \upsilon_{10}]_q = (q + q^{-1})\upsilon_{10} \qquad [\upsilon_{11}, \upsilon_{01}]_q = -(q + q^3)\upsilon_{11}$$

In the classical limit $q \mapsto 1$, $\upsilon_{00} = \overline{K} \mapsto 0$, $\upsilon_{10} \mapsto E$, $\upsilon_{01} \mapsto F$ and $\upsilon_{11} \mapsto [E, F] = H$ one can recover $\mathfrak{sl}(2)$ relations as well as the flip braiding $\tau_1(\upsilon_1 \otimes \upsilon_2) = \upsilon_2 \otimes \upsilon_1$.

κ-Poincaré Hopf algebra

Let's introduce κ-Poincaré Hopf algebra by a system of generators $H = \mathrm{gen}\{\Pi_0,\ \Pi_0^{-1},\ P_j,\ N_j,\ M_j | j = 1,2,3\}$ (AB, A. Pachol)

$$
\begin{array}{lll}
[\mathcal{P}_i, \Pi_0] = 0, & [\mathcal{P}_j, \mathcal{P}_k] = 0, & [M_j, M_k] = i\epsilon_{jkl} M_l, \\
[M_j, \Pi_0] = 0, & [M_j, \mathcal{P}_k] = i\epsilon_{jkl}\mathcal{P}_l, & [N_j, M_k] = i\epsilon_{jkl} N_l, \\
[N_j, \Pi_0] = \frac{i}{\kappa}\mathcal{P}_j, & [N_j, \mathcal{P}_k] = -i\delta_{jk}\mathcal{P}_0, & [N_j, N_k] = -i\epsilon_{jkl} M_l.
\end{array}
$$

$$
\begin{aligned}
\Delta\Pi_0 &= \Pi_0 \otimes \Pi_0, \\
\Delta\mathcal{P}_j &= \mathcal{P}_j \otimes \Pi_0 + 1 \otimes \mathcal{P}_j, \\
\Delta M_j &= M_j \otimes 1 + 1 \otimes M_j, \\
\Delta N_j &= N_j \otimes 1 + \Pi_0^{-1} \otimes N_j - \frac{1}{\kappa}\epsilon_{jkl}\mathcal{P}_k \Pi_0^{-1} \otimes M_l.
\end{aligned}
$$

$$
\mathcal{P}_0 \doteq \frac{\kappa}{2}\left(\Pi_0 - \Pi_0^{-1}(1 - \frac{1}{\kappa^2}\vec{\mathcal{P}}^2)\right).
$$

Lowest dimensional bicovariant FOCC

It is 5-dimensional with its right-free module basis [2]:
$\Upsilon = \langle \upsilon_C, \upsilon_0, \upsilon_j \rangle \subset \bar{H}_L$.

$$\delta(\upsilon_0) = \Pi_0 - 1, \qquad \delta(\upsilon_j) = \mathcal{P}_j, \qquad \delta(\upsilon_C) = \kappa^2(\Pi_0 + \Pi_0^{-1} - 2) - \vec{\mathcal{P}}^2 \Pi_0^{-1}.$$

$$\Delta_L(\upsilon_C) = \Pi_0^{-1} \otimes \upsilon_C + 2(\kappa \mathcal{P}_0 - \vec{\mathcal{P}}^2 \Pi_0^{-1}) \otimes \upsilon_0 - 2\mathcal{P}_k \Pi_0^{-1} \otimes \upsilon_k,$$
$$\Delta_L(\upsilon_0) = \Pi_0 \otimes \upsilon_0, \qquad \Delta_L(\upsilon_j) = \mathcal{P}_j \otimes \upsilon_0 + 1 \otimes \upsilon_j.$$

$$[N_j, \upsilon_0] = -\tfrac{i}{\kappa}\upsilon_j, \qquad [M_j, \upsilon_k] = i\epsilon_{jkl}\upsilon_l,$$
$$[N_j, \upsilon_k] = i\delta_{jk}(\tfrac{1}{2\kappa}\upsilon_C - \kappa\upsilon_0).$$

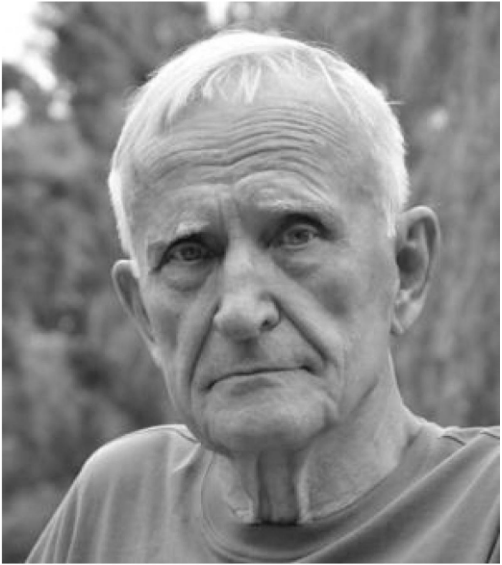
! Corresponds to known results by A. Sitarz'95.

[2] $\upsilon_0 = \bar{\Pi}_0, \upsilon_i = \bar{P}_i, \upsilon_C = \bar{C}_\kappa,\ C_\kappa = \kappa^2(\Pi_0 + \Pi_0^{-1} - 2) - P_i P^i \Pi_0^{-1}$.

Summary and conclusions

- Bicovariant FOCCi are dual counterpart of Woronowicz bicovariant FODCi over arbitrary Hopf algebras. This argues that such codifferential calculi are better suited to Drinfeld-Jimbo type quantized enveloping algebras, as they are dual to Woronowicz' bicovariant calculi over matrix quantum groups (quantum-deformed Lie groups). Such approach provides a proper classical limit.
- Their classification reduces to the classification of left-left Y-D submodules in $\bar{H}_L \in {}^H_H\mathfrak{Y}\mathfrak{D}$. Submodules generated by singletons are non-decomposable.
- $\bar{H}_L$ bears the structure of a quantum Lie algebra. Its Y-D submodules are quantum Lie subalgebras.
- $\bar{H}_L$ plays also a role of quantum covariant vector fields for the universal FODC over the dual Hopf algebra H°. It is generated by universal quantum Killing (fundamental) vector fields satisfying quantum Lie algebra relations.

THANK YOU !



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