Symbology, Cluster Algebras and Amplitudes+

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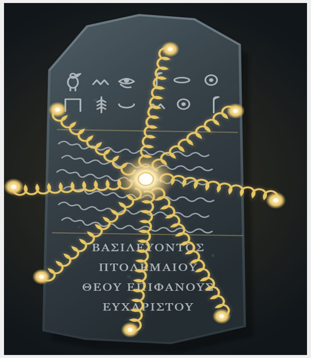
Outline

- Introduction
- N=4 Yang-Mills Amplitudes
- Tools: Symbology and Cluster Algebras
- Applications: - Feynman integrals
  - Energy correlator integrals
  - Cosmological correlator integrals
  (Mark's talk)
- Outlook

N=4 SYM: Quantum Rosetta Stone

Planar N=4 Yang-Mills (SYM) amplitudes have the most amazing mathematical structures:



- **Dual Conformal/Yangian Symmetry**
- **Amplituhedron**
- **Cluster Algebras**

It is a simple model that (likely) can be solved exactly, and provides general tools and intuition.

Arkani-Hamed, Dixon, McLeod, Spradlin, Trnka, AV 2207.10636

n-point L-loop amplitudes in N=4 SYM: status

- all n, all k L=0
- n<6 all L
- n=6 L=8
- n=7 L=5
- all n MHV L=2
- n=8 MHV L=3
- n=8, 9 NMHV L=2

Britto, Cachazo, Feng, Witten (BCFW) 0501052

Bern, Dixon, Smirnov (BDS) 0505205

Caron-Huot, Dixon, Drummond, Dulat, Foster, Gurdogan, von Hippel, McLeod, Papathanasiou, review: 2005.06735
Dixon, Liu 2308.08199 He et all 2511.09669

Caron-Huot 1105.5606

Li, Zhang 2110.00350

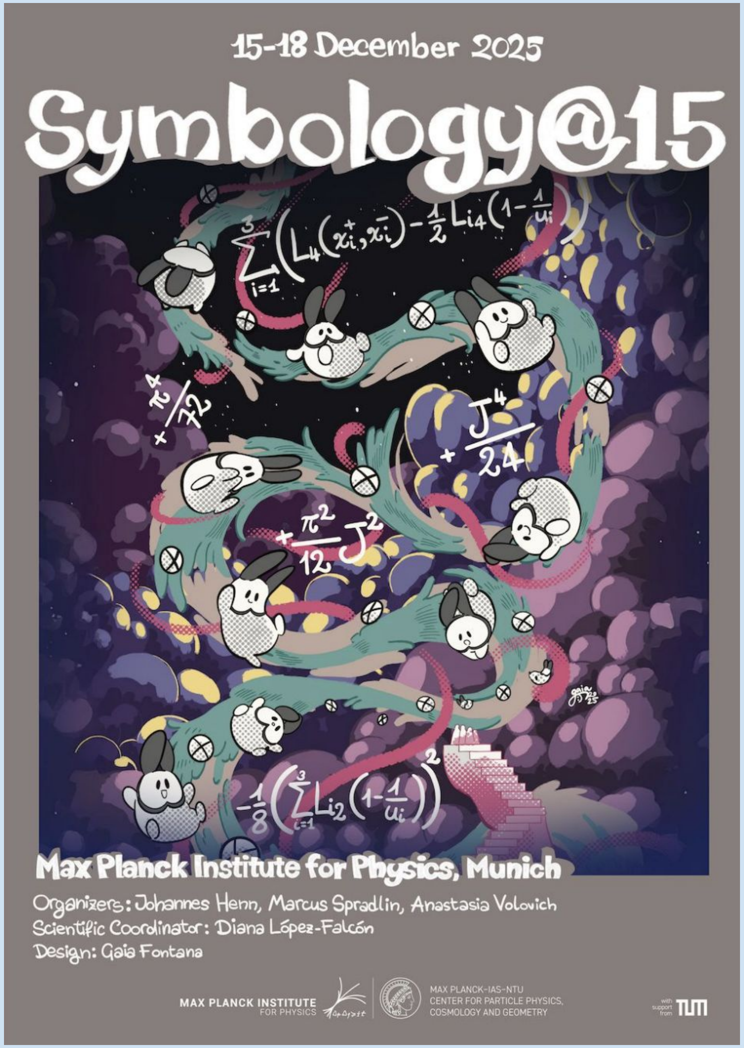
He, Li, Zhang 1911.01290
2009.11471

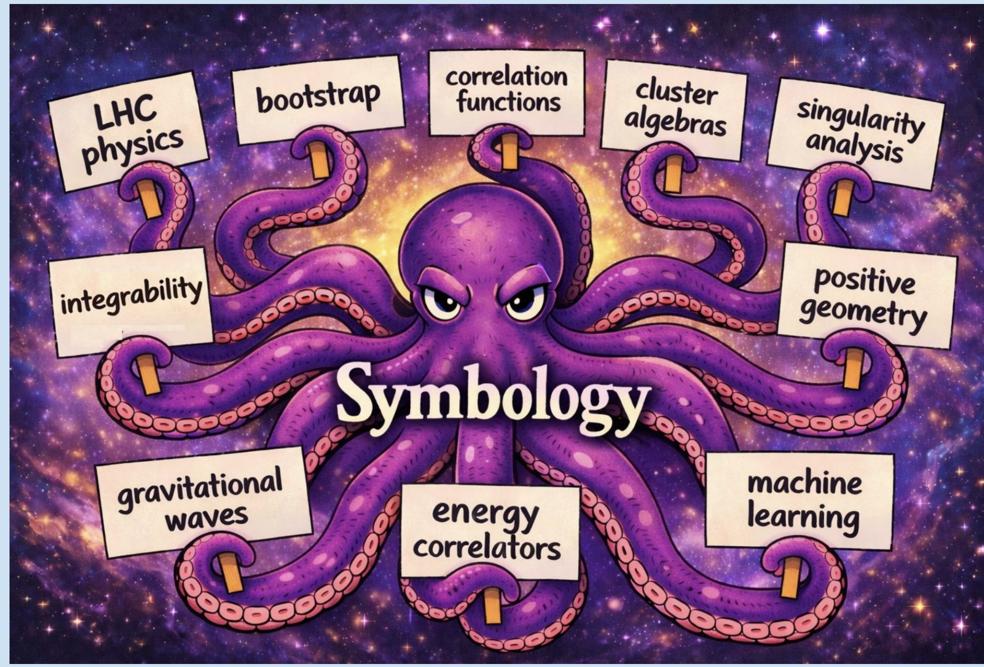
As of today, Dixon and Liu 8-loop results stand at the forefront of perturbative calculations in N=4 SYM (and quantum field theory in general) and, in my opinion, serve as a benchmark - or a "challenge" - against which any "AI takeover" should be measured.

How are there computations done?

There are two important ingredients:

- **6- and 7-point amplitudes are MPLs, i.e. they can be studied via SYMBOLOGY**
- **symbol alphabet for n-point amplitudes is described by Gr(4,n) CLUSTER ALGEBRA**


15-18 December 2025
Symbology@15
Max Planck Institute for Physics, Munich
Organizers: Johannes Henn, Marcus Spradlin, Anastasia Volovich
Scientific Coordinator: Diana López-Falcón
Design: Gaia Fontana
MAX PLANCK INSTITUTE FOR PHYSICS
MAX PLANCK-IAS-NTU CENTER FOR PARTICLE PHYSICS, COSMOLOGY AND GEOMETRY
with support from TUM


LHC physics
bootstrap
correlation functions
cluster algebras
singularity analysis
integrability
positive geometry
Symbology
gravitational waves
energy correlators
machine learning

SYMBOLOGY

- Uniform Transcendentality m=2L @ L-loops
- Polylogarithms

$$Li_m(x) = \int_0^x \frac{dt}{t} Li_{m-1}(t) \qquad Li_1(x) = -\log(1-x)$$

- Generalized Polylogarithms (MPLs)

$$G(a_1,\ldots,a_m;x) = \int_0^x \frac{dt}{t-a_1} G(a_2,\ldots,a_m;t) \qquad G(;x) = 1$$

- Elliptic Polylogarithms (eMPLs) n=10 L=2 k=3

Bourjaily, Broedel, Chaubey, Duhr, Frellesvig, Hidding, Marzucca, McLeod, Spradlin, Tancredi, Vergu, Volk, AV, Hippel, Weinzierl, Wilhelm, Zhang 2203.07088

SYMBOLOGY

Goncharov, Spradlin, Vergu, AV 1006.5703

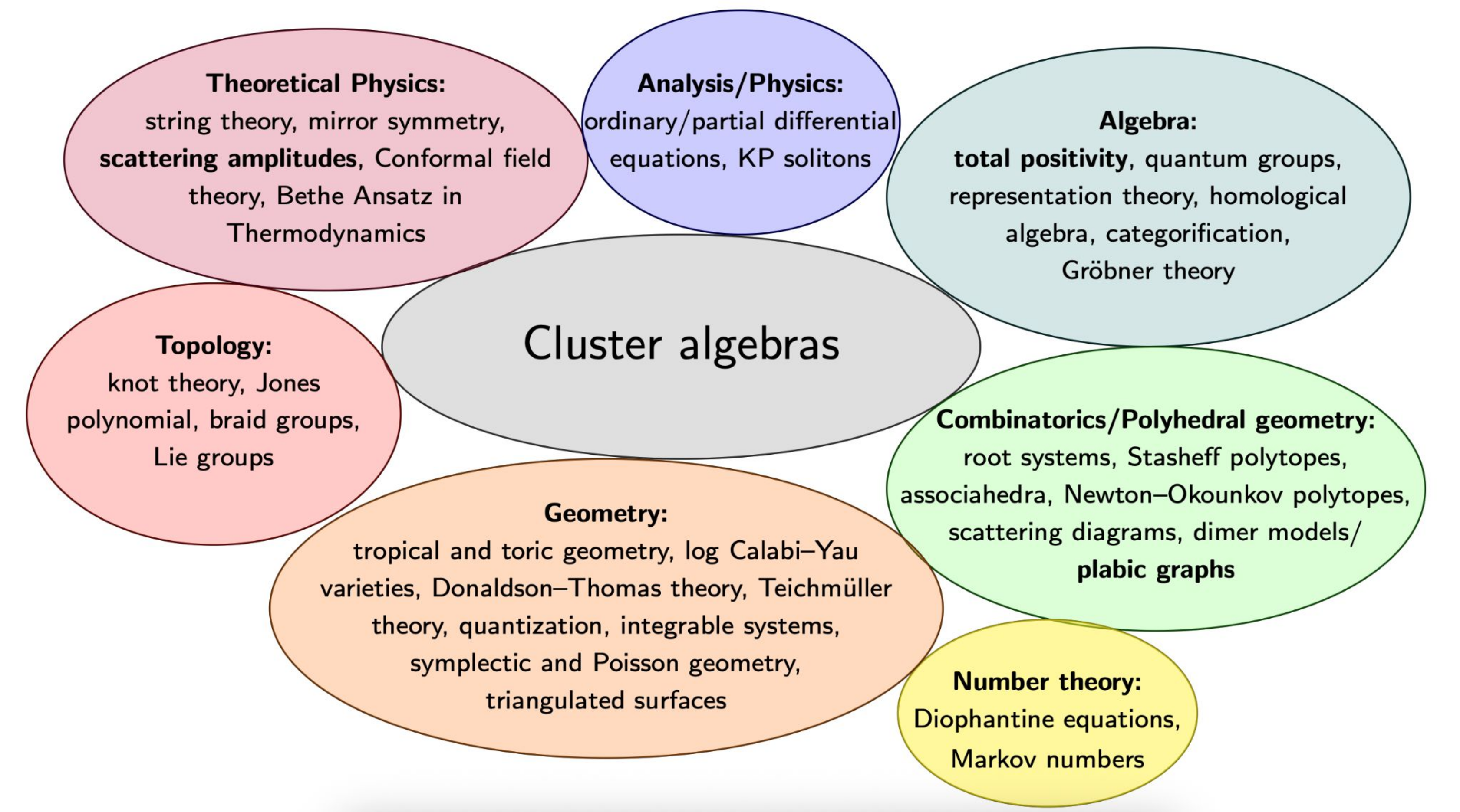
$$dF^{(m)} = \sum_i F_i^{(m-1)}\, d\log x_i$$

$$S(F^{(m)}) := \sum_i S(F_i^{(m-1)}) \otimes x_i$$

$$S(\log x) = x \qquad S[Li_2(x)] = -(1-x) \otimes x$$

The arguments that appear in a symbol are called symbol letters and collectively they are called symbol alphabet.

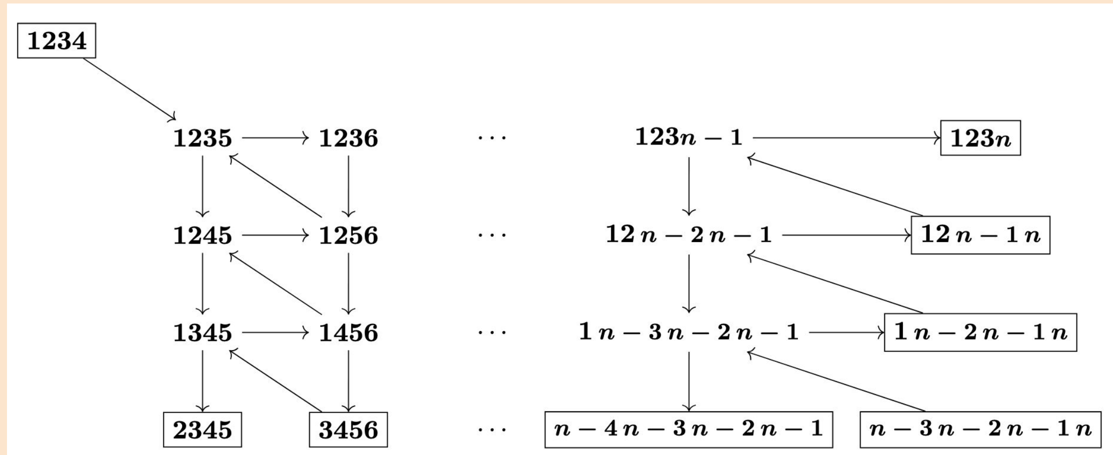
CLUSTER ALGEBRAS
MSC13F60



<u>Slide</u> from Bossinger, Harvard CMSA Summer School (2025)

Grassmannian Cluster Algebra Gr(4,n)

Initial Seed



Mutation Rule

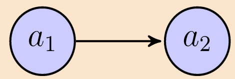
$$a_k \to a_k' = \frac{1}{a_k}\left(\prod_{\text{arrows } i\to k} a_i + \prod_{\text{arrows } k\to j} a_j\right)$$

Cluster Variables

$$\{a_k\}$$

Fomin, Zelevinsky 0104151, Scott 0311148, Gekhtman, Shapiro, Vainshtein 0208033, Williams 1212.6262, Fomin, Williams, Zelevinsky "Introduction to Cluster Algebras"

A2 Cluster Algebra



$$a_k \to a'_k = \frac{1}{a_k}\left(\prod_{\text{arrows } i\to k} a_i + \prod_{\text{arrows } k\to j} a_j\right)$$

$$a_1,\ a_2,\ a_3 = \frac{1+a_2}{a_1},\ a_4 = \frac{1+a_1+a_2}{a_1 a_2},\ a_5 = \frac{1+a_1}{a_2}$$

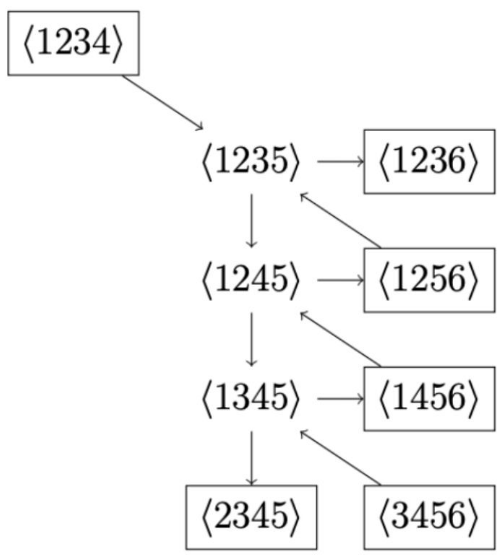
Fomin, Williams, Zelevinsky “Introduction to Cluster Algebras”

So far,
I have explained to you two main ingredients:

- 6-, 7-point amplitudes are MPLs, i.e. they can be studied via SYMBOLOGY
- symbol alphabet for n-point amplitudes is described by Gr(4,n) CLUSTER ALGEBRA

Next,
let us look how these appear in
N=4 SYM amplitudes.

Gr(4,6) Cluster Algebra for 6-point SYM



Golden, Goncharov
Spradlin, Vergu, AV
1305.1617

Start with the initial seed

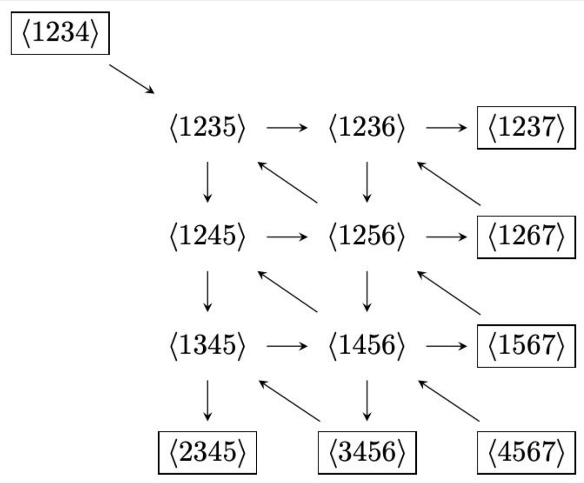
Perform all mutations

Obtain 15 cluster variables

$$\{\langle \mathbf{a}\,\mathbf{a{+}1}\,\mathbf{b}\,\mathbf{c}\rangle\}$$

Match 2-loop 6-point symbol alphabet!

Gr(4,7) Cluster Algebra for 7-point SYM



Golden, Goncharov
Spradlin, Vergu, AV
1305.1617

Start with the initial seed

Perform all mutations

Obtain 35+7+7 cluster variables

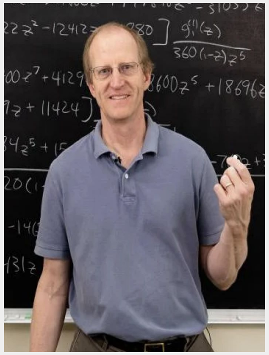
$$\{\langle \mathbf{a\,a{+}1\,b\,c}\rangle, \langle \mathbf{1(23)(45)(67)}\rangle, \langle \mathbf{1(27)(34)(56)}\rangle\}$$

Match 2-loop 7-point symbol alphabet!

Cluster Bootstrap for N=4 SYM

Dixon et al have computed 6- and 7-point amplitudes to higher number of loops (8 and 5, respectively) by starting with the symbol alphabet presented on the last two slides and using a bootstrap approach, based on mathematical and physical inputs.

Constraint	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$
1. $\mathscr{H}_6$	6	27	105	372	1214	3692?
2. Symmetry	(2,4)	(7,16)	(22,56)	(66,190)	(197,602)	(567,1795?)
3. Final-entry	(1,1)	(4,3)	(11,6)	(30,16)	(85,39)	(236,102)
4. Collinear	(0,0)	(0,0)	$(0^*,0^*)$	$(0^*,2^*)$	$(1^{*3},5^{*3})$	$(6^{*2},17^{*2})$
5. LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*,0^*)$	$(1^{*2},2^{*2})$
6. NLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*,0^*)$	$(1^*,0^{*2})$
7. NNLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	$(1,0^*)$
8. N^3LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
9. Full MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
10. T^1 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
11. T^2 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)



Caron-Huot, Dixon, Drummond, Dulat, Foster, Gurdogan, von Hippel, McLeod, Papathanasiou, review: 2005.06735
Dixon, Liu 2308.08199
He, Jiang, Li, Liu 2511.09669

New Features at n>7

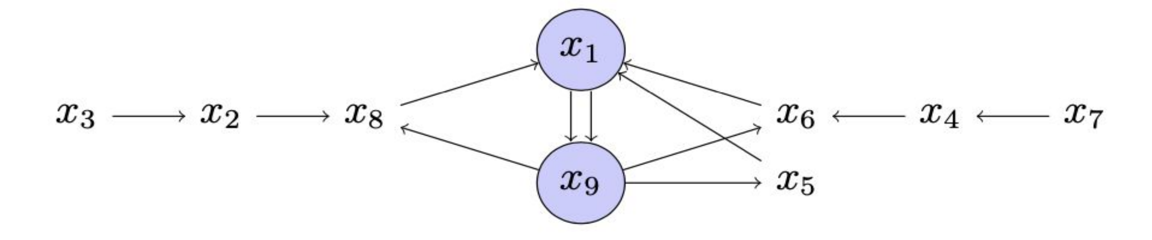
- Gr(4,n) cluster algebra is infinite for n>7
- Some symbol letters are not cluster coordinates, but involve square roots when written in terms of Pluckers

Li, Zhang 2110.00350

He, Li, Zhang 1911.01290
2009.11471

Algebraic Letters from Infinite Paths

All algebraic symbol letters known to appear in 8 and 9-point amplitudes are known to be associated to infinite mutation sequences inside Gr(4,8) and Gr(4,9).



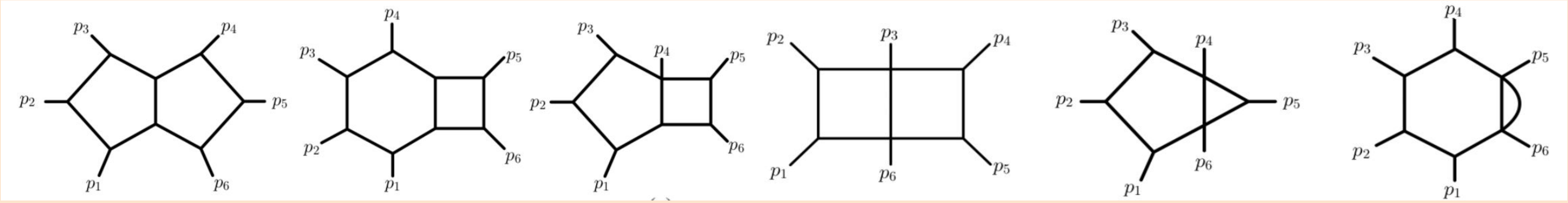
Drummond, Foster, Gürdoğan, Kalousios 1912.08217

Applications:

1. Feynman integrals

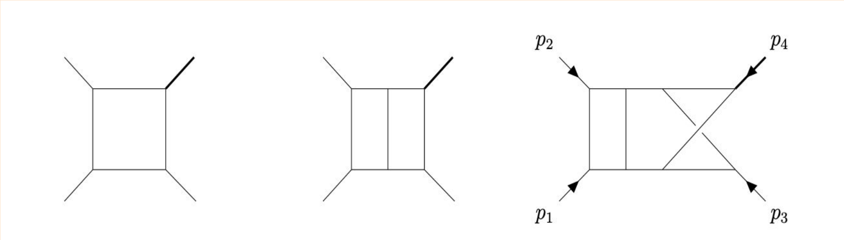
Feynman integrals: status

- **n=4 L=3 one mass** Henn, Lim, Torres Bobadilla 2302.12776; Gehrmann, Henn, Jakubcik, Lim, Mella, Syrrakos, Torres Bobadilla 2410.19088
- **n=5 L=2 massless** Gehrmann, Henn, Lo Presti 1511.05409; Chicherin, Henn, Mitev 1712.09610; Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia 1812.11160; Abreu, Dixon, Herrmann, Page, Zeng 1812.08941, 1902.08563
- **n=5 L=3 massless** Liu, Matijasic, Miczajka, Xu, Xu, Zhang 2411.18697
- **n=5 L=2 one mass** Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia 2306.15431
- **n=5 L=2 two masses** Abreu, Chicherin, Sotnikov, Zoia 2406.05201
- **n=6 L=2 massless** Henn, Matijasic, Miczajka, Peraro, Xu, Zhang 2403.19742 2501.01847; Abreu, Monni, Page, Usovitsch 2412.19884



Cluster algebras for 4-point integrals

Chicherin, Henn, Papathanasiou <u>2012.12285</u>, Aliaj, Papathanasiou <u>2408.14544</u> found cluster algebras structure in certain Feynman integrals relevant to QCD.



A_2,C_2,G_2 for L=1,2,3

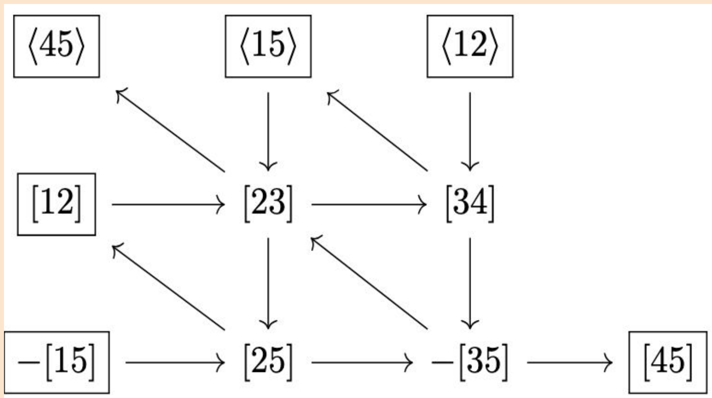
$$\{\mathbf{a_1}, \mathbf{a_2}\}$$

$$\mathbf{a_{m+1}} = \begin{cases} \dfrac{1+\mathbf{a_m}}{\mathbf{a_{m-1}}} & m \ \textbf{odd} \\ \dfrac{1+\mathbf{a_m^L}}{\mathbf{a_{m-1}}} & m \ \textbf{even} \end{cases}$$

5,6,8 cluster variables match symbol alphabet after the change of variables

$$\mathbf{a_1} = \frac{\mathbf{s} - \mathbf{p_4^2}}{\mathbf{t}} \qquad \mathbf{a_2} = \frac{\mathbf{s}}{\mathbf{t}} \frac{\mathbf{p_4^2} - \mathbf{s} - \mathbf{t}}{\mathbf{p_4^2}}$$

F(2,4;5) Flag Cluster Algebra for 5-point integrals



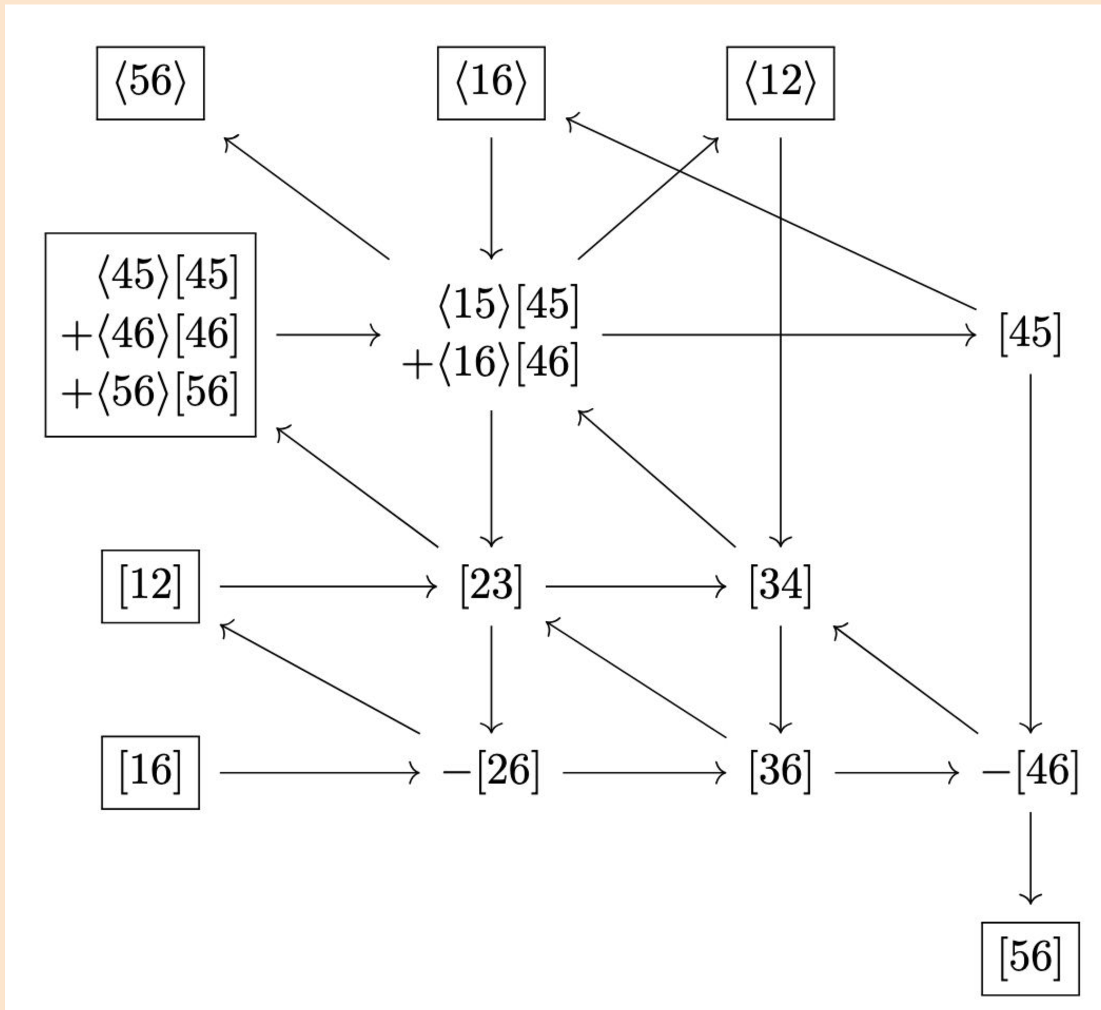
22 cluster variables

$$\{\langle i\,j\rangle, [i\,j], \langle 2\,3\rangle[2\,3] - \langle 4\,5\rangle[4\,5], \langle 1\,2\rangle[1\,2\rangle - \langle 3\,4\rangle[3\,4]\}$$

If we take the union under cyclic permutation (or under all permutations) it produces exactly 25 planar (and 5 non-planar) letters (except for one letter $\sqrt{\Delta}$ that drops out of all known amplitudes in four dimensions).

Pokraka, Spradlin, AV, Weng [2506.11895]

F(2,4;6) Flag Cluster Algebra for 6-point integrals



F(2,4;6) Cluster Algebra & 2-loop 6-point Symbol Alphabet

245 symbol letters through all orders in dim reg

- **7 are analogs of $\sqrt{\Delta}$ in the 5-point case**
- **135 are F(2,4;6) cluster variables (including permutations)**
- **40 are algebraic and we reproduce all of them from infinite mutation sequences**
- **27 of them are rational, are NOT cluster variables, but only appear at O(epsilon) so are not relevant in four dimensions**
- **36 remain somewhat mysterious…**

(24 of 36 appear in Carrolo, Chicherin, Henn, Yang, Zhang 2505.01245)

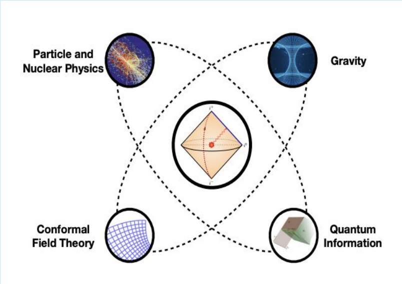
Pokraka, Spradlin, AV, Weng 2506.11895

Applications:

2. Energy correlators integrals

Energy Correlators

- Energy Correlators are of central importance in both collider phenomenology and in the formal high energy theory.



- They are defined as energy-weighted cross sections and measure energy deposited in detectors as a function of the angular separation between detector pairs.

- The study of energy correlators has a very long history… see Moult-Zhu 114 pages review 2506.09119

Energy Correlators: A Journey From Theory to Experiment

Ian Moult[1,*] and Hua Xing Zhu[2,3,†]
[1] *Department of Physics, Yale University, New Haven, CT 06511*
[2] *School of Physics, Peking University, Beijing, 100871, China*
[3] *Center for High Energy Physics, Peking University, Beijing 100871, China*

2506.09119

[hep-ph] 10 Jun 2025

Collider experiments offer a unique opportunity to explore the Standard Model (SM), and to search for new physics, new interactions, and new principles of nature. The theoretical abstraction of a collider, namely the study of correlations in asymptotic fluxes, provides one of the most basic examples of an observable in quantum field theory (QFT) and quantum gravity. Energy flux is described in QFT by energy flow operators, a particular example of light-ray operators. In addition to their central role in the theoretical description of collider physics, energy flow operators play an important role in diverse areas of formal QFT and gravity, providing a connection between real world collider phenomenology, and the deep underlying principles of QFT.
Recently it has become possible to measure correlation functions of energy flow operators in a wide variety of collider experiments, providing an exciting new connection between collider physics and formal theory. In this review, we provide a survey of recent progress in our understanding of energy operators and their correlators, highlighting their importance in both formal theory and collider phenomenology, and in particular, their great potential for bridging these areas to provide new ways to understand the real world. We intend this article as a resource for both formal theorists interested in understanding how light-ray operators are being applied in particle and nuclear physics, as well as for experimentalists interested in the theoretical motivation for these observables. Most importantly, we aim to stimulate further interaction between the formal, phenomenological and experimental communities through the common lens of energy correlators.

Energy Correlators: Four-Point

The Collinear Limit of the Four-Point Energy Correlator in $\mathcal{N} = 4$ Super Yang-Mills Theory

Dmitry Chicherin,[a] Ian Moult,[b] Emery Sokatchev,[a] Kai Yan,[c,d] Yunyue Zhu[c]

[a] *LAPTh, Universite Savoie Mont Blanc, CNRS, B.P.110, F-74941 Annecy-le-Vieux, France*

[b] *Department of Physics, Yale University, New Haven, CT 06511, USA*

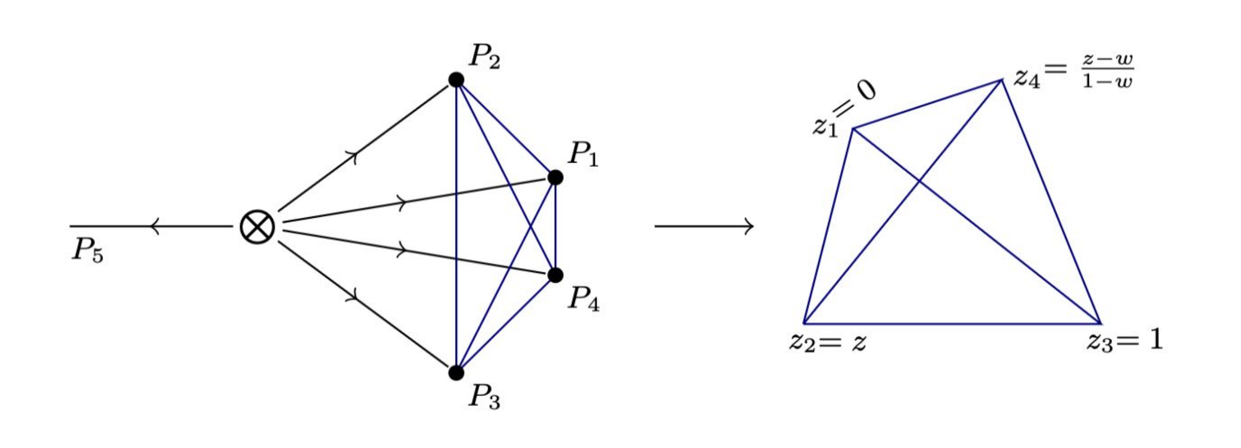
[c] *School of Physics and Astronomy, Shanghai Jiao Tong University, Shanghai 200240, China*

[d] *Key Laboratory for Particle Astrophysics and Cosmology (MOE), Shanghai 200240, China*

We present a compact formula, expressed in terms of classical polylogarithms up to weight three, for the leading order four-point energy correlator in maximally supersymmetric Yang-Mills theory, in the limit where the four detectors are collinear. This formula is derived by combining a simplified, manifestly dual conformal invariant form of the $1 \to 4$ splitting function obtained from the square of the tree-level five-particle form factor of stress-tensor multiplet operators, with a novel integration-by-parts algorithm operating directly on Feynman parameter integrals. Our results provide valuable data for exploring the structure of physical observables in perturbation theory, and for calculations of jet substructure observables in quantum chromodynamics.

2401.06463

Energy Correlators: Four-Point



$$\mathrm{E}^4\mathrm{C}^{\mathrm{LO,coll}}_{\mathcal{N}=4}(z_{ij}) = \frac{\sum_{I=1}^{51} R_I(z,\omega)\, F_I(z,\omega)}{|z_{12}|^2 |z_{23}|^2 |z_{34}|^2} + \mathrm{perm}(1,2,3,4)$$

```
{F[1] -> (2*froot[a])/3 + (2*froot[b])/3 + (2*froot[c])/3 +
   (2*Pi^2*G[0, v/(-1 + v)])/3 - 2*G[0, 1/(u*ub)]*G[0, u/(u - v)]*
    G[0, v/(-1 + v)] - G[0, u/(u - v)]^2*G[0, v/(-1 + v)] +
   (G[0, 1/(u*ub)]*G[0, v/(-1 + v)]^2)/2 + (2*G[0, v/(-1 + v)]^3)/3 +
   G[0, 1/(u*ub)]*G[0, v/(-1 + v)]*G[0, 1/(v*vb)] +
   2*G[0, u/(u - v)]*G[0, v/(-1 + v)]*G[0, 1/(v*vb)] +
   G[0, v/(-1 + v)]^2*G[0, 1/(v*vb)] + (2*Pi^2*G[0, vb/(-1 + vb)])/3 +
   G[0, 1/(u*ub)]*G[0, 1/(v*vb)]*G[0, vb/(-1 + vb)] +
   (G[0, 1/(u*ub)]*G[0, vb/(-1 + vb)]^2)/2 +
```

```
{R[1] -> -2 + u + ub + v + vb + (-1 + v)^2*(-1 + vb)*q[1] - v^2*vb*q[2] +
   (-1 + u)^2*(-1 + ub)*q[3] - u^2*ub*q[4] + (-1 + v)*(-1 + vb)^2*q[7] -
   v*vb^2*q[8] + (-1 + u)*(-1 + ub)^2*q[13] - u*ub^2*q[14],
```

```
{q[1] -> (-1 + u)^(-1), q[2] -> u^(-1), q[3] -> (-1 + v)^(-1),
 q[4] -> v^(-1), q[5] -> (u - ub)^(-1), q[6] -> (-ub + v)^(-1),
 q[7] -> (-1 + ub)^(-1), q[8] -> ub^(-1), q[9] -> (-1 + u*ub)^(-1),
```

Energy Correlators: Four-Point

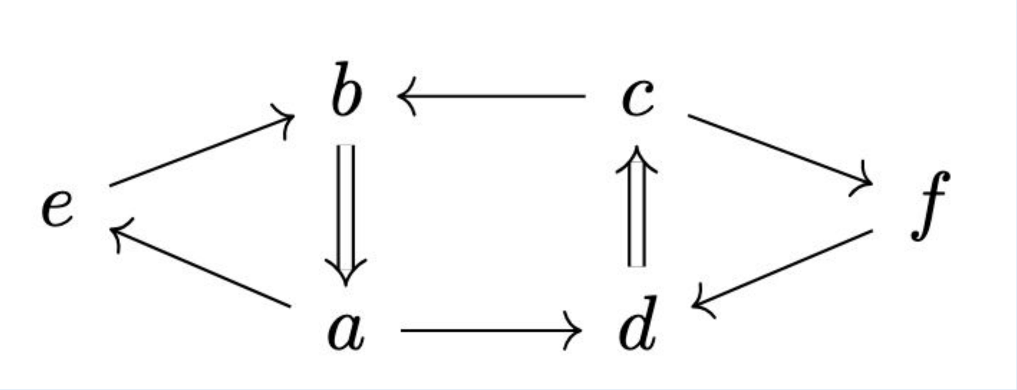
MPLs up to weight 3 with Symbol Alphabet that involves CUBIC roots

$$z,\ \bar{z},\ 1-z,\ 1-\bar{z},\ 1-|z|^2,\ z-\bar{z},\ w\bar{z}-\bar{w}z,$$
$$w\bar{z}-\bar{w},\ z\bar{w}-w,\ 1-w-\bar{w}+z\bar{w},\ 1-w-\bar{w}+w\bar{z}$$

$$|z|^2-|w|^2,\ z-|w|^2,\ \bar{z}-|w|^2,\ \frac{a}{b},\ \frac{a+|w|^2}{b+|w|^2},\ \frac{a+w\bar{b}+\bar{w}}{b+w\bar{a}+\bar{w}}$$

$$a\,b\,c=-|z|^2|w|^2,\quad a+b+c=1-w-\bar{w}-z-\bar{z}$$
$$a^{-1}+b^{-1}+c^{-1}=1-z^{-1}-\bar{z}^{-1}-w^{-1}-\bar{w}^{-1}.$$

Cube Roots and Cluster Algebra



Cluster Algebras, Cube Roots, and Energy Correlators in $\mathcal{N} = 4$ SYM

Dani Kaufman,[1] Marcus Spradlin,[2,3] Anastasia Volovich,[2] and Kai Yan[4,5]

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[2] *Department of Physics, Brown University, Providence, RI 02912, USA*
[3] *Brown Center for Theoretical Physics and Innovation, Brown University, Providence, RI 02912, USA*
[4] *School of Physics and Astronomy, Shanghai Jiao Tong University, Shanghai 200240, China*
[5] *Key Laboratory for Particle Astrophysics and Cosmology (MOE), Shanghai 200240, China*

We provide a cluster algebraic construction of the cube root symbol letters that appear at leading order in the near-collinear expansion of the four-point energy correlator in $\mathcal{N} = 4$ SYM theory.

2608.03717

Applications:

3. Cosmological correlators integrals

Mark’s talk

Conclusions

Cluster Algebras appear–for reasons that are still mysterious–in a growing number of amplitude related contexts:
Feynman integrals,
Cosmological correlator integrals,
Energy correlator integrals.

They enable us to make progress in previously untrackable computations.

They point towards richer structure yet to be understood and explored.

Thank you!