Graded Lie Groups with Examples

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Motivation

Lie groups

- Groups with a smooth structure, group operations smooth.
- Very well-understood smooth manifolds.
- Abstract nonsense: group objects in $\mathbf{Man}^{\infty}$.
- Essential for understanding symmetries in geometry and physics.

Example

Let V be a finite-dimensional real vector space.

1. Linear automorphisms of V form a **general linear group** $\mathrm{GL}(V)$.
2. If $g : V \times V \to \mathbb{R}$ is a metric (pseudo-scalar product), a set of A satisfying $(g^{-1}A^{T}g)A = \mathbb{1}_V$ forms the **orthogonal group** $\mathrm{O}(V, g)$.
3. If $\omega : V \times V \to \mathbb{R}$ is a symplectic form, one gets the **symplectic group** $\mathrm{Sp}(V, \omega)$ in the same way.

Main goal: We want these examples in $\mathbb{Z}$-graded geometry.

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Linear algebra

Definition

- A **graded vector space** is a sequence $V = (V_k)_{k \in \mathbb{Z}}$ of vector spaces. We write $v \in V$ and $|v| = k$, if $v \in V_k$ for some $k \in \mathbb{Z}$.
- A **graded linear map** $A : V \to W$ of **degree** $|A|$ is a sequence $A = (A_k)_{k \in \mathbb{Z}}$, where $A_k : V_k \to W_{k+|A|}$
- We say that V is finite-dimensional, if $\sum_{k \in \mathbb{Z}} V_k < \infty$.

- **gVect** - the category of *real finite-dimensional* graded vector spaces and *degree zero* graded linear maps.
- $\underline{\mathrm{Lin}}(V, W) \in$ **gVect** - *all* graded linear maps from V to W.
- We write $\mathfrak{gl}(V) := \underline{\mathrm{Lin}}(V, V)$.

Observation

$\mathfrak{gl}(V)$ together with the graded commutator

$$[A, B] := AB - (-1)^{|A||B|} BA$$

forms a graded Lie algebra (of degree 0).

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A **degree ℓ metric** on $V \in \mathbf{gVect}$ is bilinear $g : V \times V \to \mathbb{R}$, such that

1. $|g(v,w)| = \ell + |v| + |w|$;
2. $g(v,w) = (-1)^{(|v|+\ell)(|w|+\ell)} g(w,v)$;
3. the induced map $g : V \to V^*$ is an isomorphism.

$V^* := \underline{\mathrm{Lin}}(V, \mathbb{R})$ and $\mathbb{R}$ is viewed as a trivially graded GVS.

The involution

If g is a degree ℓ metric on V, we define $\tau : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ by

$$\tau(A) := (-1)^{|A|\ell} g^{-1} A^T g.$$

1. τ is graded linear of degree 0;
2. $\tau^2 = \mathbb{1}_{\mathfrak{gl}(V)}$ and it thus has eigenvalues ± 1;
3. Its eigenspace decomposition is $\mathfrak{gl}(V) = \mathrm{Sym}(V,g) \oplus \mathfrak{o}(V,g)$

Going from metric g to **symplectic** ω - add one minus in the definition and relabel $\mathfrak{o}(V,g)$ to $\mathfrak{sp}(V,\omega)$.

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Graded manifolds

Definition

Graded manifold $\mathcal{M}$ is a pair $(M, \mathcal{C}^\infty_{\mathcal{M}})$, where

1. M is a smooth manifold (underlying manifold, body of $\mathcal{M}$)
2. $\mathcal{C}^\infty_{\mathcal{M}}$ assigns to each $U \in \mathbf{Op}(M)$ a graded commutative associative algebra $\mathcal{C}^\infty_{\mathcal{M}}(U)$ of **functions on** $\mathcal{M}$ **over** U.
3. $\mathcal{C}^\infty_{\mathcal{M}}$ has to form a sheaf - this is not important.
4. Locally there is something happening - this is not important.

Definition

There is a notion of a **graded smooth map** $\varphi : \mathcal{M} \to \mathcal{N}$.

1. They can be associatively composed, there is the identity $\mathbb{1}_{\mathcal{M}}$;
2. There is an underlying smooth map $\underline{\varphi} : M \to N$.
3. Graded manifolds form a category $\mathbf{gMan}^\infty$.
4. There is a *body functor* $\mathfrak{B} : \mathbf{gMan}^\infty \to \mathbf{Man}^\infty$.

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Diamond functor

- $\mathcal{M}$ has a **graded dimension** $\text{gdim}(\mathcal{M}) = (n_k)_{k\in\mathbb{Z}}$, where n_k is a **number of coordinates of degree** k.
- $V \in \textbf{gVect}$ has a graded dimension $\text{gdim}(V) = (\dim(V_k))_{k\in\mathbb{Z}}$.
- For any sequence $(a_k)_{k\in\mathbb{Z}}$ write $\neg(a_k)_{k\in\mathbb{Z}} := (a_{-k})_{k\in\mathbb{Z}}$.

Proposition

1. *For any $V \in \textbf{gVect}$, there is $V_\diamond \in \textbf{gMan}^\infty$, such that*
$$\text{gdim}(V_\diamond) = \neg\,\text{gdim}(V).$$
2. *Underlying manifold is V_0 with the usual smooth structure.*
3. *To any $A : V \to W$ of degree 0, there is $A_\diamond : V_\diamond \to W_\diamond$.*
4. *We obtain a functor $\diamond : \textbf{gVect} \to \textbf{gMan}^\infty$.*

- To any basis $(t_\lambda)_{\lambda=1}^n$ of V there are coordinates $(\mathbb{z}^\lambda)_{\lambda=1}^n$ on $V_\diamond$. One has $|\mathbb{z}^\lambda| = -|t_\lambda|$. This explains the “flip”.

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Observation

$\mathbf{gMan}^\infty$ has products $\mathcal{M} \times \mathcal{N}$ and a terminal object $\{*\}$.

Definition

A **graded Lie group** is a group object $(\mathcal{G}, \mu, \iota, e)$ in $\mathbf{gMan}^\infty$, that is $\mathcal{G} \in \mathbf{gMan}^\infty$ and graded smooth maps

1. $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ (the multiplication)
2. $\iota : \mathcal{G} \to \mathcal{G}$ (the inverse)
3. $e : \{*\} \to \mathcal{G}$ (the unit)

Operations satisfy group axioms - formulated as commutative diagrams.

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To any graded Lie group $\mathcal{G}$, there is an **associated graded Lie algebra** $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$*, where $\mathfrak{g} \in$* **gVect** *is $T_e\mathcal{G}$.*

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Functor of points

Observation

By applying the functor $\mathfrak{B}$, see that $(G, \underline{\mu}, \underline{\iota}, \underline{e})$ is an ordinary Lie group.

- Let $\mathcal{G} \in \mathbf{gMan}^{\infty}$ be fixed.
- To each $\mathcal{S} \in \mathbf{gMan}^{\infty}$ assign a set $\mathfrak{P}(\mathcal{S}) = \mathbf{gMan}^{\infty}(\mathcal{S}, \mathcal{G})$.
- $\mathcal{S} \mapsto \mathfrak{P}(\mathcal{S})$ defines a **functor of points** $\mathfrak{P} : (\mathbf{gMan}^{\infty})^{\text{op}} \to \mathbf{Set}$
- Graded smooth maps μ, ι, e induce set maps
  1. $\mathbf{m}_{\mathcal{S}} : \mathfrak{P}(\mathcal{S}) \times \mathfrak{P}(\mathcal{S}) \to \mathfrak{P}(\mathcal{S})$;
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$\mathcal{G}$ is a graded Lie group, iff $(\mathfrak{P}(\mathcal{S}), \mathbf{m}_{\mathcal{S}}, \mathbf{i}_{\mathcal{S}}, \mathbf{e}_{\mathcal{S}})$ is an ordinary group (object in **Set***) for all $\mathcal{S} \in \mathbf{gMan}^{\infty}$.*

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General linear group

Let $V \in \mathbf{gVect}$. $\mathrm{GL}(V_\bullet) := \{A \in \mathfrak{gl}(V)_0 \mid A \text{ is invertible}\}$ is open.
Let $\mathrm{GL}(V) := \mathfrak{gl}(V)_\diamond|_{\mathrm{GL}(V_\bullet)}$, an open submanifold of $\mathfrak{gl}(V)_\diamond$.

1. A map $A \otimes B \mapsto AB$ defines a degree zero linear map

$$\beta : \mathfrak{gl}(V) \otimes_{\mathbb{R}} \mathfrak{gl}(V) \to \mathfrak{gl}(V).$$

One can apply the $\diamond$ functor to get a graded smooth map

$$\beta_\diamond : (\mathfrak{gl}(V) \otimes_{\mathbb{R}} \mathfrak{gl}(V))_\diamond \to \mathfrak{gl}(V)_\diamond$$

2. There is a canonical $\alpha_\diamond : \mathfrak{gl}(V)_\diamond \times \mathfrak{gl}(V)_\diamond \to (\mathfrak{gl}(V) \otimes_{\mathbb{R}} \mathfrak{gl}(V))_\diamond$
3. Let $\mu := \alpha_\diamond \circ \beta_\diamond$. It restricts to the appropriate open subsets, hence

$$\mu : \mathrm{GL}(V) \times \mathrm{GL}(V) \to \mathrm{GL}(V).$$

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- $\iota : \mathrm{GL}(V) \to \mathrm{GL}(V)$ can be constructed in coordinates.
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$\mathcal{S} \mapsto \mathfrak{F}(\mathcal{S})$ defines a functor naturally isomorphic to $\mathfrak{P}$.
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Main goal: for any metric g of degree ℓ, construct a graded Lie group $\mathrm{O}(V,g)$ and $j : \mathrm{O}(V,g) \to \mathrm{GL}(V)$.

1. j is a closed embedding and a morphism of GLG's.
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The construction closely follows the classical construction, albeit using maybe more abstract wording.

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$$\tau^\times : \mathrm{GL}(V) \to \mathrm{GL}(V).$$
This map is an **anti-automorphism** of $\mathrm{GL}(V)$. Classically this corresponds to $\tau(AB) = \tau(B)\tau(A)$.
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Classicaly $\varphi^{\times}(A) = \tau(A)A = (g^{-1}Ag)A$ is invertible and symmetric.

4. φ is transversal to $e^{\times} : \{*\} \to \mathrm{Sym}^{\times}(V,g)$ so we can form a pullback diagram

$$\begin{array}{ccc} \mathrm{O}(V,g) & \dashrightarrow & \{*\} \\ \downarrow j & & \downarrow e^{\times} \\ \mathrm{GL}(V) & \xrightarrow{\varphi} & \mathrm{Sym}^{\times}(V,g) \end{array}$$

Transversality means that $\mathbb{1}_V$ is a regular value of φ, we can thus form the level set submanifold

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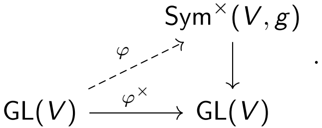

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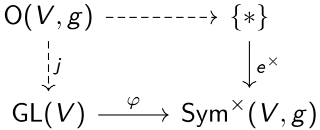

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   One has $\tau(AB)(AB) = \mathbb{1}_V$ and $\tau(A^{-1})A^{-1} = \mathbb{1}_V$ for any $A, B \in \mathrm{O}(V,g)$.

6. One has $T_{e'}(\mathrm{O}(V,g)) \cong \ker(T_e\varphi) \subseteq T_e(\mathrm{GL}(V))$. Under the identification $\mathfrak{gl}(V) \cong T_e(\mathrm{GL}(V))$, this corresponds to

$$\{A \in \mathfrak{gl}(V) \mid \tau(A) = -A\} \equiv \mathfrak{o}(V,g).$$

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Observation

By replacing g with a symplectic form ω, the whole construction works in the same way to give a **graded symplectic group** $\mathrm{Sp}(V,\omega)$ with a graded Lie algebra $\mathfrak{sp}(V,\omega)$.

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- For any $\mathcal{S} \in \mathbf{gMan}^{\infty}$, a metric g induces a $\mathcal{C}^{\infty}_{\mathcal{S}}(S)$-bilinear form

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It has degree ℓ and it is graded symmetric.

- We can thus consider a *subset* $\mathfrak{F}'(\mathcal{S}) \subseteq \mathrm{Aut}(\mathfrak{M}(\mathcal{S}))$ given by

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Isomorphisms

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$$\eta(A) := (-1)^{|M||A|} MAM^{-1}$$

is a degree 0 linear isomorphism $\eta : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$. $\eta_\diamond$ restricts to an isomorphism $\eta^\times : \mathrm{GL}(V) \to \mathrm{GL}(W)$.

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