Star-products for Lie-algebraic noncommutative Minkowski space-times

MARIS Valentine

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(based on 2503.07176 with F.Pozar and J-C.Wallet)

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Motivations

- Quantization of space time

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- Gelfand Naimark theorem:
  Manifold equivalent to commutative C^* algebra

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- Quantization of space time
- Geland Naimark theorem:
  Manifold equivalent to commutative C^* algebra
- Quantum version: Noncommutative C^* algebra

Definitions

- $\star$ Algebra :
  An algebra $(\mathcal{A}, +, \star)$ equipped with an involutive antihomomorpshim $\dagger : \mathcal{A} \to \mathcal{A}$
- Banach $\star$ algebra:
  Banach space $(\mathcal{A}, ||.||)$ such that
  - ✓ It is an $*$-algebra for a given product $\star$ and involution $\dagger$
  - ✓ Its norm satisfies $||fg|| \leq ||f|| \, ||g||, \quad \forall f, g \in \mathcal{B}$
  - ✓ $\dagger$ is an isometry for the norm: $||f^\dagger|| = ||f||$

A Banach $*$-algebra is called a $C\star$-algebra if

$$||ff^\dagger|| = ||f||^2, \quad \forall f \in \mathcal{A}$$

Context

- Deformed Minkowski space-time and Poincaré symmetry Hopf algebra
- Well known examples: κ/ρ-Minkowski [Lukierski, Ruegg, Nowicki, Tolstoy,1991],[Majid, Ruegg,1994][Dimitrijevic, Konjik,Samsarov,2018]

$$[x_0, x_j] = \frac{i}{\kappa} x_j, \quad [x_0, x_1] = \rho x_2, \ [x_0, x_2] = -\rho x_1$$

- Classification of deformed Poisson structures of the Poincaré group [Zakrzewski,1997]
- Deformed Minkowski space-time [Lukierski,Woronowicz,2005][Mercati,2024]
- Lie algebraic deformations:

$$[x^\mu, x^\nu] = i f_\rho^{\mu\nu} x^\rho \tag{1}$$

- Convolution algebra and Weyl quantization map [von Neumann,1931]

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Construction of $*$-algebra

- Start from Lie algebra $\mathfrak{g}$ with locally compact Le group $\mathcal{G}$.
- Derive its (right and left) Haar measures and its convolution algebra $\mathbb{C}(\mathcal{G}) = (L^1(\mathcal{G})(,\circ))$
- Study and characterize representations of the convolution algebra $\pi : \mathbb{C}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$
- Define quantization map, $Q := \pi \circ \mathcal{F}$, where $\mathcal{F}$ is the Fourier transform of the convolution algebra.
- Define star product $\star$ as $Q(f \star g) = Q(f)Q(g)$ and involution $Q(f)^* = Q(f^\dagger)$, where $*$ is involution of $\pi(\mathcal{B}(\mathcal{H})$

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Convolution algebra and Weyl quantization map (I)

- General structure of Lie groups :

$$\mathcal{G} = H \ltimes \mathbb{R}^3, \tag{2}$$

where $H \subset GL(n, \mathbb{R})$ one parameter abelian subgroup.

- $(a(p^M), \vec{p}) \in \mathcal{G} = H \ltimes \mathbb{R}^3 \Longrightarrow (p^M, \vec{p}) \in \mathbb{R} \ltimes \mathbb{R}^3$

The group laws of $\mathcal{G}$ are given by:

$$(a_1, \vec{p}_1)(a_2, \vec{p}_2) = (a_1 a_2,\ \vec{p}_1 + a_1 \vec{p}_2), \tag{3}$$

$$(a, \vec{p})^{-1} = (a^{-1}, -a^{-1}\vec{p}), \qquad \mathbb{I}_{\mathcal{G}} = (\mathbb{I}_H, 0) \tag{4}$$

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Convolution algebra and Weyl quantization map (II)

Right and left invariante Haar measures and modular function for $(a, \vec{p}) \in \mathcal{G}$:

$$d\nu_{\mathcal{G}}((a,\vec{p}) = \Delta_{\mathcal{G}}((a,\vec{p})^{-1}) d\mu_{\mathcal{G}}((a,\vec{p}) \qquad \begin{cases} d\mu_{\mathcal{G}}((a,\vec{p})) = d^3\vec{p}\,|\det(a)|^{-1}\, d\mu_H(a) \\ \Delta_{\mathcal{G}}((a,\vec{p})) = |\det(a)|^{-1} \underbrace{\Delta_H(a)}_{=1,\ H \text{ abelian}} \end{cases}$$

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- Expression of the convolution product and involution on $(L^1(\mathcal{G}, \circ, *)$:

$$(F \circ G)(s) = \int_{\mathcal{G}} d\nu_{\mathcal{G}}(t) F(st^{-1}) G(t), \quad t, s \in \mathcal{G}$$

$$F^*(s) = \overline{F}(s^{-1}) \Delta_{\mathcal{G}}(s)$$

This convolution algebra can be completed to a $C\star$ algebra, with $C\star$ norm $||.||_{C\star}$

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- Derivation of $\star$-product and involution :

$$(F \circ G)(s) = \int_{\mathcal{G}} d\nu_{\mathcal{G}}(t) F(st^{-1}) G(t) \quad \xrightarrow[\text{Inverse}]{\text{Fourier}} \quad f \star g = \mathcal{F}^{-1}(\mathcal{F}f \circ \mathcal{F}g)$$

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Explicit formulation of star product and involution

$$(f \star g)(x^M, \vec{x}) = \int \frac{dp^M dy^M}{(2\pi)} e^{-ip^M y^M} f(x^M + y^M, \vec{x}) g(x^M, a(p^M)x)$$
$$f^\dagger(x^M, \vec{x}) = \int \frac{dp^M dy^M}{(2\pi)} e^{-ip^M y^M} \overline{f(x^M + y^M, a(p^M)\vec{x})} \tag{5}$$

where $x = (x^M, \vec{x}) \in \mathbb{R} \times \mathbb{R}^3$

⇝ Construction of 10 such $\star$ product $\star$-algebras $\mathcal{M}_\lambda$

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Commutators

Commutators of lie algebra of coordinates $\mathfrak{g}$ reads:

$$\left[x^M, x^\mu\right]_\star = x^M \star x^\mu - x^\mu \star x^M = -i \left[\partial_{p^M} a(p^M)|_{p^M=0}\right]^\mu{}_\sigma x^\sigma \,, \qquad (6)$$

x^M can be a:

- timelike coordinate
- spacelike coordinate
- lightlike coordinate

Rescaling $p^M \to \lambda p^M$ and thus (6) become

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Nonunimodular group and twisted trace

$\mathcal{G}$ nonunimodular $\Longrightarrow \begin{cases} \Delta_{\mathcal{G}}(p^M) = e^{-n\lambda p^M} & \Delta_{\mathcal{G}}, \text{ modular function} \\ x^M = x^0 & x^M \text{ is timelike} \\ \mathrm{tr}(f \star g) = \mathrm{tr}((\sigma \triangleright g) \star f), & \mathrm{tr} := \int dx^\mu \quad \text{twisted trace} \end{cases}$

where the twist $\sigma \in \mathrm{Aut}(\mathcal{M}_\lambda)$ reads:

$$(\sigma \triangleright g)(x^0, \vec{x}) = (e^{in\lambda\partial_0} \triangleright g)(x^0, \vec{x}) = g(x^0 + in\lambda, \vec{x}) \quad (7)$$

note that such twist already appear in κ Minkowski [Poulain,Wallet 1994]

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KMS weight

With (7) we can define a one-parameter group of $\star$-automorphisms of $\mathcal{M}_\lambda$:

$$\{\sigma_t := e^{itn\lambda\partial_0}\}_{t\in\mathbb{R}}, \tag{8}$$

- $\{\sigma_t\}_{t\in\mathbb{R}}$ **modular group** for the **KMS weight** given by the twisted trace

A KMS weight ω on a $*$ algebra $\mathcal{A}$ for a modular group of $\star$-automorphism $\{\sigma_t\}_{t\in\mathbb{R}}$

- A weight $\omega : \mathcal{A} \to \mathbb{R}$ densely define and lower semi-continuous.
- $\{\sigma_t\}_{t\in\mathbb{R}}$ admit analytic extension $\{\sigma_z\}_{z\in\mathbb{C}}$
- It exist a function $F : \mathbb{C} \to \mathbb{C}$ holomorphic on the strip $\{0 \leq \mathcal{I}(z) \leq \beta\}_{z\in\mathbb{C}}$, $\beta > 0$ which satisfy:

$$F(t) = \omega(f\sigma_t(g)), \quad F(t+i\beta) = \omega(\sigma_t(g)f), \ \forall t \in \mathbb{R}$$

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- Why KMS weight ω?
- Equilibrium in thermodynamic $\Longrightarrow$ time flow on the algebra of observable $\alpha_t(A) = e^{iHt} A e^{-Ht}$ where A is an observable.
- $\mathbb{C}\star$ algebra: Tomita/modular flow α_t given by the Tomita-Takesaki theorem and acting on the GNS representation π_ω of the $\star$-algebra

  $$\alpha_s(\pi_\omega(f)) = \Delta_T^{-is} \pi_\omega(f) \Delta^{is}$$

  where Δ_T Tomita operator.
- $\alpha_s = -\sigma_{\beta t}$ and $\omega = \mathrm{tr}$
- Thermal times hypothesis: [Connes,Rovelli 1994]

  The physical time depends on the state. When the system is in a state ω, the physical time is given by the modular group σ_t of ω.
- Time translation $(\sigma_t \triangleright f)(x^0, \vec{x}) = f(x^0 + it\lambda t, \vec{x}), \quad f \in \mathcal{M}_\lambda$

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Conclusion and Outlook

- What we did:
  - ✓ Construction of several C^*-algebras
  - ✓ Investigation of nonunimodular groups and KMS weights. Derivation of a natural time flow.
- What we want to do:
  - ✍ Construction of associated Poincaré symmetry Hopf algebra (already done for half of them)
  - ✍ Implementation of gauge theory

Thank you for your attention

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