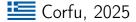
Frobenius Theorem for Graded Manifolds

Joint work with Jan Vysoký

Rudolf Šmolka





In this talk

► Vector Fields on Z-graded Manifolds

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- ▶ Distributions and Integral Submanifolds

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- ► Frobenius Theorem

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$$f=\sum_{\vec{p}}\,f_{\vec{p}}(x^{\mu})\,\xi^{\vec{p}},$$

where $f_{\vec{p}}$ are smooth functions on U, and

$$\xi^{\vec{p}} = (\xi^1)^{p_1} \cdots (\xi^{\hat{m}})^{p_{\hat{m}}}.$$

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- each term has the same degree. This is the degree $|f| \in \mathbb{Z}$ of the graded function $f \in C^{\infty}_{\mathcal{M}}(U)$.
- ► There is |fg| = |f| + |g|, $fg = (-1)^{|f||g|}gf$.

Vector Fields on ${\mathcal M}$

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. $X \in \mathscr{X}_{\mathcal{M}}(U)$, $X(f + \lambda g) = X(f) + \lambda X$

for $f,g \in C^{\infty}_{\mathcal{M}}(U)$, $U \subseteq M$ open.

 $X(fg) = X(f)g + (-1)^{|X||f|}fX(g),$

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- \triangleright Vector fields on $\mathcal M$ form a sheaf of $\mathcal C^\infty_{\mathcal M}$ -modules \mathscr{X}_{M} .
 - \triangleright $\mathscr{X}_{\mathcal{M}}$ is locally freely and finitely generated by coordinate vector fields.

 $X = X(x^{\mu}) \frac{\partial}{\partial x^{\mu}} + X(\xi^{a}) \frac{\partial}{\partial \xi^{a}}.$

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- $|\frac{\partial}{\partial x^{\mu}}| = 0$ and $|\frac{\partial}{\partial \xi^a}| = -|\xi^a|$.
- $\mathscr{X}_{\mathcal{M}}$ is the sheaf of sections of a graded vector bundle $T\mathcal{M}$.

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▶ Let $X \in \mathscr{X}_{\mathcal{M}}(M)$, then $X|_{p} \in T_{p}\mathcal{M}$ is given by

$$X|_{p}(f) = [X(f)](p).$$

Or in coordinates.

$$|X|_p = X^\mu(p) \frac{\partial}{\partial x^\mu}|_p + X^a(p) \frac{\partial}{\partial \xi^a}|_p.$$

Value of Graded Function

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In particular, $|f| \neq 0 \implies f(p) = 0$ for all $p \in M$.

Example: Graded Euler Vector Field

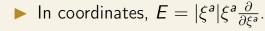
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- ▶ In coordinates, $E = |\xi^a| \xi^a \frac{\partial}{\partial \xi^a}$.
- ▶ Thus $E|_p = 0$ for all $p \in M$.

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 $\iota: \mathcal{S} \to \mathcal{M}$ is an **immersion** if $T_p\iota$ is injective for all $p \in \mathcal{S}$. The pair (\mathcal{S}, ι) is then called an **immersed submanifold** of \mathcal{M} .

Distributions and Integral Submanifolds

Distributions

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Let X_1, \ldots, X_r be a frame for $\mathcal{D}(U)$. Then $(r_j)_{j\in\mathbb{Z}}$ given by

$$r_j := \#\{k : |X_k| = j\},$$
 (1)

is the **rank** of \mathcal{D} .

A distribution \mathcal{D} is **involutive** if $\mathcal{D}(M)$ is closed under the commutator

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What is a good notion of an integral submanifold of \mathcal{D} ?

Integral Submanifold

An integral submanifold of \mathscr{D} is any immersed submanifold (\mathcal{S}, ι) of \mathcal{M} such that

$$(\mathrm{T}_p\iota)(\mathrm{T}_p\mathcal{S})=\mathscr{D}_{\underline{\iota}(p)},$$

for any $p \in S$.

Integral Submanifold

An integral submanifold of \mathscr{D} is any immersed submanifold (S, ι) of \mathcal{M} such that

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► This definition clearly generalizes the non-graded one. But there are other possibilities.

Tangent Vector Fields

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Let $\iota: \mathcal{S} \to \mathcal{M}$ be an immersed submanifold and consider $X \in \mathscr{X}_{\mathcal{M}}(M)$. We say that X is tangent to \mathcal{S} if there exists a vector field $Y \in \mathscr{X}_{\mathcal{S}}(S)$ such that

$$Y \sim_{\iota} X$$
,

i.e. $Y \circ \iota^* = \iota^* \circ X$.

Strongly Integral Submanifold

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of degree 1 and η of degree 3 on \mathcal{M} . \blacktriangleright Any $\iota : \mathcal{S} \to \mathcal{M}$ is fully given by

$$\iota^*(\xi^a) = A^a{}_b \, \theta^b, \qquad \iota^*(\eta) = B \, \theta^1 \theta^2 \theta^3.$$

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▶
$$(T_{\star}\iota)(\frac{\partial}{\partial \theta^{a}}|_{\star}) = A^{a}{}_{b}\frac{\partial}{\partial \xi^{a}}|_{\star}$$
, hence ι is an immersion $\iff A^{a}{}_{b}$ is invertible $\iff (\mathcal{S}, \iota)$ is integral.

ightharpoonup For B=0 there is

$$\left((A^{-1})^a{}_b \frac{\partial}{\partial \theta^a} \right) \sim_\iota \frac{\partial}{\partial \xi^b},$$

so (S, ι) is strongly integral.

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For
$$B \neq 0$$
, consider
$$f := \xi^1 \xi^2 \xi^3 - \frac{\det(A)}{R} \eta.$$

Then $\iota^*(f) = 0$, but $(\iota^* \circ \frac{\partial}{\partial \xi^1})(f) = A^2{}_i A^3{}_j \theta^i \theta^j \neq 0$, hence (\mathcal{S}, ι) is not strongly integral.

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Integrable Distribution

A distribution \mathscr{D} on \mathcal{M} is called integrable if every point $p \in M$ has a neighborhood $p \in U$ such that there exist coordinates x^{μ}, ξ^{a} for \mathcal{M} in which $\mathscr{D}(U)$ is spanned by $\frac{\partial}{\partial x^{\mu}}$ and $\frac{\partial}{\partial \xi^{a}}$ for $\mu \leq r_{0}$ and $a \leq \hat{r}$.

A distribution is involutive if and only if it is integrable.

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 - \mathbb{Z}_2^n -manifolds (Covolo, Kwok & Poncin 2016 [1]).
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 - For Z-manifolds it seems to work as well.

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Similarly Immersed Submanifolds

Let $\mathcal{S}=(\mathcal{S},\iota)$ and $\tilde{\mathcal{S}}=(\tilde{\mathcal{S}},\tilde{\iota})$ be two immersed submanifolds of \mathcal{M} . We say that they are **similarly immersed** if $\mathcal{S}=\tilde{\mathcal{S}}$ as smooth manifolds, and there exists a graded diffeomorphism $\vartheta:\mathcal{S}\to\tilde{\mathcal{S}}$ such that $\underline{\vartheta}=\mathrm{id}_{\mathcal{S}}$ and $\iota=\tilde{\iota}\circ\vartheta$.

Global Frobenius Theorem

Let \mathscr{D} be an involutive distribution on a \mathbb{Z} -manifold \mathcal{M} . Then every point $p \in M$ is contained within a unique (upto similar immersion) strongly integral submanifold of \mathscr{D}

- Covolo, Kwok, Poncin. (2016) Frobenius theorem for \mathbb{Z}_2^n -manifolds
 - Bursztyn, Cueca, Mehta. (2025) A geometric characterization of N-graded manifolds and the Frobenius theorem. J. Noncommut. Geom.
- Vysoký, J. (2022) Global theory of graded manifolds. Reviews in Mathematical Physics.

Thank You.

Acknowledgements

The author is grateful for the support of the grant GAČR 24-10031K and SGS25/163/OHK4/3T/14.