Frobenius Theorem for Graded Manifolds

Joint work with Jan Vysoký

Rudolf Šmolka

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In this talk

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- Vector Fields on $\mathbb{Z}$-graded Manifolds

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- Distributions and Integral Submanifolds

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Vector Fields on $\mathbb{Z}$-graded Manifolds

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$$f = \sum_{\vec{p}} f_{\vec{p}}(x^\mu)\, \xi^{\vec{p}},$$

where $f_{\vec{p}}$ are smooth functions on U, and

$$\xi^{\vec{p}} = (\xi^1)^{p_1} \cdots (\xi^{\hat{m}})^{p_{\hat{m}}}.$$

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- There is $|fg| = |f| + |g|$, $fg = (-1)^{|f||g|} gf$.

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Derivations of $C^\infty_\mathcal{M}$. $X \in \mathscr{X}_\mathcal{M}(U)$,

$$X(f + \lambda g) = X(f) + \lambda X(g),$$
$$X(fg) = X(f)g + (-1)^{|X||f|} f\, X(g),$$

for $\quad f, g \in C^\infty_\mathcal{M}(U)$, $U \subseteq M$ open.

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- $\mathscr{X}_{\mathcal{M}}$ is the sheaf of sections of a graded vector bundle $T\mathcal{M}$.

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Or in coordinates,

$$X|_p = X^\mu(p)\frac{\partial}{\partial x^\mu}|_p + X^a(p)\frac{\partial}{\partial \xi^a}|_p.$$

Value of Graded Function

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- In particular, $|f| \neq 0 \implies f(p) = 0$ for all $p \in M$.

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- In coordinates, $E = |\xi^a|\xi^a \frac{\partial}{\partial \xi^a}$.
- Thus $E|_p = 0$ for all $p \in M$.

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- $\mathrm{T}_p\varphi : \mathrm{T}_p\mathcal{M} \to \mathrm{T}_{\underline{\varphi}(p)}\mathcal{N}$ is given by

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- $\iota : \mathcal{S} \to \mathcal{M}$ is an **immersion** if $\mathrm{T}_p\iota$ is injective for all $p \in S$. The pair $(\mathcal{S}, \iota)$ is then called an **immersed submanifold** of $\mathcal{M}$.

Distributions and Integral Submanifolds

Distributions

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- Let $X_1, \ldots, X_r$ be a frame for $\mathscr{D}(U)$. Then $(r_j)_{j \in \mathbb{Z}}$ given by

$$r_j := \#\{k : |X_k| = j\}, \tag{1}$$

is the **rank** of $\mathscr{D}$.

- A distribution $\mathscr{D}$ is **involutive** if $\mathscr{D}(M)$ is closed under the commutator

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- What is a good notion of an integral submanifold of $\mathscr{D}$?

Integral Submanifold

An integral submanifold of $\mathscr{D}$ is any immersed submanifold $(\mathcal{S}, \iota)$ of $\mathcal{M}$ such that

$$(\mathrm{T}_p \iota)(\mathrm{T}_p \mathcal{S}) = \mathscr{D}_{\underline{\iota}(p)},$$

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- This definition clearly generalizes the non-graded one. But there are other possibilities.

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Tangent Vector Fields

Let $\iota : \mathcal{S} \to \mathcal{M}$ be an immersed submanifold and consider $X \in \mathscr{X}_{\mathcal{M}}(M)$. We say that X **is tangent to** $\mathcal{S}$ if there exists a vector field $Y \in \mathscr{X}_{\mathcal{S}}(S)$ such that

$$Y \sim_{\iota} X,$$

i.e. $Y \circ \iota^* = \iota^* \circ X$.

Strongly Integral Submanifold

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- Not every integral submanifold is strongly integral.

Example

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- Any $\iota : \mathcal{S} \to \mathcal{M}$ is fully given by

$$\iota^*(\xi^a) = A^a{}_b\, \theta^b, \qquad \iota^*(\eta) = B\, \theta^1\theta^2\theta^3.$$

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- $(T_\star\iota)(\frac{\partial}{\partial\theta^a}|_\star) = A^a{}_b \frac{\partial}{\partial\xi^a}|_\star$, hence ι is an immersion $\iff$ $A^a{}_b$ is invertible $\iff$ $(\mathcal{S}, \iota)$ is integral.

- For $B = 0$ there is

$$\left((A^{-1})^a{}_b \frac{\partial}{\partial \theta^a}\right) \sim_\iota \frac{\partial}{\partial \xi^b},$$

so $(\mathcal{S}, \iota)$ **is strongly integral**.

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- For $B \neq 0$, consider

$$f := \xi^1 \xi^2 \xi^3 - \frac{\det(A)}{B} \eta.$$

  Then $\iota^*(f) = 0$, but $(\iota^* \circ \frac{\partial}{\partial \xi^1})(f) = A^2{}_i A^3{}_j \theta^i \theta^j \neq 0$, hence $(\mathcal{S}, \iota)$ **is not strongly integral**.

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Integrable Distribution

A distribution $\mathscr{D}$ on $\mathcal{M}$ is called integrable if every point $p \in M$ has a neighborhood $p \in U$ such that there exist coordinates x^μ, ξ^a for $\mathcal{M}$ in which $\mathscr{D}(U)$ is spanned by $\frac{\partial}{\partial x^\mu}$ and $\frac{\partial}{\partial \xi^a}$ for $\mu \leq r_0$ and $a \leq \hat{r}$.

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  - $\mathbb{Z}_2^n$-manifolds (Covolo, Kwok & Poncin 2016 [1]).
  - $\mathbb{N}$-manifolds (Bursztyn, Cueca & Mehta 2025 [2]).
  - For $\mathbb{Z}$-manifolds it seems to work as well.

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Similarly Immersed Submanifolds

Let $\mathcal{S} = (S, \iota)$ and $\tilde{\mathcal{S}} = (\tilde{S}, \tilde{\iota})$ be two immersed submanifolds of $\mathcal{M}$. We say that they are **similarly immersed** if $S = \tilde{S}$ as smooth manifolds, and there exists a graded diffeomorphism $\vartheta : \mathcal{S} \to \tilde{\mathcal{S}}$ such that $\underline{\vartheta} = \mathrm{id}_S$ and $\iota = \tilde{\iota} \circ \vartheta$.

Global Frobenius Theorem

Let $\mathscr{D}$ be an involutive distribution on a $\mathbb{Z}$-manifold $\mathcal{M}$. Then every point $p \in M$ is contained within a unique (upto similar immersion) strongly integral submanifold of $\mathscr{D}$.

- Covolo, Kwok, Poncin. (2016) Frobenius theorem for $\mathbb{Z}_2^n$-manifolds
- Bursztyn, Cueca, Mehta. (2025) A geometric characterization of N-graded manifolds and the Frobenius theorem. J. Noncommut. Geom.
- Vysoký, J. (2022) Global theory of graded manifolds. *Reviews in Mathematical Physics.*

Thank You.

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