AN INDEX FOR FLUX VACUA

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[hep-th/251x.xxxxx] (with P. Balavoine)

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EXISTENCE OF REALISTIC STRING VACUA

- Main problem for String Phenomenology:

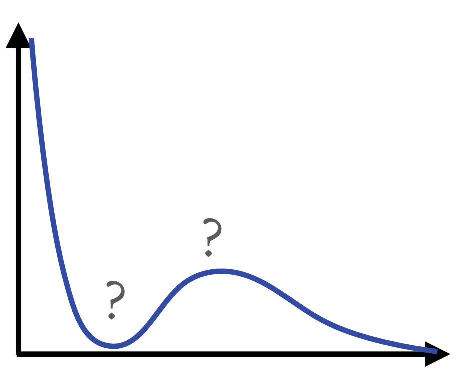
Determine the existence (and properties)
of realistic string theory vacua / backgrounds.

(labeled, e.g., by a choice of compactification space, branes, fluxes, …)

Large body of literature on the existence of *DGKT, KKLT, LVS, de Sitter vacua, scale separation, non-susy AdS,* $\mathcal{N} = 1$ *Minkowski, …*

- Lower-dimensional / EFT perspective:

Minima (or critical points) of a scalar potential
over a scalar field (moduli) space

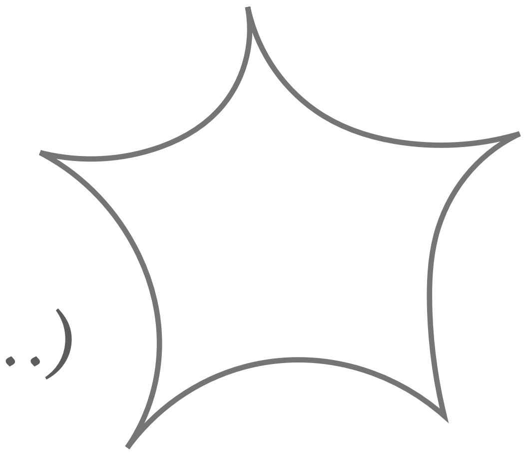


MODULI SPACES AND SUPERSYMMETRY

- Extended ($\mathcal{N} \geq 2$) supersymmetry:

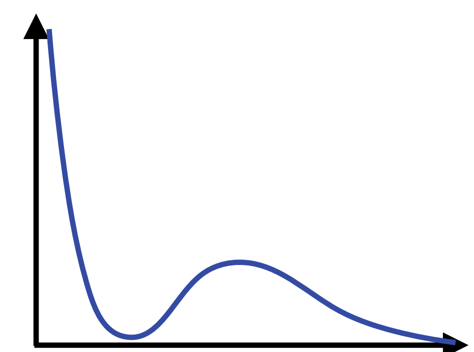
  exact moduli space

  (e.g. dilaton, geometric moduli of compactification, …)



- no or little ($\mathcal{N} \leq 1$) supersymmetry:

  Quantum effects:
  generate a potential $V(\phi)$ for moduli



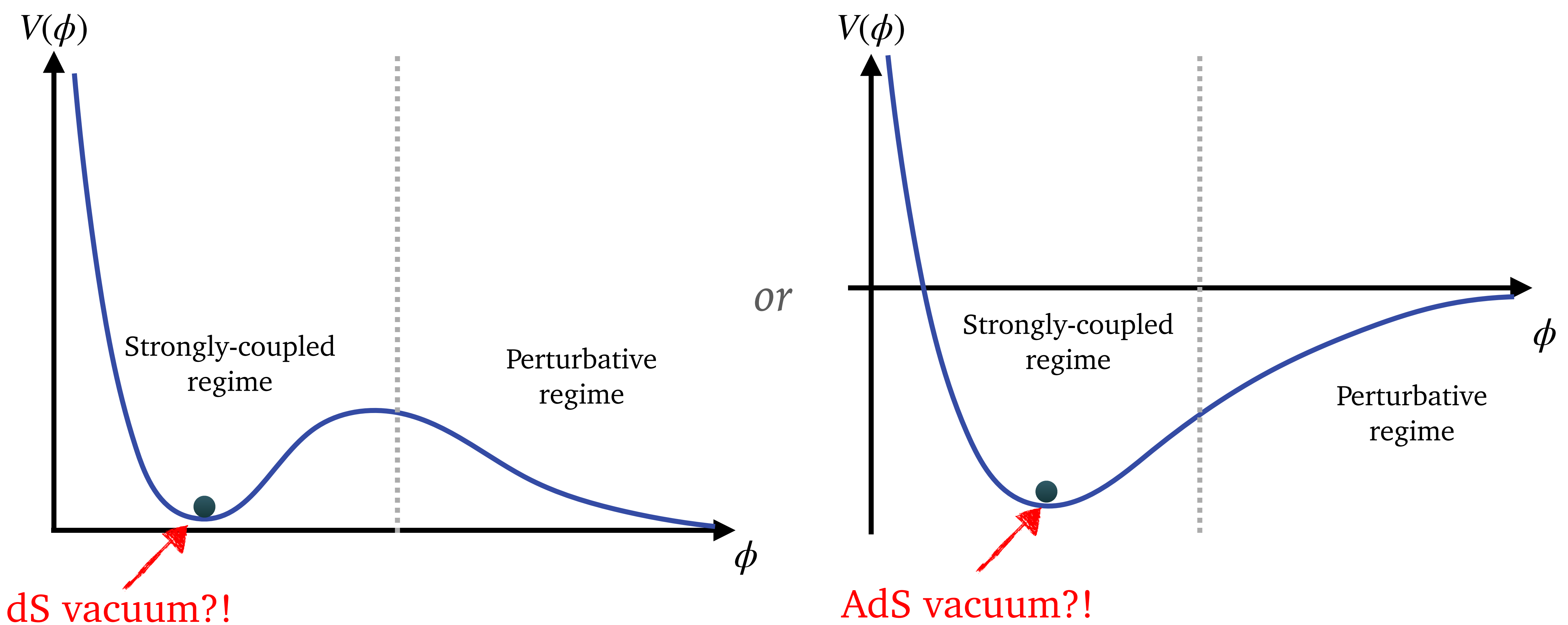
- Dine-Seiberg Problem: [Dine, Seiberg '85]
  - Quantum corrections: series expansions around asymptotic boundaries at $\phi \to \infty$
  - critical points: only in the interior / strong coupling?

DINE-SEIBERG PROBLEM

[Dine, Seiberg '85]

run-away unless one takes higher order corrections into account:

$$\lim_{\phi \to \infty} V = 0$$



at minimum of V: higher order corrections ≈ first order corrections → *strong coupling / interior!*

DE SITTER NO-GO

- Possible solution to the Dine-Seiberg problem:

  Generate a potential at the classical / tree-level → *Fluxes!*

- However: [Maldacena, Nuñez ’00] and many others:

  no smooth de Sitter compactification in two-derivative supergravity

- de Sitter vacua from String Theory must involve:

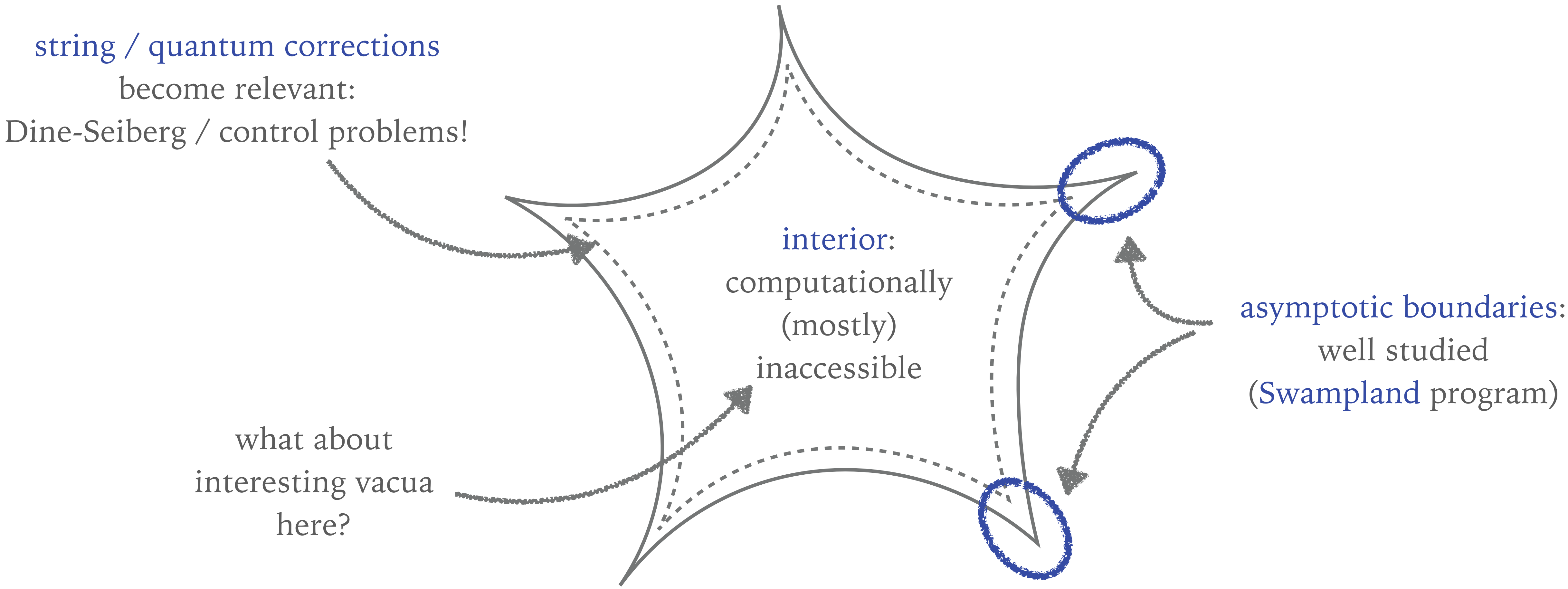
  a) quantum effects
  or
  b) stringy ingredients
  (higher-derivative terms, O-planes, …)

  → Danger of Dine-Seiberg like control issues!

→ long standing debate on consistency and control of de Sitter (and even N=1!) solutions (KKLT, LVS, DGKT, …)

ABSTRACT VIEW ON THE PROBLEM:

cartoon of the moduli space:



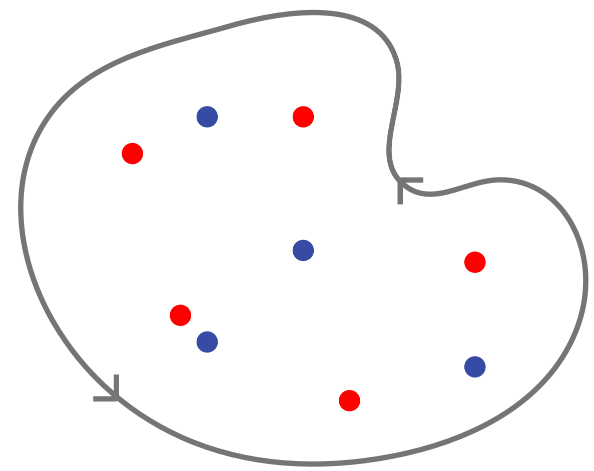
Goal / Idea:

boundary behavior of $V(\phi)$ → information on string vacua in the interior of moduli space?

NON-HOLOMORPHIC CONTOUR INTEGRALS

CAUCHY'S ARGUMENT PRINCIPLE

- f: $\mathbb{C} \to \mathbb{C}$ meromorphic function, C closed contour



- count zeros and poles inside C by contour integral:

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = \#\, zeros - \#\, poles$$

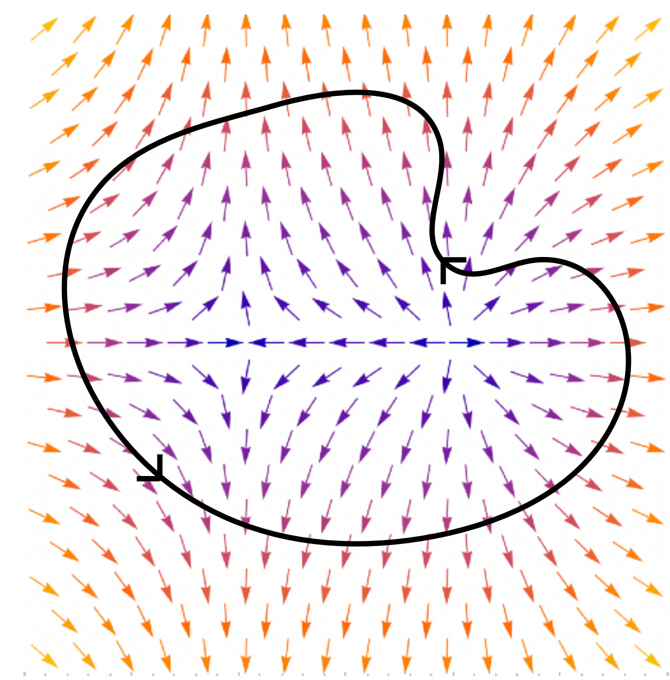
(including multiplicities)

- what about non-holomorphic / real functions?

INDEX FOR REAL VECTOR FIELDS

- $\vec{X} = (X^1, X^2)$ vector field on $U \subseteq \mathbb{R}^2$
  (differentiable; finitely many, isolated zeros / poles)
- C closed curve in U
  ($\vec{X}$ non-zero and well-defined on C)
- Define the index:

$$I_C = \frac{1}{2\pi} \oint_C \frac{\epsilon_{ij} X^i dX^j}{\left\| \vec{X} \right\|^2}$$



- Relation to complex functions:

$$z = x^1 + ix^2$$
$$f(z, \bar{z}) = X^1 + iX^2$$

$$\Rightarrow \quad I_C = \frac{1}{2\pi i} \oint_C \frac{\partial_z f\, dz + \partial_{\bar{z}} f\, d\bar{z}}{f(z)}$$

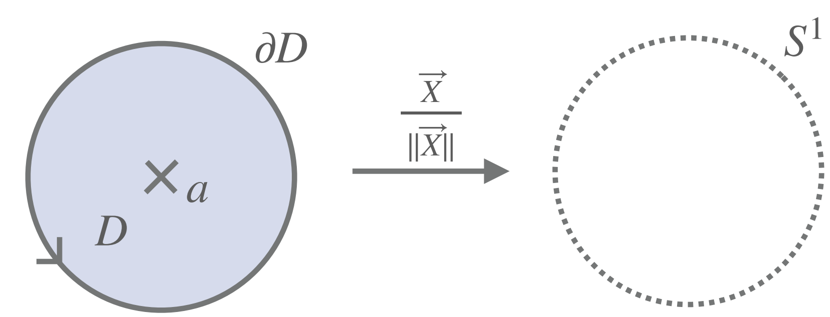
INDEX FOR REAL VECTOR FIELDS

- Counting of zeros / poles: *(see also Poincaré-Hopf theorem)*

$$I_C = \sum_i ind_{x_i}(\vec{X}) \in \mathbb{Z}$$

x_i: zeros / poles of $\vec{X}$ encircled by C

- Index of a point:
  - $D \subset U$ small disc with center a (no other zeros / poles within D)



  - $\mathrm{ind}_a(\vec{X})$: degree (= winding number) of the map $\frac{\vec{X}}{\|\vec{X}\|} : \partial D \to S^1$

- Remark: Does not distinguish between zeros and poles!

- Higher-dimensional generalisation:

$$I_C = \frac{1}{\mathrm{vol}(S^n)} \oint \frac{\epsilon_{i_1 \dots i_n} X^{i_1}\, \mathrm{d}X^{i_2} \wedge \dots \wedge \mathrm{d}X^{i_n}}{\| \vec{X} \|^n} = \sum_i ind_{x_i}(\vec{X})$$

INDEX FOR F-TERM MINIMA

[SL '24]

- Idea: apply to the gradient of the potential, $\vec{X} = \vec{\nabla} V$
  but: critical points can be minima / maxima / saddle points
- better: N=1 SUGRA F-term potential:

W: superpotential

$$V = e^K \left(g^{i\bar{j}} D_i W \overline{D_{\bar{j}} W} - 3|W|^2 \right)$$

$$DW = 0 \quad \Leftrightarrow$$ supersymmetric minimum (stable in the BF sense) of V

- Compute index I_C for $X = DW$ *(as real vector field)*

(if DW is non-singular)

$$I_C = \sum_i ind_{x_i}(DW) \neq 0 \longrightarrow DW = 0 \longrightarrow$$

(somewhere in the interior of the moduli space)

sufficient condition for susy minimum of V

APPLICATION: FLUX VACUA

CLASSICAL FLUX VACUA OF M-THEORY

- M-theory on Calabi-Yau 4-fold $M_{11} = M_4 \times CY_4$ [Becker, Becker '96]
- Flux compactification:
  (completely analogous to IIB on 3-fold & related via F-theory)

  $h^{3,1}$ complex str. moduli (volumes of 4-cycles) → stabilized by $G_4 \neq 0$ (4-form flux)

- Superpotential: $W = \int_{CY_4} G_4 \wedge \Omega$ [Gukov, Vafa, Witten '99]
- F-term condition: $D_i W = 0 \quad \Leftrightarrow \quad \star G_4 = G_4$

  complex structure moduli (→ D_i)

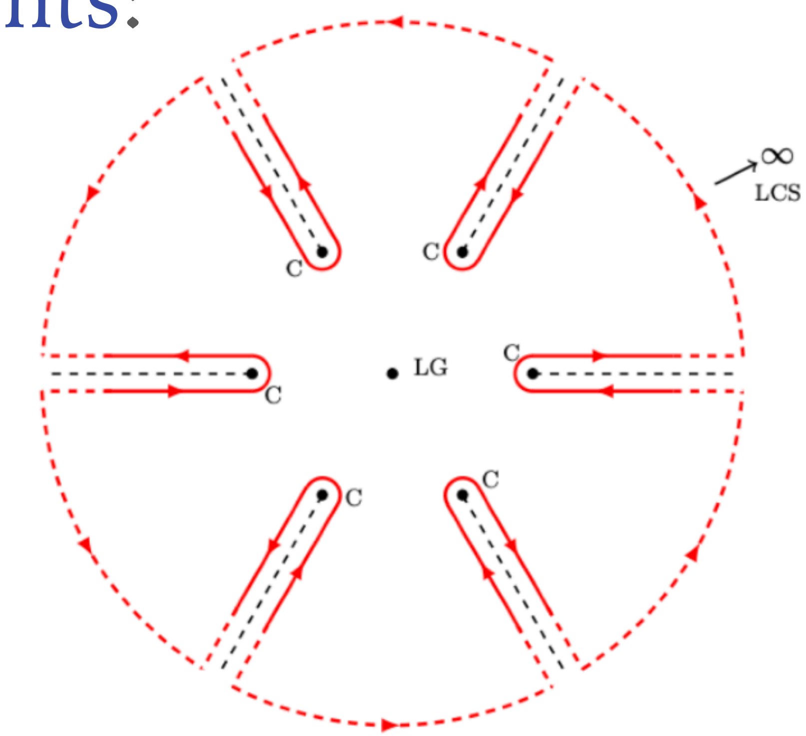
  → classical stabilization of c.s. moduli

(*) $D_i W = 0$ does not include Kähler moduli,
and does not yet guarantee vacuum of the full quantum theory
(see, e.g., [Kachru, Kallosh, Linde, Trivedi '03], [Balasubramanian, Berglund, Conlon, Quevedo '05])

COMPUTATION OF THE INDEX

[SL '24]

- Example: Flux vacua on the Fermat mirror sextic CY 4-fold
- Moduli space geometry: Period integrals: $\Pi^I = \int_{A^I} \Omega$
  [Grimm, Ha, Klemm, Klevers '09][van de Heisteeg '24]
- There are 3 critical locii / degeneration points:
  - $\psi = \infty$: large complex structure point
  - $\psi^6 = 1$: conifold point
  - $\psi = 0$: Landau-Ginzburg / Fermat point



- Periods $\Pi(\psi)$ exhibit monodromies around these points ⟶ branch cuts
- Integration contour for index: encircle all singular points and branch cuts

$$I(g^I) = \sum_{i=1}^{6} \left(I_i^{b.c.} + I_i^C + I_i^{LCS} \right)$$

BOUNDARY CONTRIBUTIONS

[SL '24]

- Large complex structure:
  - Expand periods in $t = x + iy \sim \log \psi$ for large y:

$$D_t W = -\frac{i}{2} g^1 y^3 + \left(g^2 - x g^1\right) y^2 - \left(x^3 + \frac{15x}{4} - \frac{525\zeta(3)i}{16\pi^3}\right) g^1 + \left(3x^2 - 3x + \frac{3}{4}\right) g^2 + (6x - 3)\, g^3 - g^4 + \mathcal{O}\left(y^{-1}\right)$$

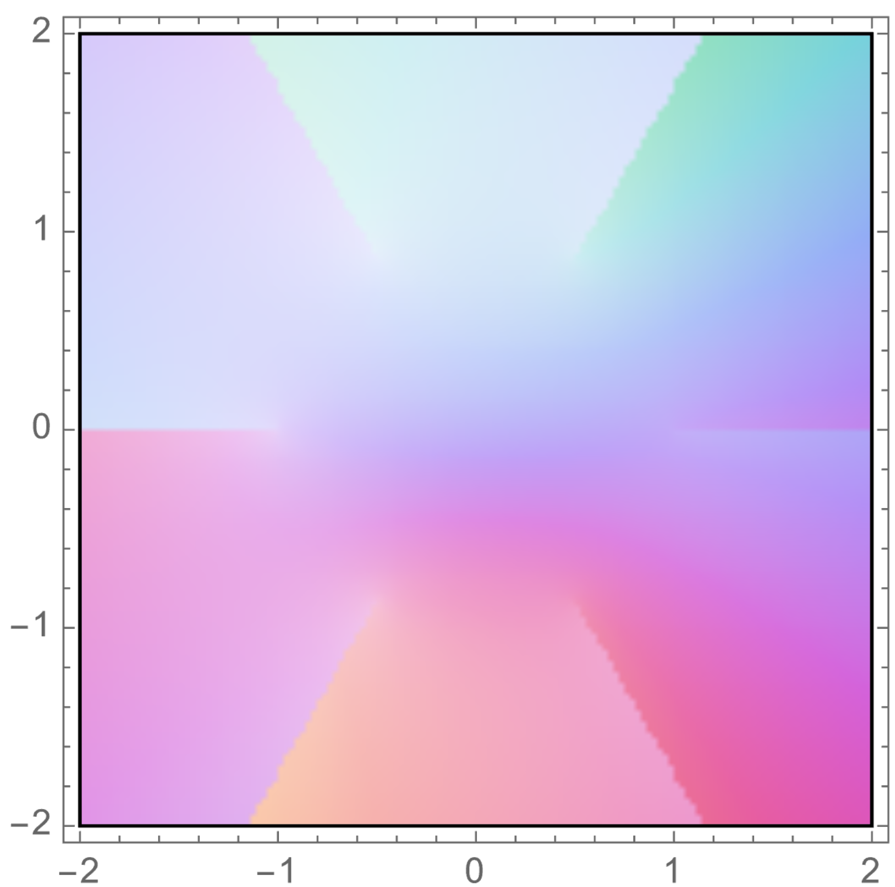
  - LCS contour: $x \in (0,1)$ at $y \to \infty$
  - Using $D_\psi W \sim \psi^{-1} D_t W$: $I_0^{\mathrm{LCS}} = -\frac{1}{3}$ unless $g^1 = g^2 = 0$
- conifold:
  - Similar procedure: $I_0^{\mathrm{C}} = 0$ unless $g^I \sim (1,0,0,0,1)$
- branch cuts:
  - $I_0^{b.c.}$ probes interior of moduli space

    ➜ requires evaluation of full periods

EXAMPLES

[SL '24]

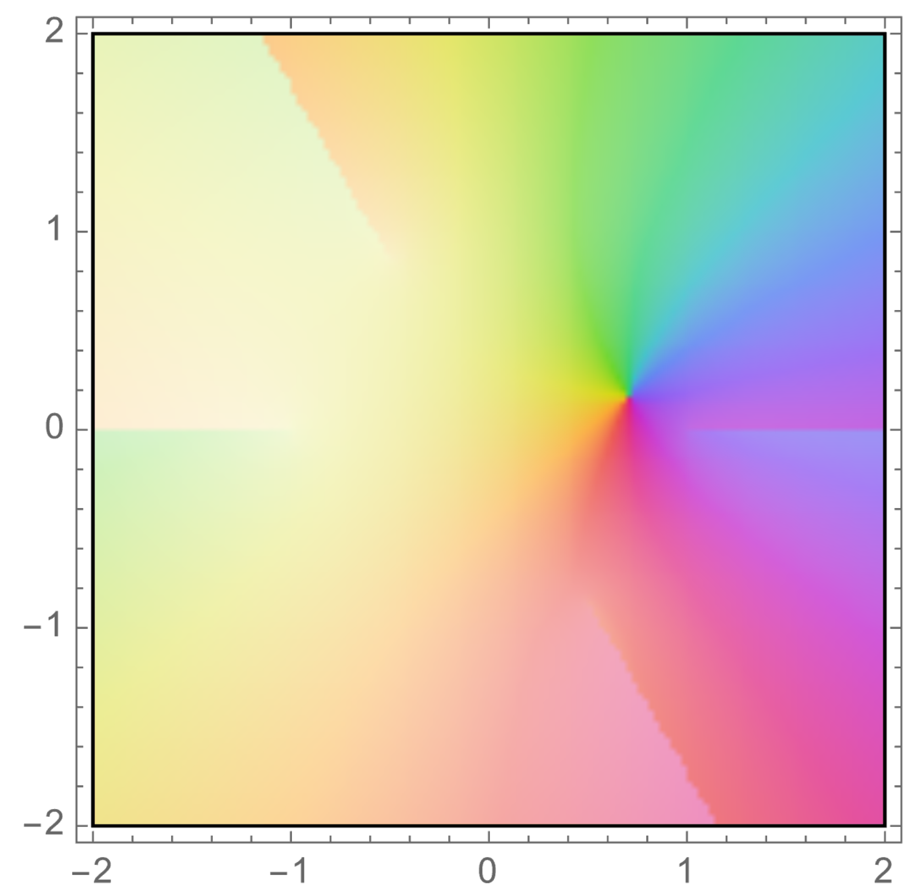
$g_a = (1,2,1,1,0)$



$I(g_a) = 0$

→ *is consistent with the absence of flux vacua!*

$g_b = (1,2,2,1,0)$



$I(g_b) = -1$

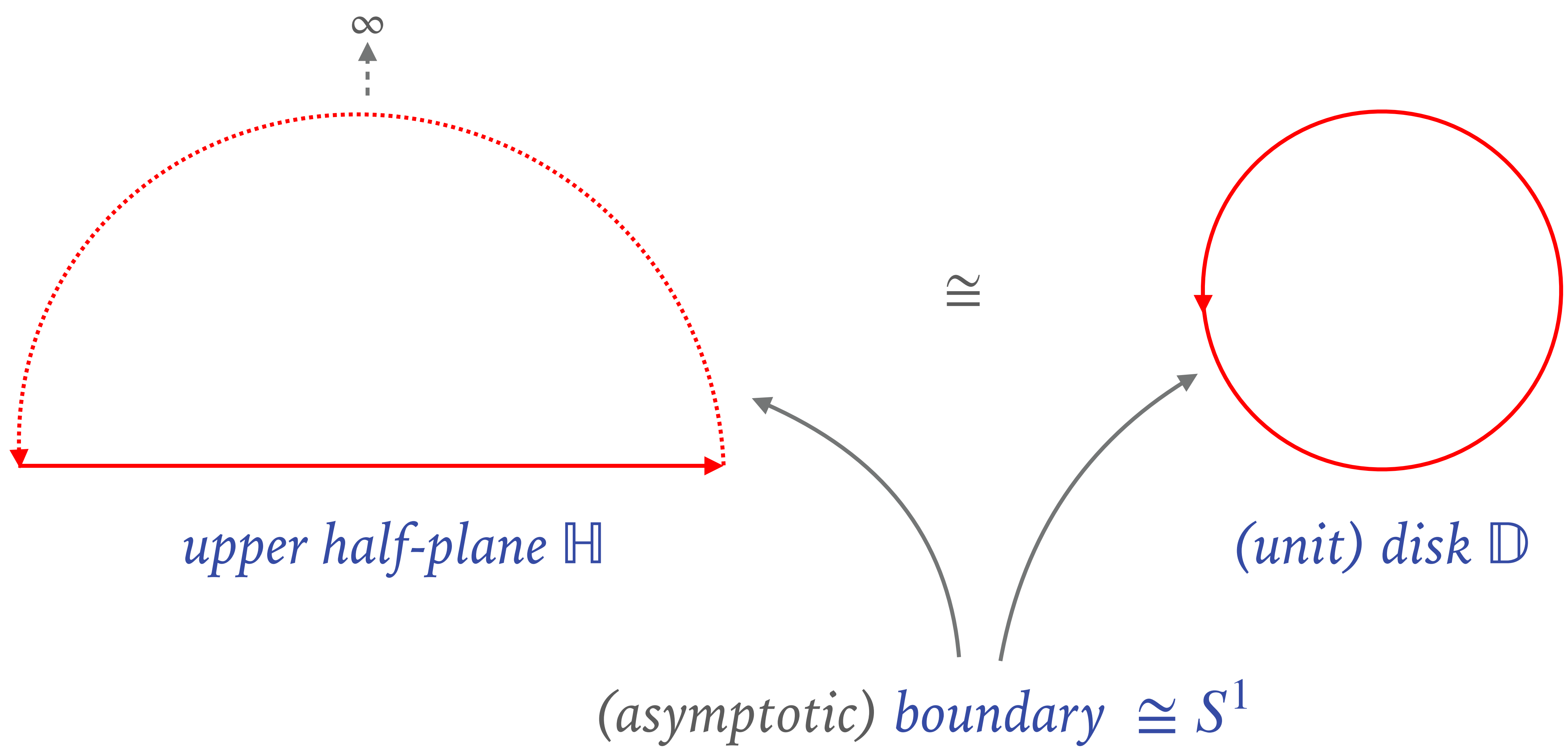
→ *predicts flux vacuum in the interior of moduli space*

THE BOUNDARY OF THE COVERING SPACE

THE COVERING SPACE

[P. Balavoine, SL, *to appear*]

- **Problem:** branch cuts probe the interior of the moduli space!
- **Solution:** work on the (universal) covering space.
  (c.f. marked moduli space [Raman, Vafa ’24][Delgado, van de Heisteeg, Raman, Torres, Vafa, Xu ’24])
- For one-parameter CY manifolds (e.g., mirror sextic):



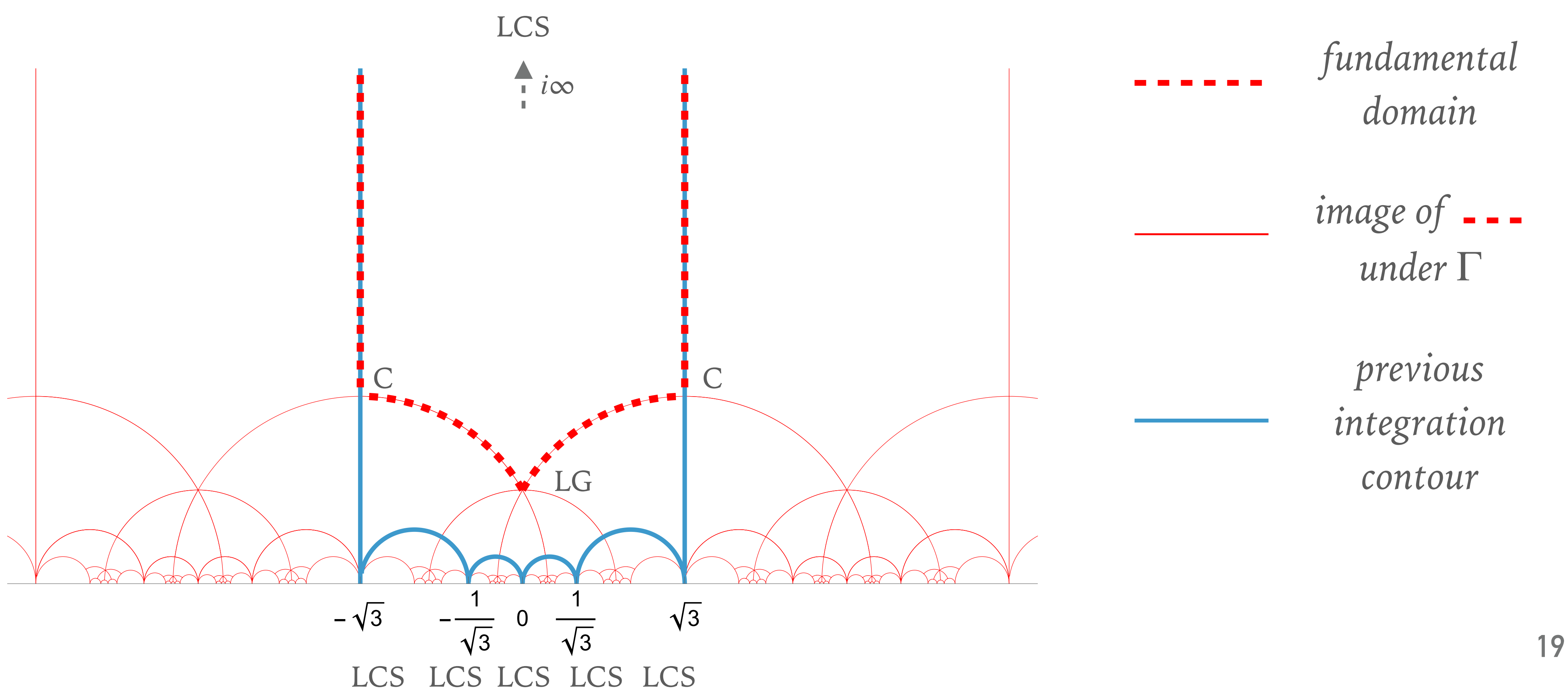
ACTION OF THE MONODROMY GROUP:

[P. Balavoine, SL, *to appear*]

- Identify the monodromy group Γ as a subgroup of $PSL(2,\mathbb{R})$:
- Mirror sextic: (see [Candelas, de la Ossa, Green, Parkes '91] for mirror quintic)

$$\gamma_{LCS} = \begin{pmatrix} 1 & 2\sqrt{3} \\ 0 & 1 \end{pmatrix} \qquad \gamma_{LG} = \frac{1}{2}\begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix} \qquad \gamma_{C} = \gamma_{LG}\cdot\gamma_{LCS} = \frac{1}{2}\begin{pmatrix} \sqrt{3} & 7 \\ -1 & -\sqrt{3} \end{pmatrix}$$

- Action on the covering space and fundamental domain:

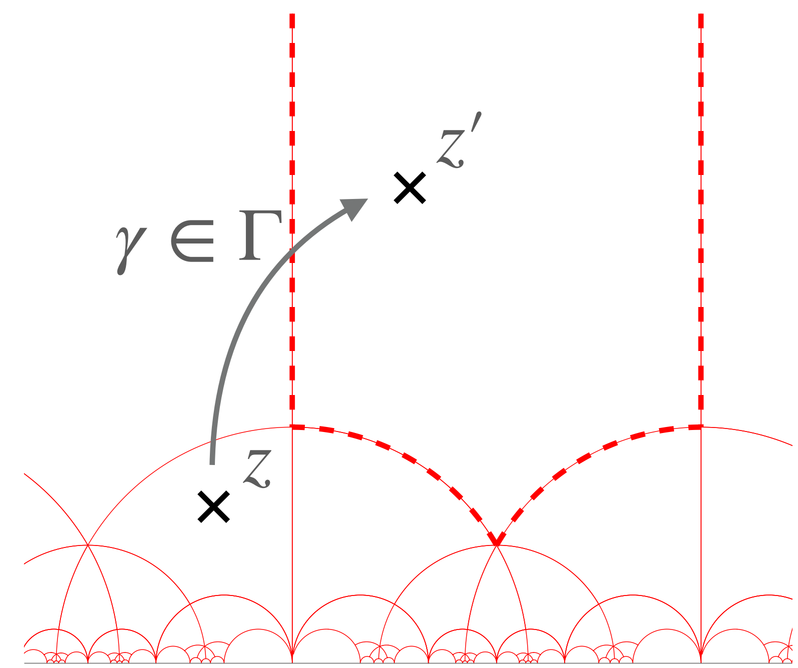


PERIODS ON THE COVERING SPACE

[P. Balavoine, SL, *to appear*]

- Monodromy group Γ: map periods from fundamental region onto full upper half-plane:

$$\Pi^i(z) = M(\gamma)^i{}_j \Pi^j(z')$$



- Problem: Periods diverge at LCS

$$\textit{for} \quad \frac{\log \psi}{2\pi i} \sim t \to i\infty : \qquad \Pi^n \sim t^n \quad (n = 0,\ldots,4)$$

- Normalized periods:

$$\overline{\Pi}^n \equiv \frac{\Pi^n}{\Pi^0}$$

→

- meromorphic on (open) upper half-plane
- continuous on closed upper half-plane (including boundary / real axis!)

THE INDEX ON THE COVERING SPACE

[P. Balavoine, SL, *to appear*]

- Γ: Fuchsian group of first kind:

  maps LCS point at $i\infty$ densely onto the real line

- Periods on the boundary $\partial\mathbb{H} \cong S^1$:

  defined purely algebraically in terms of:

  a) monodromy group Γ

  b) asymptotic periods at LCS ($\Pi^n \sim t^n$)

- Index as contour integral along $\partial\mathbb{H} = i\infty \cup \mathbb{R}$:

$$I(\partial\mathbb{H}) = ind_{i\infty}\big(DW\big) + \sum_{x\in\mathbb{R}:\ DW(x)=0} ind_x\big(DW\big)$$

map back to $i\infty$ with Γ, and use LCS expressions to compute

possible complication:
what about zeros not in the Γ-image of $i\infty$?

CONCLUSIONS

- Index defined by contour integral:
  Sufficient condition for critical points of effective potentials

- Particularly well-suited for N=1 F-term vacua
  (critical points of W are minima of V)

- Example: classical flux vacua on mirror sextic
  - on fundamental domain:
    branch cuts probe interior of the moduli space
  - on the covering space:
    purely algebraic evaluation of the contour integral in terms of LCS boundary information possible

THANK YOU!