Effective normalization of sub-Riemannian connections

joint work with Erlend Grong

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Cartan sub-Riemannian

<u>Filtration</u> on M: $E \subset TM$, $E^j = \{[X,Y], X \in E^{j+1}, Y \in E\}$, $j = -2, \dots$

<u>Grading</u> on M: $gr_- = gr_{-step} \oplus \dots \oplus gr_{-1} \cong TM$

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$$\mathfrak{g}_- = \mathfrak{g}_{-stp} \oplus \cdots \oplus \mathfrak{g}_{-1}, \quad \mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0$$

metric

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<u>sub-Riemannian with constant symbol</u>:

$G_0 \to \mathcal{F} \to M$ ← principal f.b., $T_x M \simeq \mathfrak{g}_-$

<u>Cartan connection</u> on $\mathcal{F}$: $\omega : T\mathcal{F} \to \mathfrak{g}$

(i) absolute parallelism (ii) right equiv. w.r.t. Ad (iii) reproduces ζ_X fundamental v.f.

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Tanaka, Tatsui, Morimoto $\longrightarrow$ unique ω

Cartan $\mathcal{G}$ $\longleftrightarrow$ affine ∇ on M !
$+$ $\mathfrak{g}\mathfrak{r}_- = TM$

<u>Normalization ?</u>

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$+ \ \mathrm{gr}_- = TM$

Normalization ?

curvature $K = d\psi + \frac{1}{2}[\psi,\psi] = \kappa(\psi,\psi)$

$C^\infty(\mathcal{F}, \mathfrak{g}\otimes\Lambda^2\mathfrak{g}_-^*)^{G_0}$

$\mathcal{C} = \bigoplus \mathcal{C}^k, \quad \mathcal{C}^k = \mathfrak{g}\otimes\Lambda^k\mathfrak{g}_-^*$

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$\uparrow$

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differentials :

$$\partial\alpha(A_0,\dots,A_k) = \sum_{i=0}^{k}(-1)^i A_i\cdot\alpha(A_0,\overset{\wedge i}{\dots},A_k) + \sum_{i<j}(-1)^{i+j}\alpha([A_i,A_j],\dots)$$

for $\mathfrak{g}$ as trivial rep

$$\partial_b\alpha(A_0,\dots,A_k) = \sum_{i<j}(-1)^{i+j}\alpha([A_i,A_j],\dots)$$

Filtration of forms

∂_b^{-1} ... pseudoinverse of ∂_b (on $\mathcal{C}$)

$$\mathcal{C} = \operatorname{im} \partial_b \oplus_\perp \operatorname{im} \partial_b^* \oplus_\perp \ker \Box_b , \quad \Box_b = \partial_b \partial_b^* + \partial_b^* \partial_b$$

$$= I \oplus_\perp \partial_b I \oplus_\perp \mathcal{C}_0 , \qquad \mathcal{C}_0 = \ker \Box_b$$

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$= \mathcal{I} \oplus_\perp \partial_b \mathcal{I} \oplus_\perp \mathcal{C}_0$, $\mathcal{C}_0 = \ker \Box_b$

Want to compare with ∂!

$P = \mathrm{id} - \partial_b^{-1}\partial - \partial\partial_b^{-1}$ $\quad P^s = \underbrace{P\circ\ldots\circ P}_{s \text{ times}}$ stabilizes... P^∞

$\mathcal{P} = \mathrm{im}\,P^\infty$, $\Pi = \mathrm{id} - \partial_b^{-1}\partial_b - \partial_b\partial_b^{-1}$ orthogonal

$\mathcal{C} = \mathcal{I} \oplus \partial\mathcal{I} \oplus \mathcal{P} = \mathcal{I} \oplus \partial_b\mathcal{I} \oplus \mathcal{P}$, $\ker P^\infty = \mathcal{I} \oplus \partial\mathcal{I}$

$P^\infty \circ \Pi = P^\infty$, $\Pi \circ P^\infty = \Pi$

effective normalization

Theorem There is unique $\psi : T\mathcal{F} \to \mathfrak{g}$ with curvature function $\kappa \in C^\infty(\mathcal{F}, \mathcal{C}^2)^{G_0}$:

(i) $\partial_b^{-1}\kappa = 0$ extension condition

(ii) $\partial^*\kappa_1 = \partial^*\partial_b^{-1}\partial_b\kappa_1$. normalization

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remarks:

- partial Cartan connection $\psi^E : T^{-1}\mathcal{F} \to \mathfrak{g}_{-1} \oplus \mathfrak{g}$
(i) $\boxed{\text{uniquely extends}}$ any ψ^E to ψ
- (ii) deals with homogeneity 1 of κ only and says: $\boxed{\partial^*\kappa_1 \text{ vanishes on } \ker \partial_b}$.

Consequences of the normalization

on TM, the choice of $\mathcal{H}$ gives ∇ and grading

- affine connection ∇ is uniquely determined by $(\nabla_{|E})|_E$ on the horizontal bundle E.
- the extension condition (i) ensures that $Hol_x \nabla$ coincides with the horizontal holonomy $Hol_{E,x} \nabla$. (trivial along horizontal loops => trivial)
- normalization of the choice of ∇^E then involves only κ_1. (makes sense via the unique extension)

Normalization seen from M

sub-Riemannian (M, E, g), E equiregular

E-grading : $I : TM \to \mathfrak{gr}_-$, $I(E^{-j}) = \mathfrak{gr}_{-j} \oplus \dots \oplus \mathfrak{gr}_{-1}$

minimal torsion : $T_0(v, w) = -I^{-1}[\![I(v), I(w)]\!]$

bracket in $\mathfrak{gr}_-$

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bracket on gr_-

∇ is <u>strongly compatible</u> with (M, E, g, I):

$\boxed{\text{compatible with } E, g}$, all $\boxed{TM_{-j} \text{ parallel}}$, $\boxed{\nabla T_0 = 0}$

$$\{\text{strongly compatible } \nabla\} \overset{1-1}{\longleftrightarrow} \text{regular Cartan } \mathcal{G}$$

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∇ is strongly compatible with (M, E, g, I):

$\boxed{\text{compatible with } E, g}$, all $\boxed{TM_{-j} \text{ parallel}}$, $\boxed{\nabla T_0 = 0}$

$\{\text{strongly compatible } \nabla\} \overset{1-1}{\longleftrightarrow}$ regular Cartan $\mathcal{G}$

Indeed, $\mathcal{G}$ defines linear ∇^{gr} on the grading, $\nabla = I^{-1} \nabla^{gr} I$

Extension

Lemma For each partial Cartan $\mathcal{H}_E$ on $\mathcal{F}$, there is a unique Cartan $\mathcal{H}$ with curvature K,

$$\mathcal{H}|_{\pi_*^{-1}E} = \mathcal{H}_E, \qquad \partial_b^{-1} K = 0.$$

Further, if $\mathcal{H}_E$ regular, then $\mathcal{H}$ regular.

Extension

Lemma For each partial Cartan ω_E on $\mathcal{F}$, there is a unique Cartan ω with curvature κ,

$$\omega|_{\pi_*^{-1}E} = \omega_E \,, \qquad \partial_b^{-1}\kappa = 0 .$$

Further, if ω_E regular, then ω regular.

Construction based on

$$\mathfrak{g}_{-2} \xrightarrow{-(\partial_b^{-1})^\vee} \Lambda^2 \mathfrak{g}_{-1} \xrightarrow{\Lambda^2 \omega_E^{-1}} \Lambda^2 \Gamma(T\mathcal{F}) \xrightarrow{\text{Lie b.}} \Gamma(T\mathcal{F})$$

$$B \longmapsto \sum_i A_{1,i} \wedge A_{2,i} \longmapsto H_B = \sum_i [H_{A_{1,i}}, H_{A_{2,i}}]$$

building iteratively the grading and ω with the right properties

Seen on M:

$\boxed{\text{set } \chi : TM \to \Lambda^2 TM, \ \chi = -T_0^{-1}}$

Lemma ∇^E partial affine with horizontal holonomy in $G_0|_E$. Then there is unique E-grading I and strongly compatible ∇ such that

$$R(\chi(\cdot)) = 0, \quad (T - T_0)(\chi(\cdot)) = 0.$$

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Remark: Following earlier work by Erlend Grong, the conditions ensure the holonomy properties!

... and the normalization

for $\mathcal{F} = (\mathrm{id}+\alpha)\tilde{\mathcal{F}}$ with the regular Cartan φ's

$$\kappa_1 = \partial\alpha_1 + \tilde{\kappa}_1$$

Now $\Pi\kappa_1$ determines κ_1, $\Pi(\kappa_1 - \tilde{\kappa}_1) \in \Pi\partial P^{\infty}\varphi_1^1$
$$\overset{\Pi\partial\varphi_1^1}{\parallel}$$

We require $0 = \langle \Pi\kappa_1, \Pi\partial\varphi^1 \rangle = \langle \Pi\kappa_1, \partial\varphi^1 \rangle$

$$= \langle \partial^*(\mathrm{id} - \partial_0^{-1}\partial_0)\kappa_1, \varphi^1 \rangle$$

$\Rightarrow$ our normalization.

Theorem (viewed on M)

Let (M, E, g) sub-Riemannian with constant symbol. There is unique E-grading I and strongly compatible ∇ with:

$$R(X(\cdot)) = 0 \qquad (T - T_0)(X(\cdot)) = 0$$

and $\langle T_{Jac}(v, \cdot), s \rangle = 0 \qquad v \in E, s \in \mathrm{iso}$

$$\langle T_{Jac}(v, \cdot), T_0(w, \cdot) \rangle = 0 \qquad v \in TM_{-j-1}, w \in TM_{-j}$$

$\left(\Omega^2_{Jac}(M) \subset \Omega^2(M), \quad \alpha(T_0(v_1, v_2), v_3) + \alpha(T_0(v_3, v_1), v_2) + \ldots = 0 \right)$