Regularization of matrices in the covariant derivative interpretation of the type IIB matrix model

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Background

- Matrix models are expected to give a nonperturbative formulation of superstring theory.
- The type IIB matrix model: [Ishibashi, Kawai, Kitazawa, Tsuchiya (1997)]

$$S = -\frac{1}{g^2}\mathrm{Tr}\left(\frac{1}{4}[A_a, A_b][A^a, A^b] - \frac{1}{2}\bar{\psi}\Gamma^a[A_a, \psi]\right)$$

$A_a\ (a = 1, \ldots, 10)$, ψ : $N \times N$ Hermitian matrices

  - Spacetime emerges from the degrees of freedom of matrices.
  - This model is expected to include gravity.

$$\Downarrow$$

Curved spacetime should be described in the type IIB matrix model.

Background

- The Covariant derivative interpretation of matrix models:

[Hanada, Kawai, Kimura (2006)]

$$\boxed{A_{(a)} = i\nabla_{(a)}}$$

$$\Downarrow$$

- Curved spacetime can be described in matrix models.
- Einstein eq. can be obtained from EOM of the type IIB matrix model.

- Problem: the size of the matrices $A_{(a)} = i\nabla_{(a)}$ is infinite.

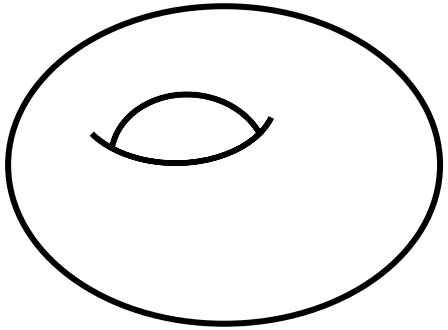
$$\Downarrow$$

It is necessary to make the size of the matrices finite to apply the covariant derivative interpretation to numerical simulations.

Research overview

- We regularize the matrices $A_{(a)} = i\nabla_{(a)}$ to finite-size matrices.

manifold M



infinite-size matrices

$$A_{(a)} = i\nabla_{(a)}$$

finite-size matrices

$$\begin{pmatrix} p_{11} & \cdots & p_{1N} \\ \vdots & \ddots & \vdots \\ p_{N1} & \cdots & p_{NN} \end{pmatrix}$$

covariant derivative interpretation

our study

- M: a closed connected $2n$-dimensional Kahler manifold
- We use Berezin-Toeplitz (BT) quantization for regularization.
- Our method generalizes [Hattori, Mizuno, Tsuchiya (2024)] for the 2-dimensional case to $2n$ dimensions.

Outline

- Background
- Research overview
- Covariant derivative interpretation
- BT quantization

Preparation (Review)

- **Regularization of covariant derivatives**

Our study

- Summary and future work

Covariant derivative interpretation

- Covariant derivative interpretation: $A_{(a)} = i\nabla_{(a)}$
- $\nabla_{(a)}\ (a = 1, \dots, 2n)$

[Hanada, Kawai, Kimura (2006)]

$$\boxed{\nabla_{(a)}\varphi(x,g) \coloneqq {R_{(a)}}^{b}(g^{-1})\nabla_b\varphi(x,g)} \qquad \nabla_b \coloneqq e_b^{\mu}\left(\partial_\mu + \frac{1}{2}{\Omega_\mu^{cd}}\mathcal{O}_{cd}\right)$$

① $\nabla_{(a)}$ acts on a regular rep. field $\varphi(x,g)$ $\big(x \in M,\ g \in \mathrm{Spin}(2n)\big)$.

$\longrightarrow$ For $h \in \mathrm{Spin}(2n)$, $\hat{h}\varphi(x,g) = \varphi(x, h^{-1}g)$

② $\nabla_{(a)}$ is ∇_b multiplied by ${R_{(a)}}^{b}(g^{-1})$.

$\longrightarrow$ the vector rep. matrix

The index (a) of $\nabla_{(a)}$ does not transform under $\mathrm{Spin}(2n)$.

⇨ $\nabla_{(a)}$ can be regarded as matrices.

BT quantization

- BT quantization is a method for regularizing a field to a finite-size matrix.

ϕ : a field on a Kahler manifold M

(a section of a homomorphism bundle)

→ a bundle whose fiber is a vector space of linear maps $V \to V'$

projecting onto the spaces of zero modes of the Dirac op. D & D'

→ By Atiyah-Singer index theorem, these spaces are finite-dimensional.

D acts on $\psi \in \Gamma\left(S \otimes L^{\otimes p} \otimes E_V\right)$ and D' acts on $\psi' \in \Gamma\left(S \otimes L^{\otimes p} \otimes E_{V'}\right)$.

$T(\phi)$: finite-size $(\dim\mathrm{Ker}D' \times \dim\mathrm{Ker}D)$ matrix

- p: a topological charge of the gauge field in D

$p \to \infty$ ⇨ the matrix size of $T(\phi) \to \infty$

BT quantization

- $T(\phi)$'s behavior for large p [Adachi, Ishiki, Kanno (2023)]

- $\lim_{p\to\infty} |T(\phi)T(\phi') - T(\phi\phi')| = 0$
- $\lim_{p\to\infty} \left| i\hbar_p^{-1}[T(f), T(\phi)] - T(\{f,\phi\}) \right| = 0$

Here,

- $\hbar_p \propto \frac{1}{p}$
- $f \in C^\infty(M)$
- $[T(f), T(\phi)] \coloneqq T(f)T(\phi) - T(\phi)T'(f)$
- $\{f, \phi\} \coloneqq W^{ab}(\partial_a f)(\nabla_b \phi)$, W^{ab}: Poisson tensor

Outline

- Background
- Overview of our study
- Covariant derivative interpretation
- BT quantization

  Preparation (Review)
- **Regularization of covariant derivatives**

  Our study
- Summary and future works

Regularization of covariant derivatives

- We will regularize the matrices $i\nabla_{(a)}$ to finite-size matrices.
- $\mathcal{P}_{(a)}$: matrix regularization of $i\nabla_{(a)}$

① $\mathcal{P}_{(a)}$ acts on $T(\varphi)$, matrix regularization of a regular rep. field φ.

② $\mathcal{P}_{(a)}T(\varphi) \sim T\left(i\nabla_{(a)}\varphi\right)$ for large p.

③ $\mathcal{P}_{(a)}$ is Hermitian for finite p.

$$\mathcal{P}_{(a)}T(\varphi) := -\hbar_p^{-1}T\left(\partial_{(a)}X^A\right)[T(X^A),T(\varphi)] + \frac{1}{2}\hbar_p^{-1}\left[T\left(\partial_{(a)}X^A\right),T(X^A)\right]T(\varphi)$$

- $X^A(x)\ (A = 1, \dots, D)$:
  - embedding coordinate functions of $2n$-dim Kahler manifold M into D-dim Euclidean space R^D
  - $(\partial_b X^A)(\partial_c X^A) = \delta_{bc}$

Regularization of covariant derivatives

- $\mathcal{P}_{(a)}$: matrix regularization of $i\nabla_{(a)}$

$$\mathcal{P}_{(a)}T(\varphi) \coloneqq -\hbar_p^{-1}T\left(\partial_{(a)}X^A\right)[T(X^A),T(\varphi)] + \frac{1}{2}\hbar_p^{-1}\left[T\left(\partial_{(a)}X^A\right),T(X^A)\right]T(\varphi)$$

3 steps

① Obtain $T(\varphi)$, matrix regularization of a regular rep. field φ

② Obtain $T(X^A)$, matrix regularization of embedding coordinate functions X^A

③ Show that $\mathcal{P}_{(a)}T(\varphi) \sim T\left(i\nabla_{(a)}\varphi\right)$ for large p

Regularization of covariant derivatives

- $T(\varphi)$: matrix regularization of a regular rep. field $\varphi(x,g)$

By Peter-Weyl theorem,

$$V_{reg} = \bigoplus_{r:\,irr.} \underbrace{(V_r \oplus \cdots \oplus V_r)}_{d_r}$$

- V_{reg}: regular rep. of Spin$(2n)$
- V_r: irreducible rep. r of Spin$(2n)$
- d_r: dimension of rep. r

$$\varphi(x,g) = \sum_{r:\,irr.} \underbrace{\tilde{\varphi}^{\langle r \rangle}_{i(j)}(x)}_{\text{field of rep. } r} \underbrace{\sqrt{d_r}\, R^{\langle r^* \rangle}_{i(j)}(g)}_{\text{basis}}$$

Under Spin$(2n)$, the index i transforms as rep. r, while (j) does not.

Regularization of covariant derivatives

- $T(\varphi)$: matrix regularization of a regular rep. field $\varphi(x,g)$

By Peter-Weyl theorem,

$$\overset{\varphi}{V_{reg}} = \bigoplus_{r:\,irr.} \underbrace{\left(\overset{\tilde{\varphi}^{\langle r\rangle}_{i(1)}}{V_r} \oplus \cdots \oplus \overset{\tilde{\varphi}^{\langle r\rangle}_{i(d_r)}}{V_r}\right)}_{d_r}$$

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$$T(\varphi) = \begin{array}{c} \\ d_r\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \\ d_{r'}\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \\ \\ \end{array} \begin{pmatrix} \vdots \\ T\left(\tilde{\varphi}^{\langle r\rangle}_{(1)}\right) \\ \vdots \\ T\left(\tilde{\varphi}^{\langle r\rangle}_{(d_r)}\right) \\ T\left(\tilde{\varphi}^{\langle r'\rangle}_{(1)}\right) \\ \vdots \\ T\left(\tilde{\varphi}^{\langle r'\rangle}_{(d_{r'})}\right) \\ \vdots \end{pmatrix}$$

Under Spin$(2n)$, the index i transforms as rep. r, while (j) does not.

Regularization of cova

・ Consider rep. r whose Casimir is less than Ξ.
・ Take the limit in which $\Xi \to \infty$ & $p \to \infty$ while keeping $\Xi \ll p$.

・ $T(\varphi)$: matrix regularization of a regula

By Peter-Weyl theorem,

$$\overset{\varphi}{V_{reg}} = \bigoplus_{r:\,irr.} \underbrace{(\overset{\tilde{\varphi}^{\langle r\rangle}_{i(1)}}{V_r} \oplus \cdots \oplus \overset{\tilde{\varphi}^{\langle r\rangle}_{i(d_r)}}{V_r})}_{d_r}$$

・ V_{reg}: regular rep. of Spin$(2n)$
・ V_r: irreducible rep. r of Spin$(2n)$
・ d_r: dimension of rep. r

$$\varphi(x,g) = \sum_{r:\,irr.} \underbrace{\tilde{\varphi}^{\langle r\rangle}_{i(j)}(x)}_{\text{field of rep. } r} \underbrace{\sqrt{d_r}\, R^{\langle r^*\rangle}_{i(j)}(g)}_{\text{basis}}$$

cutoff : Ξ

$$T(\varphi) = \begin{pmatrix} \vdots \\ \left.\begin{matrix} T\left(\tilde{\varphi}^{\langle r\rangle}_{(1)}\right) \\ \vdots \\ T\left(\tilde{\varphi}^{\langle r\rangle}_{(d_r)}\right) \end{matrix}\right\} d_r \\ \left.\begin{matrix} T\left(\tilde{\varphi}^{\langle r'\rangle}_{(1)}\right) \\ \vdots \\ T\left(\tilde{\varphi}^{\langle r'\rangle}_{(d_{r'})}\right) \end{matrix}\right\} d_{r'} \\ \vdots \end{pmatrix}$$

Under Spin$(2n)$, the index i transforms as rep. r, while (j) does not.

Regularization of covariant derivatives

- $T(X^A)$: matrix regularization of embedding coordinate functions X^A

$$\mathcal{P}_{(a)}T(\varphi) := -\hbar_p^{-1}T\big(\partial_{(a)}X^A\big)\underline{[T(X^A),T(\varphi)]} + \frac{1}{2}\hbar_p^{-1}\big[T\big(\partial_{(a)}X^A\big),T(X^A)\big]T(\varphi)$$

$$:= T(X^A)T(\varphi) - T(\varphi)T^{(1,1)}(X^A)$$

$$T(X^A) = \begin{pmatrix}
\ddots & & & & & & & \\
& T^{(r,r)}(X^A) & & & & & \mathbf{0} & \\
& & \ddots & & & & & \\
& & & T^{(r,r)}(X^A) & & & & \\
& & & & T^{(r',r')}(X^A) & & & \\
& \mathbf{0} & & & & \ddots & & \\
& & & & & & T^{(r',r')}(X^A) & \\
& & & & & & & \ddots
\end{pmatrix}$$

Regularization of covariant derivatives

- Proof that $\mathcal{P}_{(a)}T(\varphi) \sim T\left(i\nabla_{(a)}\varphi\right)$ for large p.

$$\mathcal{P}_{(a)}T(\varphi) \coloneqq -\hbar_p^{-1}T\left(\partial_{(a)}X^A\right)[T(X^A),T(\varphi)] + \underbrace{\frac{1}{2}\hbar_p^{-1}\left[T\left(\partial_{(a)}X^A\right),T(X^A)\right]T(\varphi)}_{O(1/p)}$$

$$= T\left(\partial_{(a)}X^A\right)T(i\{X^A,\varphi\}) + O(1/p)$$

$$= T\left(\partial_{(a)}X^A\right)T\left(iW^{cd}(\partial_c X^A)(\nabla_d\varphi)\right) + O(1/p)$$

$$= T\left(i\underbrace{\left(\partial_{(a)}X^A\right)W^{cd}(\partial_c X^A)}_{=R_{(a)}{}^d(g^{-1})}(\nabla_d\varphi)\right) + O(1/p)$$

$$= T\left(i\nabla_{(a)}\varphi\right) + O(1/p)$$

⇨ $\mathcal{P}_{(a)}$ gives matrix regularization of $i\nabla_{(a)}$.

Summary and future works

- Summary
  - We have regularized $i\nabla_{(a)}$ on closed connected $2n$-dimensional Kahler manifold M to finite-size matrices $\mathcal{P}_{(a)}$ by BT quantization.
  - $\mathcal{P}_{(a)}$ are given as follows:

$$\mathcal{P}_{(a)}T(\varphi) := -\hbar_p^{-1}T\left(\partial_{(a)}X^A\right)[T(X^A),T(\varphi)] + \frac{1}{2}\hbar_p^{-1}\left[T\left(\partial_{(a)}X^A\right),T(X^A)\right]T(\varphi)$$

- Future work
  - Calculation of 1-loop effective action (the mass of higher-spin fields)
  - Applying the covariant derivative interpretation to numerical simulations

BT quantization

- $\phi \in \Gamma\big(\mathrm{Hom}(E_V, E_{V'})\big) \leftarrow$ linear map: $\Gamma(E_V) \to \Gamma(E_{V'})$
  - E_V , $E_{V'}$: vector bundles whose connections are A^{E_V} , $A^{E_{V'}}$ respectively
- The Dirac operator $D^{(E_i)}$ on $\Gamma\big(S \otimes L^{\otimes p} \otimes E_i\big)$ $(i = V, V')$:

  $$D^{(E_i)} = i\gamma^a \nabla_a^{S\otimes L^{\otimes p}\otimes E_i} = i\gamma^a e_a^\mu \left(\partial_\mu + \frac{1}{4}\Omega_{bc\mu}\gamma^b\gamma^c + pA_\mu^L + A_\mu^{E_i}\right)$$

  - S : spinor bundle
  - L : complex line bundle (connection : A^L ($dA^L = -ik\omega$, ω :symplectic form))
- By the Atiyah-Singer index theorem, $\dim\big(\mathrm{Ker}D^{(E_i)}\big)$ is finite
- $\psi_I^{(E_i)} \in \mathrm{Ker}D^{(E_i)}$ $\quad \big(I = 1, \dots, \dim\big(\mathrm{Ker}D^{(E_i)}\big)\big)$
- BT quantization of ϕ : $T^{(E_1, E_2)}(\phi)_{IJ} \coloneqq \int_M \mu \left(\psi_I^{(E_1)}(x)\right)^\dagger \cdot \phi(x)\, \psi_J^{(E_2)}(x)$

BT quantization

- $\tilde{\varphi}_i^{\langle r \rangle}$: field in representation r of Spin$(2n)$ $\longrightarrow$ $\tilde{\varphi}_i^{\langle r \rangle} \in \Gamma\big(\mathrm{Hom}(E_{\mathrm{trivial}}\,,\, E_r\,)\big)$
  - E_r : vector bundle whose fiber is representation r of Spin$(2n)$
  - E_{trivial} : trivial bundle
- The Dirac operator $D^{(E_i)}$ on $\Gamma\big(S \otimes L^{\otimes p} \otimes E_i\big)$:

$$D^{(E_r)} = i\gamma^a e_a^\mu \left(\partial_\mu + \frac{1}{4}\Omega_{bc\mu}\gamma^b\gamma^c + pA_\mu^L + A_\mu^{E_r}\right)$$

$$D^{(E_{\mathrm{trivial}})} = i\gamma^a e_a^\mu \left(\partial_\mu + \frac{1}{4}\Omega_{bc\mu}\gamma^b\gamma^c + pA_\mu^L\right)$$

- $\psi_I^{(E_i)} \in \mathrm{Ker}D^{(E_i)} \quad \left(I = 1, \ldots, \dim\big(\mathrm{Ker}D^{(E_i)}\big)\right)$
- BT of $\tilde{\varphi}_i^{\langle r \rangle}$: $T^{(E_r\,,\,E_{\mathrm{trivial}})}\big(\tilde{\varphi}^{\langle r \rangle}\big)_{IJ} \coloneqq \int_M \mu\, \left(\psi_I^{(E_r)}(x)\right)_i^\dagger\, \tilde{\varphi}_i^{\langle r \rangle}(x)\, \psi_J^{(E_{\mathrm{trivial}})}(x)$

Matrix size of $T^{(E_{V'}, E_V)}(\varphi)$

- matrix size of $T^{(E_{V'}, E_V)}(\phi)$

  $= \dim\left(\mathrm{Ker} D^{(E_{V'})}\right) \times \dim\left(\mathrm{Ker} D^{(E_V)}\right)$

- $\dim\left(\mathrm{Ker} D^{(E_i)}\right)$

$$\dim\left(\mathrm{Ker} D^{(E_i)}\right) = \mathrm{Ind} D^{(E_i)} = \int_M \mathrm{Td}(TM) \wedge \mathrm{ch}\left(L^{\otimes p} \otimes E_i\right)$$

  - $\mathrm{Td}(\cdot)$: Todd class
  - $\mathrm{ch}(\cdot)$: Chern character
  - TM : tangent bundle

- Expanding in p,

$$\dim\left(\mathrm{Ker} D^{(E_i)}\right) = \mathrm{rank}(E_i) \int_M \exp\left(\frac{ip}{2\pi} dA^L\right) + O(p^{n-1}) = \frac{\mathrm{rank}(E_i)}{\left(2\pi\hbar_p\right)^n} \int_M \mu + O(p^{n-1})$$

$T(X^A)\ , T\left(\partial_{(a)}X^A\right)$

- (I,J) component of $\left((r,j)\ ,\ (r',l)\right)$ block of matrix $T(X^A)$

  $:= \delta^{\langle r\rangle\langle r'\rangle}\delta_{jl}T^{(r,r)}\left(X^A\mathbf{1}_{E_r}\right)_{IJ}$, $\mathbf{1}_{E_r} \in \Gamma(\mathrm{End}(E_r))$

- (I,J) component of $\left((r,j)\ ,\ (r',l)\right)$ block of matrix $T\left(\partial_{(a)}X^A\right)$

  $:= T^{(r,r')}\left(\sqrt{\frac{d_{r'}}{d_r}}C^{v,\,r'^*;\,r}_{b(a),\,(l);\,(j)}\omega_{cb}\partial^c X^A\right)_{IJ}$

  - $R^{\langle v\rangle}_{b(a)}(g)R^{\langle r'^*\rangle}_{k(l)}(g) = \sum_{r:\,irr.}C^{v,\,r'^*;\,r}_{b(a),\,k(l);\,i(j)}R^{\langle r^*\rangle}_{i(j)}(g)$
  - If the representation r does not appear in the decomposition of the representation $v \otimes r'^*$, then the $\left((r,j)\ ,\ (r',l)\right)$ block of $T(\partial_{(a)}X^A)$ vanishes.

$T\left(i\nabla_{(a)}\varphi\right)$

- $\nabla_{(a)}\varphi(x,g)$

$$= R^{\langle v\rangle}_{(a)b}(g^{-1})\nabla_b \sum_{r':\,irr.} \tilde{\varphi}^{\langle r'\rangle}_{k(l)}(x)\sqrt{d_{r'}}R^{\langle r'^*\rangle}_{k(l)}(g)$$

$$= \sum_{r':\,irr.}\left(\nabla_b\tilde{\varphi}^{\langle r'\rangle}_{k(l)}(x)\right)\sum_{r:\,irr.} C^{v,\,r'^*;\,r}_{b(a),\,k(l);\,i(j)}\sqrt{d_{r'}}R^{\langle r^*\rangle}_{i(j)}(g)$$

$$= \sum_{r:\,irr.}\left(\sum_{r':\,irr.}\sqrt{\frac{d_{r'}}{d_r}}C^{v,\,r'^*;\,r}_{b(a),\,k(l);\,i(j)}\left(\nabla_b\tilde{\varphi}^{\langle r'\rangle}_{k(l)}(x)\right)\right)\sqrt{d_r}R^{\langle r^*\rangle}_{i(j)}(g)$$

- (I,J) component of (r,j) block of matrix $T\left(i\nabla_{(a)}\varphi\right)$

$$:= T^{(r,1)}\left(i\sum_{r':\,irr.}\sqrt{\frac{d_{r'}}{d_r}}C^{v,\,r'^*;\,r}_{b(a),\,k(l);\,(j)}\left(\nabla_b\tilde{\varphi}^{\langle r'\rangle}_{k(l)}\right)\right)_{IJ}$$