

CORFU , SEPTEMBER 21ST
CALISTA GENERAL MEETING

JORDANIAN-TWISTED STRINGS AND SPINS: INTEGRABILITY & SPECTRUM

SIBYLLE DRIEZEN

WITH RICCARDO BORSATO, FEDOR LEVKOVICH-MASLYUK AND
ADRIEN MOLINES (POSTER!)

MOTIVATION: USING INTEGRABILITY

- BIG QUESTION: EXACT CONTROL OVER QFT'S AT ANY COUPLING

- GOOD TRACK RECORD: INTEGRABLE MODELS

→ IN 2 DIMENSIONS: FANTASTIC RESULTS [EXACT MASS GAP: HASENFRATZ, MAGGIORE, NIEDERMAYER '90'S]

→ BEYOND 2D? IN AdS/CFT CONTEXT

ONLY 240 CITATIONS...



2D INTEGRABLE QFT: WORLDSHEET

INTEGRABLE STRUCTURES: SPIN CHAIN

MOTIVATION: USING INTEGRABILITY

- BIG QUESTION: EXACT CONTROL OVER QFT'S AT ANY COUPLING

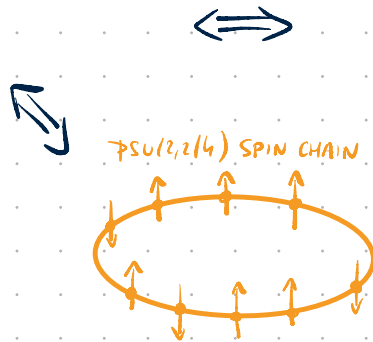
- GOOD TRACK RECORD: INTEGRABLE MODELS

→ IN 2 DIMENSIONS: FANTASTIC RESULTS [EXACT MASS GAP: HASENFRATZ, MAGGIORE, NIEDERMAYER '90'S]

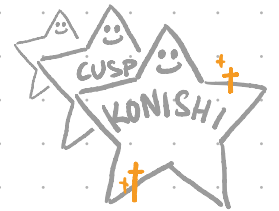
→ BEYOND 2D? IN AdS/CFT CONTEXT

ONLY 240 CITATIONS...

AdS₅/CFT₄ AT LARGE N: [EXACT PLANAR SPECTRUM OF A 4D QFT: A LOT OF PEOPLE '00'S-...]



$\mathcal{N}=4$ SYM
 $\text{Tr}(F_{\mu\nu} \cdot F^{\mu\nu})$



HIGHLY SYMMETRIC: PSU(2,2|4) (CFT, MAX. SUSY) + INTEGRABILITY (YANGIAN)

MOTIVATION: USING INTEGRABILITY

- BIG QUESTION: EXACT CONTROL OVER QFT'S AT ANY COUPLING

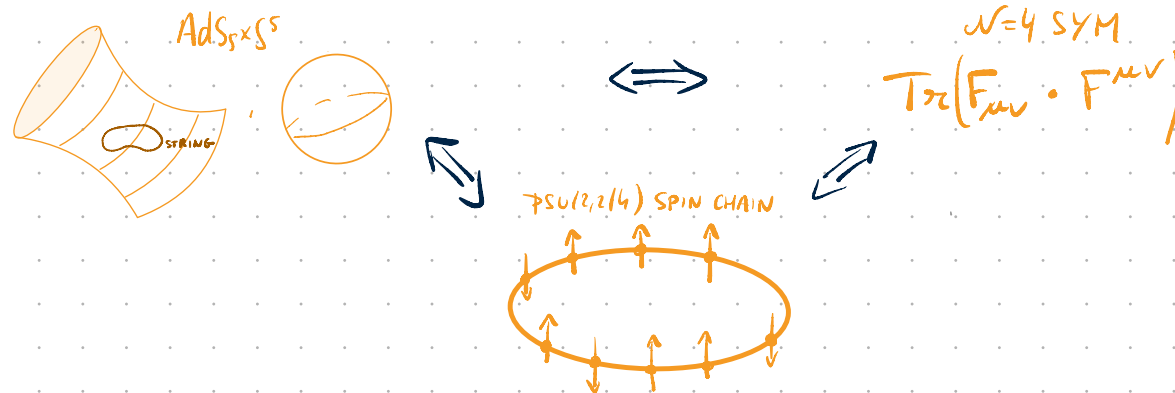
- GOOD TRACK RECORD: INTEGRABLE MODELS

→ IN 2 DIMENSIONS: FANTASTIC RESULTS [EXACT MASS GAP: HASENFRATZ, MAGGIORE, NIEDERMAYER '90'S]

→ BEYOND 2D? IN AdS/CFT CONTEXT

ONLY 240 CITATIONS...

AdS₅/CFT₄ AT LARGE N: [EXACT PLANAR SPECTRUM OF A 4D QFT: A LOT OF PEOPLE '00'S-...]



HIGHLY SYMMETRIC: $PSU(2,2|4)$ (CFT, MAX. SUSY) + INTEGRABILITY (YANGIAN)

⇒ DEFORM THIS STORY? DIFFERENT SOLVABLE STRUCTURES? WHAT DO WE LEARN?

DEFORM AdS/CFT AND USE INTEGRABILITY

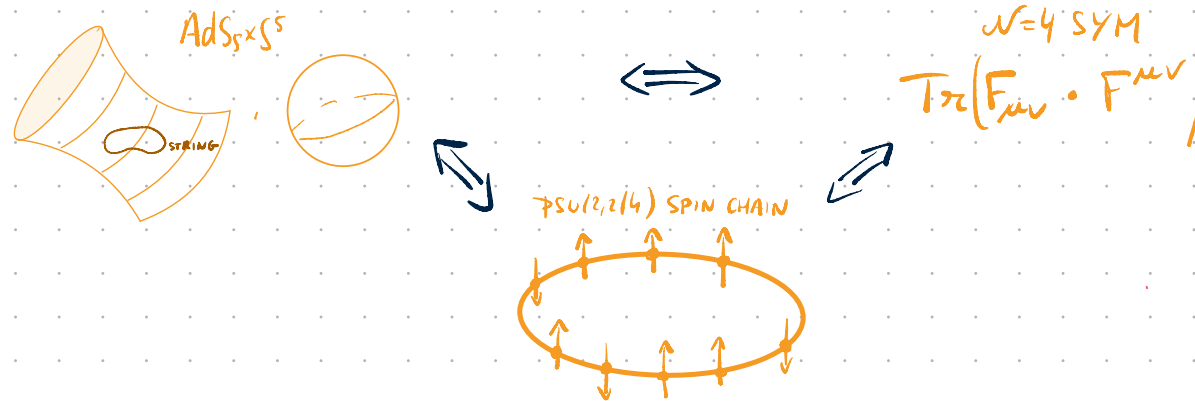
- SIMPLE ALGEBRAIC REALISATION OF DEFORMED SPIN CHAINS:

"DRINFEL'D TWIST" OF THE INTEGRABLE (HOPF) FRAMEWORK [DRINFEL'D '80's]

$$\Delta \rightarrow \Delta^F = F \Delta F^{-1} \quad \& \quad R \rightarrow R^F = F \circ P \cdot R \cdot F^{-1}$$

- CONJECTURED [VAN TONGEREN '15] TO DEFORM STRING WORLDSHEET AS A

"HOMOGENEOUS YANG-BAXTER DEFORMATION" [KLIMCHIK '02, '08]



DEFORM AdS/CFT AND USE INTEGRABILITY

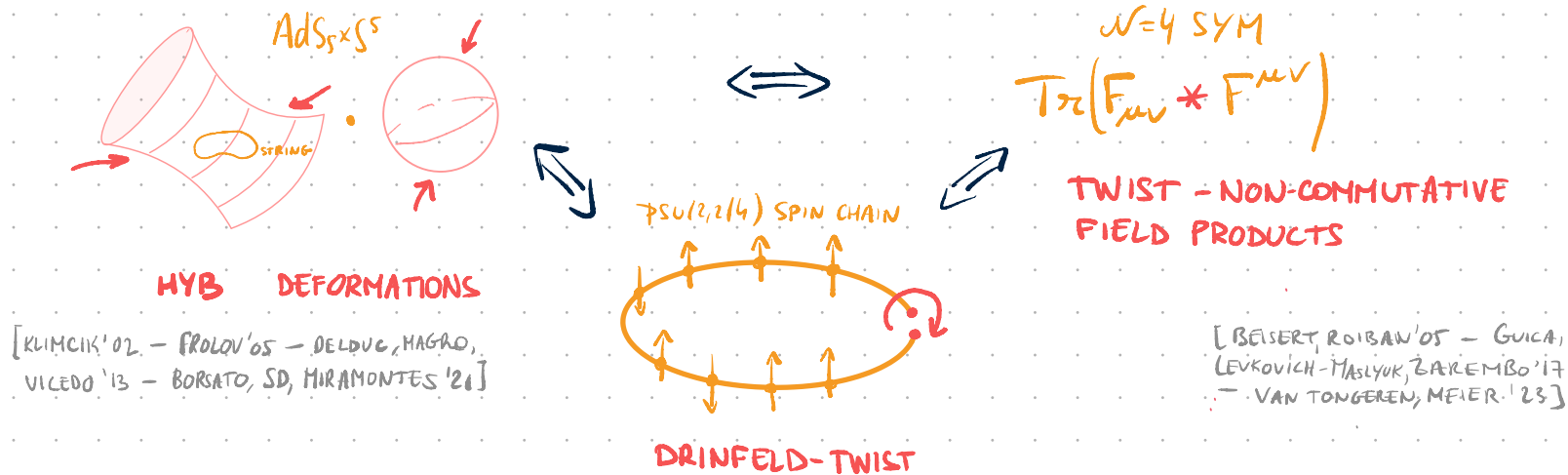
- SIMPLE ALGEBRAIC REALISATION OF DEFORMED SPIN CHAINS :

"DRINFEL'D TWIST" OF THE INTEGRABLE (HOPF) FRAMEWORK [DRINFEL'D '80's]

$$\Delta \rightarrow \Delta^F = F \Delta F^{-1} \quad \& \quad R \rightarrow R^F = F \circ P \cdot R \cdot F^{-1}$$

- CONJECTURED [VAN TONGEREN '15] TO DEFORM STRING WORLDSHEET AS A

"HOMOGENEOUS YANG-BAXTER DEFORMATION" [KLIMCHIK '02, '08]



⇒ FOR THE "JORDANIAN" SUBCLASS IT WILL FORCE US TO RETHINK WHAT AdS/CFT INTEGRABILITY MEANS & HOW TO REORGANISE ITS PHYSICS

DEFORM AdS/CFT AND USE INTEGRABILITY

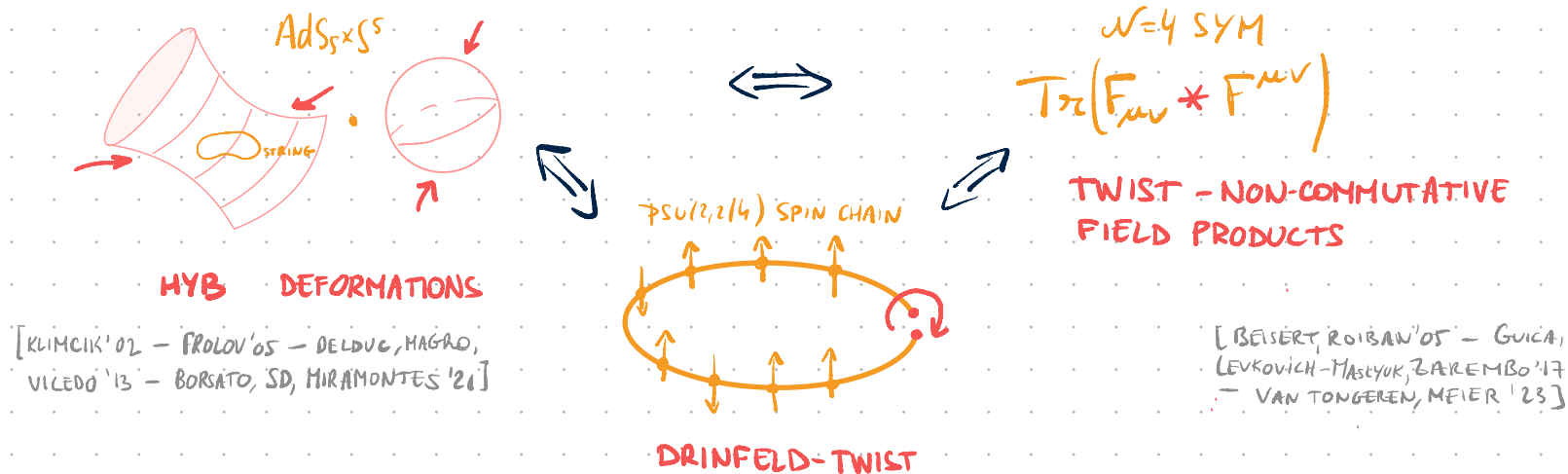
- SIMPLE ALGEBRAIC REALISATION OF DEFORMED SPIN CHAINS :

"DRINFEL'D TWIST" OF THE INTEGRABLE (HOPF) FRAMEWORK [DRINFEL'D '80's]

$$\Delta \rightarrow \Delta^F = F \Delta F^{-1} \quad \& \quad R \rightarrow R^F = F \circ R \circ F^{-1}$$

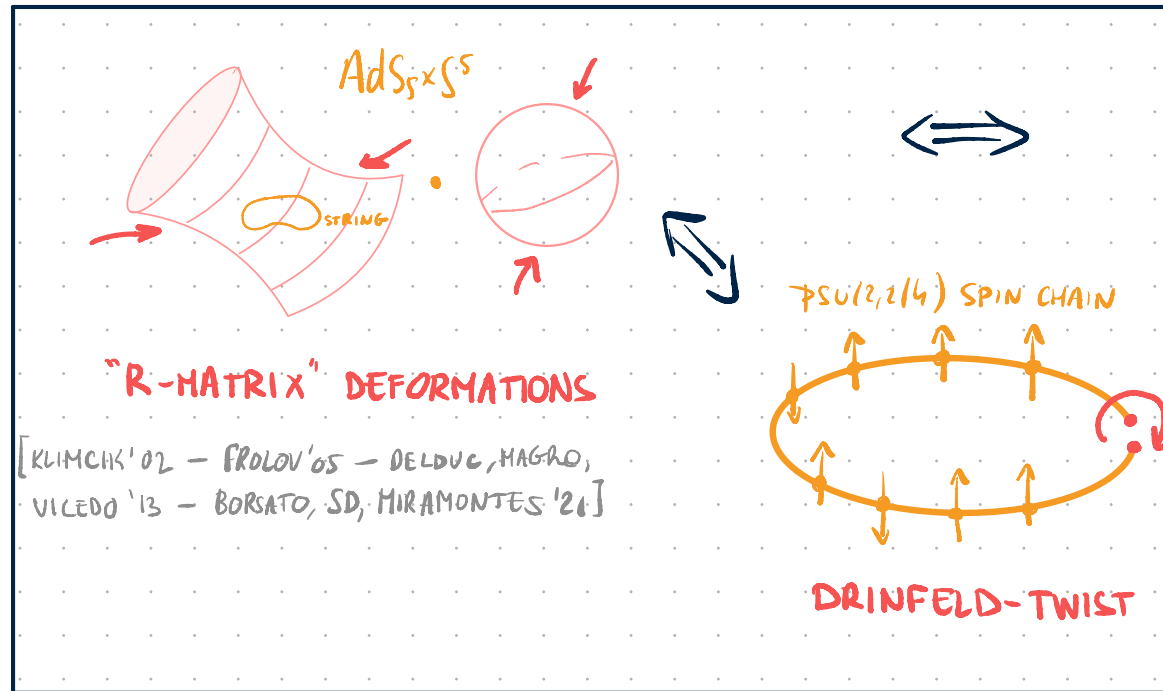
- CONJECTURED [VAN TONGEREN '15] TO DEFORM STRING WORLDSHEET AS A

"HOMOGENEOUS YANG-BAXTER DEFORMATION" [KLIMCHIK '02, '08]



⇒ FOR THE "JORDANIAN" SUBCLASS IT WILL FORCE US TO RETHINK WHAT AdS/CFT INTEGRABILITY MEANS & HOW TO REORGANISE ITS PHYSICS

DEFORM ADS/CFT AND USE INTEGRABILITY



THIS TALK

$\Rightarrow$ FOR THE "JORDANIAN" SUBCLASS IT WILL FORCE USE TO RETHINK WHAT ADS/CFT INTEGRABILITY MEANS & HOW TO REORGANISE ITS PHYSICS

→ STRING WORLD SHEET: LAX VS. S-MATRIX

≤ 2412.08411

→ SPIN CHAIN: SPECTRUM FROM BAXTER

≥ 2507.13911

YANG-BAXTER WORLDSHEET MODELS

DEF.: INTEGRABLE DEFORMATIONS OF 2D SIGMA-MODELS WITH ISOMETRIES G
 $\rightarrow$ STRINGS IN $AD S_5 \times S^5$: $G = PSU(2, 2 | 4)$

$$S[g] = \frac{\sqrt{\lambda}}{8\pi} \int d^2x (\sqrt{-h} h^{\alpha\beta} + \varepsilon^{\alpha\beta}) \text{Tr} \left(j_\alpha P (1 - \eta R_g P)^{-1} j_\beta \right)$$

• FIELDS $g: \Sigma \rightarrow G$; $g(x^M)$ w. $M = 1, \dots, \dim(G)$; $j_\alpha = g^{-1} \partial_\alpha g$

• DEFORMATION PARAMETER η & OPERATOR $R: \mathfrak{g} \rightarrow \mathfrak{g}$ SATISFYING

$$\left. \begin{array}{l} * [R_x, R_y] = R([R_x, Y] - [X, R_Y]) ; x, y \in \mathfrak{g} \quad (\text{CYBE}) \\ * R^T = -R \quad (\text{ANTISYMMETRIC}) \\ * R^{AB} F_{AB}{}^C = 0 \quad (\text{UNIMODULAR}) \end{array} \right\}$$

INTEGRABLE

[KLIMCHIK '08; DELDUK, MAGRO, VICEDO '13]

+

IIB SUPERGRAVITY

[BORSATO, WULFF '16]

YANG-BAXTER WORLDSHEET MODELS

DEF.: INTEGRABLE DEFORMATIONS OF 2D SIGMA-MODELS WITH ISOMETRIES G
 $\rightarrow$ STRINGS IN $AD S_5 \times S^5$: $G = PSU(2, 2|4)$

$$S[g] = -\frac{\sqrt{-1}}{8\pi} \int d^2x (\sqrt{-h} h^{\alpha\beta} - \xi^{\alpha\beta}) \text{Tr} (j_\alpha P (1 - \eta R_g P)^{-1} j_\beta)$$

• FIELDS $g: \Sigma \rightarrow G$; $g(x^M)$ w. $M = 1, \dots, \dim(G)$; $j_\alpha = g^{-1} \partial_\alpha g$

• DEFORMATION PARAMETER η & OPERATOR $R: \mathfrak{g} \rightarrow \mathfrak{g}$ SATISFYING

$$\left. \begin{aligned} * [R_x, R_y] &= R([R_x, Y] - [X, R_Y]) ; x, y \in \mathfrak{g} \quad (\text{CYBE}) \\ * R^T &= -R \quad (\text{ANTISYMMETRIC}) \\ * R^{AB} F_{AB}^C &= 0 \quad (\text{UNIMODULAR}) \end{aligned} \right\}$$

INTEGRABLE

[KLIMCHIK '08; DELDUZ, MAGRO, VICEDO '13]

+ IIB SUPERGRAVITY
 [BORSATO, WULFF '16]

CORRESPOND TO DRINFEL'D TWISTS F CONNECTED TO THE IDENTITY

$$F = 1 \otimes 1 + \frac{\eta R}{\sqrt{\lambda}} + \mathcal{O}(\eta^2) \quad (\Delta_F(x) = F \Delta(x) F^{-1}, \dots) \quad [\text{DRINFEL'D '83}]$$

$\rightarrow$ HAS A LOT OF SOLUTIONS

$\rightarrow$ CLASSIFIED FOR $\mathfrak{g} = \mathfrak{psu}(2, 2|4)$ IN [BORSATO, WULFF '16, BORSATO, SD '22]

SUPER ROUGH OVERVIEW

TST - TRANSFORMATIONS

- BASED ON ABELIAN T-DUALITY :

T-DUALITY · SHIFT · T-DUALITY

[OSTEN, VAN TONGEREN '16]

- DONE OVER ABELIAN SUBALGEBRA OF ISOMETRIES :

$$\text{IM}(\mathbf{R}) = \text{SPAN}(h_1, h_2) \subset \mathfrak{g}$$

WITH

$$[h_1, h_2] = 0$$

- REMAINING SYMMETRIES : $[T_a, R(X)] = R([T_a, X])$,
ALWAYS INCLUDES :

$$\{h_1, h_2\} + \mathbb{Z}_{\mathfrak{g}}(\text{IM}(\mathbf{R})) + \dots$$

JORDANIAN DEFORMATIONS

- EQUIV. TO NON-ABELIAN T-DUALITY

[ORLANDO, REFFERT, SAKOMOTO, YOSHIDA '16,
HOARE, TSEYTLIN '16, BORSATO, WULF '16]

- DONE OVER NON-ABELIAN BOREL SUBALGEBRA OF ISOMETRIES :

$$\text{IM}(\mathbf{R}) = \text{SPAN}(h, e) \subset \mathfrak{sl}(2, \mathbb{R}) \subset \mathfrak{g}$$

WITH

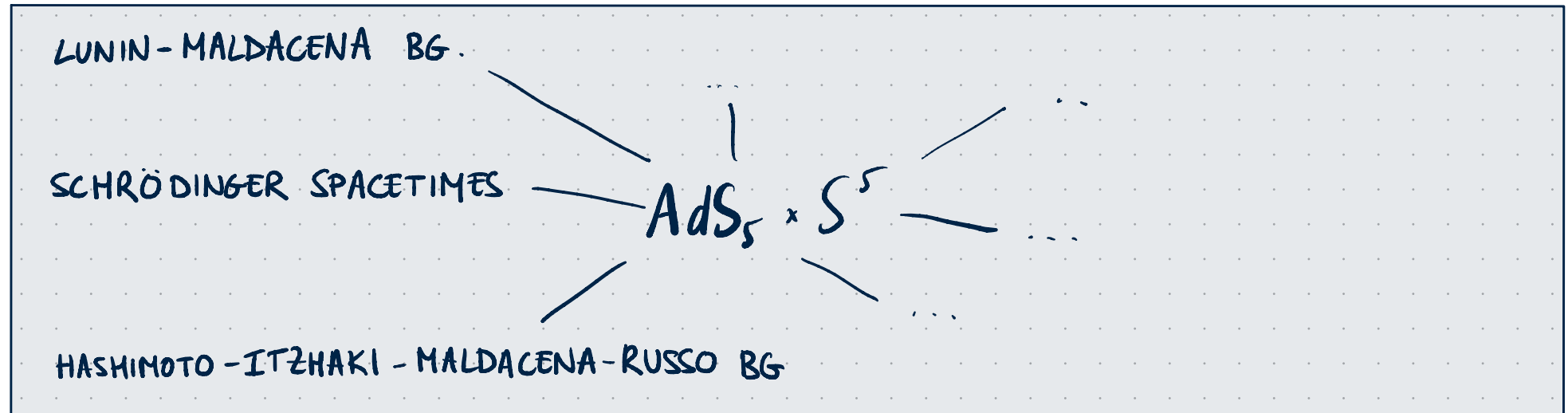
$$[h, e] = e$$

↓ ↓
CARTAN ROOT

- REMAINING SYMMETRIES : NEVER h BUT

$$\{e\} + \mathbb{Z}_{\mathfrak{g}}(\text{IM}(\mathbf{R}))$$

SUPER ROUGH OVERVIEW



SUPER ROUGH OVERVIEW

TST - TRANSFORMATIONS

NON-ABELIAN DEFORMATIONS

LUNIN-MALDACENA BG.

SCHRÖDINGER SPACETIMES

HASHIMOTO-ITZHAKI-MALDACENA-RUSSO BG

$AdS_5 \times S^5$

SEQUENCES OF TST

JORDANIAN (NATD)
DEFORMATIONS

TST - TRANSFORMATIONS

NON-ABELIAN DEFORMATIONS

LUNIN-MALDACENA BG.

$$h_1, h_2 \in \mathfrak{so}(6)$$

SCHRÖDINGER SPACETIMES

$$h_1 \in \mathfrak{so}(6), h_2 \in \mathfrak{so}(2,4)$$

$$\rightarrow h_1, h_2 \in \mathfrak{so}(2,4)$$

HASHIMOTO-ITZHAKI-MALDACENA-RUSSO BG

$$AdS_5 \times S^5$$

SEQUENCES OF TST

JORDANIAN (NATD)
DEFORMATIONS



MARGINAL DEFORMATIONS

$\rightarrow \beta$ -DEF.

$\rightarrow \gamma$ -DEF.

$$h_1, h_2 \in \mathfrak{so}(6)$$

NON-COMMUTATIVE DEFS

$\rightarrow$ DIPOLE DEF.

$\rightarrow$ GROENEWOLD-MOYAL DEF.

$$h_1, h_2 \in \mathfrak{so}(6) \oplus \mathfrak{so}(2,4)$$

$$\mathcal{N}=4 \text{ SYM}$$

NON-COMMUTATIVE DEF.

$\rightarrow$ JORDANIAN DEF??

$$h, e \in \mathfrak{sl}(2, \mathbb{R})$$

JORDANIAN DEFORMATIONS

THE UNIQUE CASE WITH CONSTANT DILATON ($g_s \sim \exp(\Phi)$) & RESTRICTED TO $SL(2, \mathbb{R})$:

$$h = \frac{D - J_{03}}{2}$$

&

$$e = p_0 + p_3$$

+ SUPERCHARGES

[SD, MOLINES '15]

• SYMMETRY = SPAN $\{ D + J_{03}, k_0 + k_3, p_0 - p_3, p_0 + p_3, J_{12} \} \oplus \underbrace{SU(3)}_{\subset SU(4)} \oplus \underbrace{U(1)}_{\subset SU(4)} \oplus 12 \text{ SUSY's}$

$$SL(2, \mathbb{R}) \oplus U(1)^2 \subset SU(2, 2)$$

$$\subset SU(4)$$

↳ NON-RELATIVISTIC CFT (SCHRÖDINGER ALGEBRA $\sim CFT_1$)

WITH "CARTAN":

• $H_V = e = p_0 + p_3$

• $H_\theta = J_{12}$

• AND $H_T = \frac{1}{2}(p_0 - k_0 - p_3 - k_3)$

OR $H_R = D + J_{03}$

[NULL MOMENTUM M]

[SPIN S]

[NISHIDA, SON '07]

[TIMELIKE: ENERGY E] $\leftarrow S=0 \rightarrow$

[RADIAL: NON-RELATIVISTIC SCALING Δ]

• $p_0 - k_0$ on D : E on Δ

• J_{12} : S_1

• J_{34} : S_2

• IIB SUGRA:
$$ds^2 = \frac{dz^2 + dP^2 + P^2 d\theta^2 + 2dTdV}{z^2} - \frac{(4z^4 + y^2)(z^2 + P^2)}{4z^6} dT^2 + ds_{S^5}$$

$$B = -y \frac{PdP \wedge dT}{2z^4}; \quad F_3(y); \quad F_5 = F_5(y=0); \quad \Phi = \Phi_0$$

[KAWAGUCHI, MATSUMOTO, YOSHIDA '14
VAN TONGEREN '19]

→ STRING WORLD SHEET: LAX VS. S-MATRIX

≤ 2412.08411

→ SPIN CHAIN: SPECTRUM FROM BAXTER

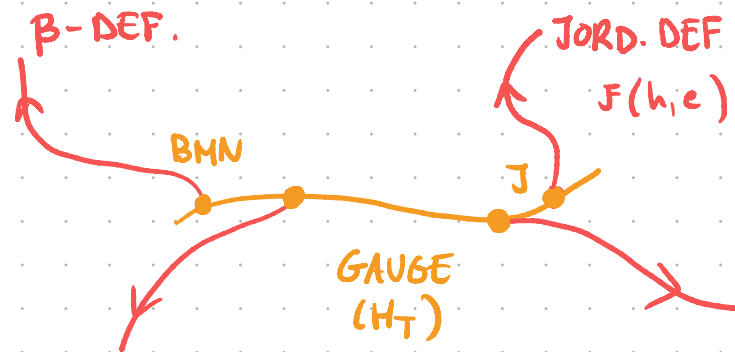
≥ 2507.13911

JORDANIAN SCATTERING

$$S^F = F^{-1} S F ?$$

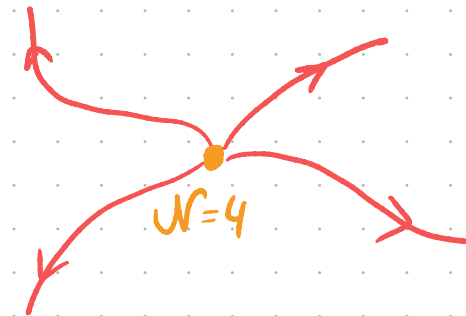
- DEFORMATIONS (SUCH AS JORDANIAN) ONLY POSSIBLE IN SPECIFIC LIGHT-CONE GAUGE

(BORSATO, SD, HOARE, KETORE, SEIBOLD '23)



S-MATRIX DESCRIPTION

VS.



DEFORMATIONS OF ADS/CFT

- POINTLIKE SOLUTION: $T = E \tau$, $V = \frac{\eta^2}{4} M \tau$, $Z = \sqrt{E/M}$, $\psi = J \tau$ ($S=0$)

$$E = \sqrt{J^2 + \frac{\eta^2}{4} M^2} = J + \frac{\eta^2 M^2}{8J} + O(\eta^4)$$

→ USE TO GAUGE FIX $T \sim \tau \Rightarrow H_T \sim$ WORLDSHEET HAMILTONIAN

→ SYMMETRIES $\mathfrak{so}(2,1) \oplus \mathfrak{so}(2) \oplus H_T$ CONTAINS h & e

JORDANIAN SCATTERING

$$S^F = F^{-1} S F ?$$

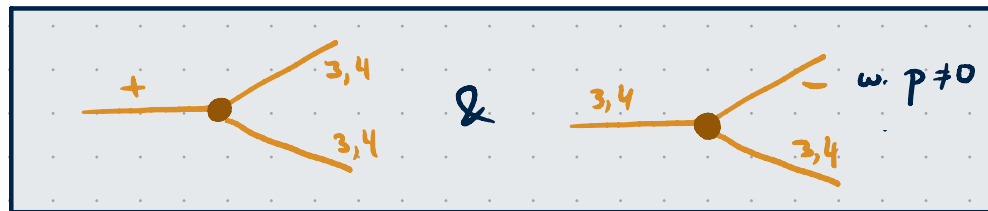
TAKE J-GAUGE AND OBTAIN PERTURBATIVE HAMILTONIAN $\tilde{H} = \hat{H}_2 + \hat{H}_3 + \hat{H}_4 + \dots$

① DISPERSION RELATIONS:

$$\omega_{\pm}^2 = \left(\pm 1 + \sqrt{(1-\eta^2)^{-2} + p^2} \right)^2 - \eta^4(1-\eta^2)^{-2}, \quad \omega_3^2 = \omega_4^2 = \frac{1+\eta^2}{1-\eta^2} + p^2, \quad \omega_{\text{SPHERE}}^2 = 1 + p^2$$

↳ **NON-RELATIVISTIC** DISP. RELS $\rightsquigarrow$ NO "KLEIN-GORDON" RULES FOR KINEMATICS

② SCATTERING:



$$\begin{aligned} T_{(+)}^{(+)(x)} &= -\frac{\sqrt{2\pi}}{6}\eta - i\frac{\sqrt{2\pi}(1-\sqrt{1+p^2})}{6p}\eta^2 - \frac{\sqrt{2\pi}(1+2p^2)(1+\sqrt{1+p^2})+2}{12p^2\sqrt{1+p^2}}\eta^3 + \mathcal{O}(\eta^4) \\ T_{(+)}^{(-)(x)} &= \frac{\sqrt{2\pi}}{6}\eta - i\frac{\sqrt{2\pi}(1+\sqrt{1+p^2})}{6p}\eta^2 + \frac{\sqrt{2\pi}(1+2p^2)(-1+\sqrt{1+p^2})}{12p^2\sqrt{1+p^2}}\eta^3 + \mathcal{O}(\eta^4). \end{aligned}$$

$\Rightarrow$ **INELASTIC** AT TREE-LEVEL: VIOLATES AXIOMS OF S-MATRIX INTEGRABILITY...?

[BORSATO, SD '24]

→ STRING WORLD SHEET: LAX VS. S-MATRIX

≤ 2412.08411

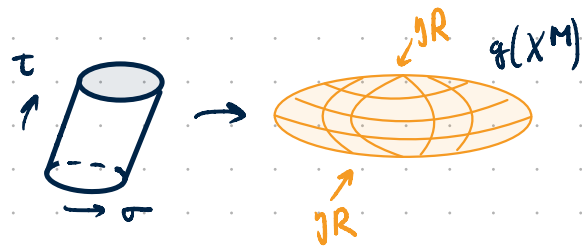
→ SPIN CHAIN: SPECTRUM FROM BAXTER

≥ 2507.13911

TWO LAX DESCRIPTIONS

[FROLOV '05 (TST)-
BORSATO, SD, MIRAMONTES '21 (ALL YB)]

DEFORMED $\tilde{H}(\gamma R)$, BUT
PERIODIC FIELDS q

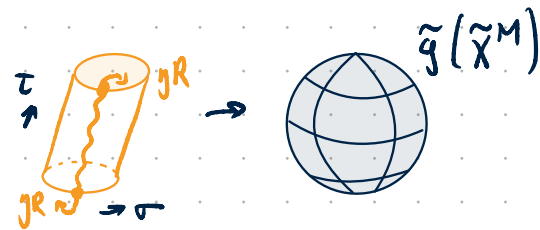


$\hookrightarrow$ LAX (γR)

NON-LOCAL
CANONICAL
TRANSFORMATION
ON THE
WORLD SHEET

UNDEFORMED H , BUT
TWISTED (γR) FIELDS $\tilde{q}$

$$q = F \tilde{q}$$



$\hookrightarrow$ FIELDS $\tilde{q}(2\pi, \tau) = e^{\gamma R(Q)} \tilde{q}(0, \tau)$

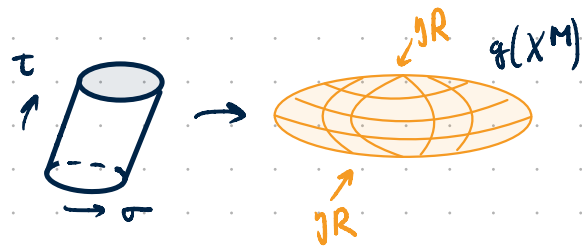
DEFORMED SIGMA-MODEL MONODROMY IS DRINFEL'D TWISTED (CLASSICALLY) [VAN TONGEREN '15]

$$T_\gamma(u) = P_{\exp}(Lax(\gamma R)) = F(2\pi; \gamma R) P_{\exp}(Lax) F^{-1}(0; \gamma R) \quad \text{OR} \quad \tilde{T}_\gamma(u) \sim e^{\gamma R(Q)} P_{\exp}(\tilde{Lax})$$

TWO LAX DESCRIPTIONS

[FROLOV '05 (TST) -
BORSATO, SD, MIRAMONTES '21 (ALL YB)]

DEFORMED $\tilde{H}(\gamma R)$, BUT
PERIODIC FIELDS q

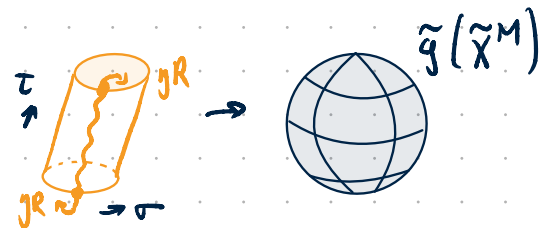


$\hookrightarrow$ LAX (γR)

NON-LOCAL
CANONICAL
TRANSFORMATION
ON THE
WORLD SHEET

UNDEFORMED H , BUT
TWISTED (γR) FIELDS $\tilde{q}$

$$q = F \tilde{q}$$



$\hookrightarrow$ FIELDS $\tilde{q}(2\pi, \tau) = e^{\gamma R(Q)} \tilde{q}(0, \tau)$

DEFORMED SIGMA-MODEL MONODROMY IS DRINFEL'D TWISTED (CLASSICALLY) [VAN TONGEREN '15]

$$T_\gamma(u) = P \exp(Lax(\gamma R)) = F(2\pi; \gamma R) P \exp(Lax) F^{-1}(0; \gamma R) \quad \text{OR} \quad \tilde{T}_\gamma(u) \sim e^{\gamma R(Q)} P \exp(\tilde{Lax})$$

$$\text{WHERE } F(\sigma) \sim 1 + \frac{1}{\gamma R} \gamma R(Q(\sigma)) + \mathcal{O}(\gamma R^{-1})$$

• ABELIAN : $F_A = \exp\left(\frac{\gamma}{\gamma R} (Q_1(\sigma) T_2 - Q_2(\sigma) T_1)\right)$

• JORDANIAN : $F_J = \exp\left(\frac{\gamma}{\gamma R} Q_h(\sigma) e\right) \exp\left(\log(1 - \frac{\gamma}{\gamma R} Q_e(\sigma)) h\right)$

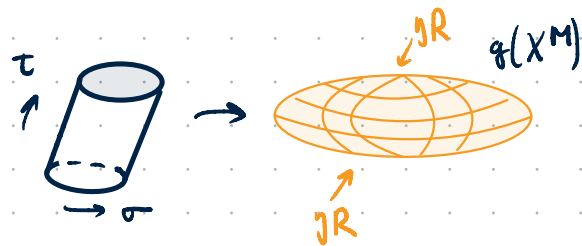
[BORSATO, SD, MIRAMONTES '21]

USED FOR SPECTRAL CURVE:
[BORSATO, SD, METO-GARCIA, WYSS '22,
SD, KAMATH '24]

TWO LAX DESCRIPTIONS

[FROLOV '05 (TST) -
BORSATO, SD, MIRAMONTES '21 (ALL YB)]

DEFORMED $\tilde{H}(\gamma R)$, BUT
PERIODIC FIELDS q



$\hookrightarrow$ LAX (γR)

NON-LOCAL
CANONICAL
TRANSFORMATION
ON THE
WORLD SHEET

UNDEFORMED H , BUT
TWISTED (γR) FIELDS $\tilde{q}$

$$q = F \tilde{q}$$



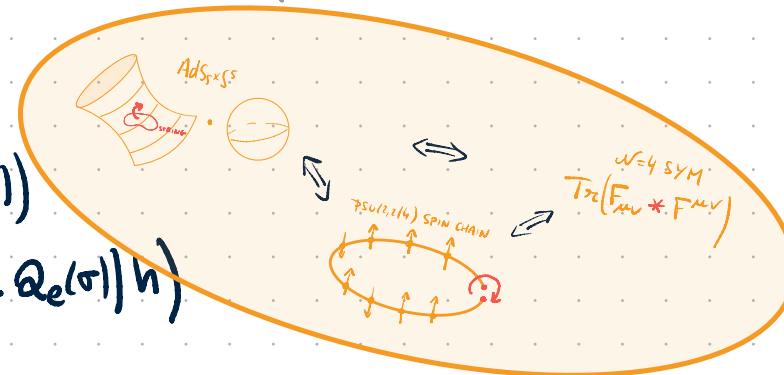
$\hookrightarrow$ FIELDS $\tilde{q}(2\pi, \tau) = e^{\gamma R(Q)} \tilde{q}(0, \tau)$

DEFORMED SIGMA-MODEL MONODROMY IS DRINFEL'D TWISTED (CLASSICALLY) [VAN TONGEREN '15]

$$T_\gamma(u) = P \exp(Lax(\gamma R)) = F(2\pi; \gamma R) P \exp(Lax) F^{-1}(0; \gamma R) \quad \text{OR} \quad \tilde{T}_\gamma(u) \sim e^{\gamma R(Q)} P \exp(\tilde{L}ax)$$

$$\text{WHERE } F(\sigma) \sim 1 + \frac{1}{\sqrt{2}} \gamma R(Q(\sigma)) + \mathcal{O}(\lambda^{-1})$$

- ABELIAN : $F_A = \exp\left(\frac{\gamma}{\sqrt{2}} (Q_1(\sigma) T_2 - Q_2(\sigma) T_1)\right)$
- JORDANIAN : $F_J = \exp\left(\frac{\gamma}{\sqrt{2}} Q_h(\sigma) e\right) \exp(\log(1 - \frac{\gamma}{\sqrt{2}} Q_e(\sigma)) h)$



→ STRING WORLD SHEET: LAX VS. S-MATRIX

≤ 2412.08411

→ SPIN CHAIN: SPECTRUM FROM BAXTER

≥ 2507.13911

DRINFEL'D - TWISTED XXX-1/2 (SL(2, R) SECTOR)

$\mathcal{A} = \mathcal{U}(\mathfrak{sl}(2, \mathbb{R}))$ WITH $\Delta(X) = \sum_j X_j$; $X \in \mathcal{A}$; AND

- $R_{aj}(u) = u 1_{aj} + i(\sigma_a \cdot S_j) \rightarrow$ INF. DIM. REP.: $\begin{cases} h_j \sim z_j \partial_j + 1/2 \\ e_j \sim \partial_j \\ f_j \sim z_j^2 \partial_j + z_j \end{cases}$ & $\Psi = \Psi(z_j)$
 $\rightarrow$ FINITE-DIM. REP. (2x2 MATRICES)

$T_a(u) = R_{a3}(u) \dots R_{a2}(u) R_{a1}(u) \rightarrow$ ENCODES COMMUTING OPERATORS

- NN-HAMILTONIAN $H_{ij} = P_{ij} \frac{d}{du} R_{ij}(u) \Big|_{u=u_0}$

DRINFEL'D - TWISTED $XX_{-1/2}$ ($sl(2, \mathbb{R})$ SECTOR)

$\mathcal{A} = \mathcal{U}(sl(2, \mathbb{R}))$ WITH $\Delta(X) = \sum_j X_j$; $X \in \mathcal{A}$; AND

• $R_{aj}(u) = u \mathbb{1}_{aj} + i \sigma_a \cdot S_j \rightarrow$ INF. DIM. REP. : $\begin{cases} h_j \sim z_j \partial_j + 1/2 \\ e_j \sim \partial_j \\ f_j \sim z_j^2 \partial_j + z_j \end{cases}$ & $\Psi = \Psi(z_j)$

$\rightarrow$ FINITE-DIM. REP. (2x2 MATRICES)

$T_a(u) = R_{a3}(u) \dots R_{a2}(u) R_{a1}(u) \rightarrow$ ENCODES COMMUTING OPERATORS

• NN-HAMILTONIAN $H_{ij} = P_{ij} \frac{d}{du} R_{ij}(u) \Big|_{u=u_0}$

DRINFEL'D TWIST : $(F \otimes \mathbb{1})(\Delta \otimes \mathbb{1})(F) = (\mathbb{1} \otimes F)(\mathbb{1} \otimes \Delta)(F)$ COCYCLE CONDITION.

THEN $\Delta(X) \rightarrow \Delta_F(X) = F \Delta(X) F^{-1} \rightarrow$ ALSO QUASI-TRIANGULAR HOPF (YBE, RTT, ...)

$R \rightarrow R_F = F^{op} R F^{-1}$

$T \rightarrow T_a^F(u) = \Omega (\mathbb{1}_a \times \Delta^{J^{-1}})(F^{op}) T_a(u) (\mathbb{1}_a \times \Delta^{J^{-1}})(F^{-1}) \Omega^{-1}$ [SB, MOLINER '25]

$H \rightarrow H_{ij}^F = F_{ij} H_{ij} F_{ij}^{-1}$ & $H_{TOT} = \Omega H^* \Omega^{-1}$ [GUICA, LEVKOVICH-MASLYUK, ZARETSKO '17, BORSATO, FERNANDEZ '25]

Ω : all physical sites $\rightarrow$ SIMILARITY OF HILBERT SPACE

JORDANIAN-TWISTED $XXZ_{-1/2}$ (J SITES)

HERE :

$$F_{ij} = \exp(h_i \otimes \log(1 + \xi e_j)) ; \quad \xi \in \mathbb{R} \quad [\text{KVLISH, ... '90's}]$$

MANIFEST SYMMETRIES :

• $\Delta_F^{J-1}(X)$ IS A SYMM. IF $[\Delta_F^{J-1}(X), H_{\text{TOT}}^F] = 0$ OR EQUIV. $[\Delta^{J-1}(X), M^*] = 0$

• FOR LEVEL-0 $X \in \mathfrak{sl}(2, \mathbb{R})$: • ALWAYS THE ROOT GENERATOR $\hat{M} := \Delta^{J-1}(e)$

WITH A CONTINUOUS SPECTRUM

(IDEM @ SIGMA-MODEL SIDE)

• NEVER THE CARTAN GENERATOR $\hat{K} = \Delta^{J-1}(h)$

WITH A DISCRETE SPECTRUM $(|k_1, k_2\rangle \sim z_1^{k_1} z_2^{k_2})$

TWISTED MONODROMY : $T_a^F(u) = \Omega \exp(\log(1 + \xi e_a) \otimes \Delta^{J-1}(h)) T_a(u) \underbrace{\exp(-h_a \otimes \log(1 + \xi \Delta^{J-1}(e)))}_{\tilde{P}_a^{-1}} \Omega^{-1}$

[SB, MOLINES '25]

$\Rightarrow$ AFTER SIMILARITY TRANSF. BY Ω WE NEED TO ORGANISE SPECTRUM BY $\hat{M}$

TWISTED-BOUNDARY COND^S

[SD, MOLINER '25]

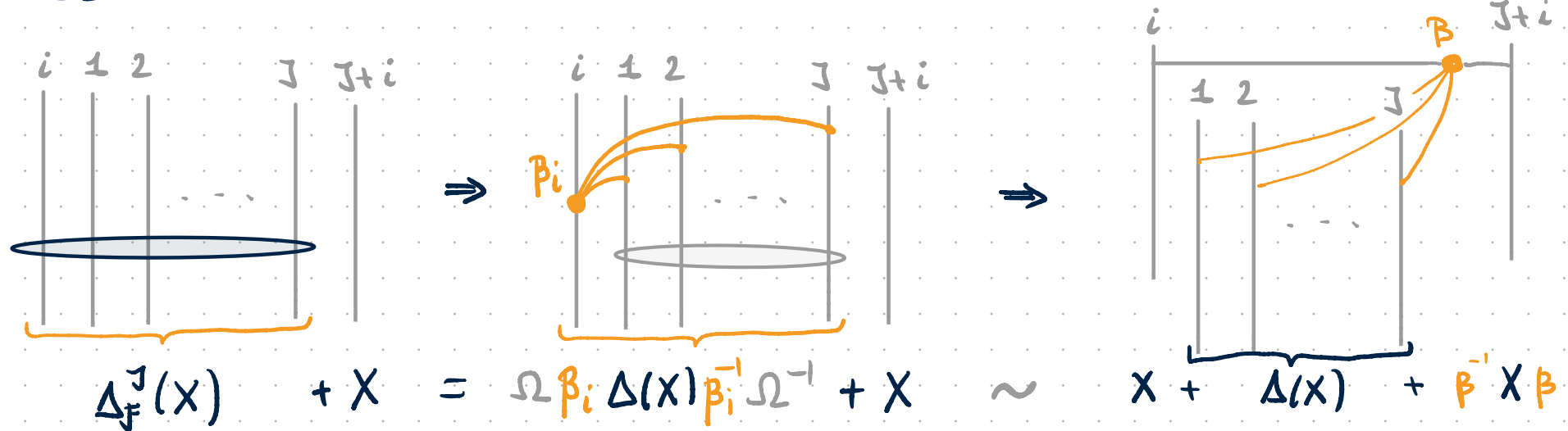
EXPECTATION: ONE CAN MAP $(\Delta_F(X), R_F, \dots)$ WITH DEF. INTERACTIONS $\tilde{H}(F)$
(CLOSED CHAINS) AND $X_{j+i} = X_i$

TO $(\Delta(X), R, \dots)$ WITH UNDEF. H

BUT $X_{j+i} = \mathcal{B}_F(X_i) \neq X_i$

↳ SYMM. ACTION 1...J

EXPLICIT MAP: BY GLUING PROCEDURE OF OPEN TO CLOSED CHAIN



$$X_{j+i} = \beta_i X_i \beta_i^{-1} \quad \text{w. } \beta_i := (\mathbb{1}_i \times \Delta^{j-1})(F)$$

HERE :

$$e_{j+i} = e_i (1 + \xi \hat{M}) , \quad h_{j+i} = h_i , \quad f_{j+i} = f_i (1 + \xi \hat{M})^{-1}$$

SPECTRAL PROBLEM

[SD, MOLINES '25]

[SD, LEVKOVICH-MASLYUK, MOLINES '25]

① THERMODYN. LIMIT ($J \rightarrow \infty$):

② LENGTH $J = 2$:

③ ALL J :

SPECTRAL PROBLEM

[SD, MOLINES '25]

① THERMODYN. LIMIT ($\gamma \rightarrow \infty$):

SPECTRAL PROBLEM

[SD, MOLINES '25]

① THERMODYN. LIMIT ($J \rightarrow \infty$): [STEFANSKI, TSEYTLIN '04]

• UNDEFORMED INTERACTIONS

$$H_{QM} \xrightarrow[\text{FIELDS}]{J \rightarrow \infty}$$

$$H_{LL} = \frac{J \tilde{\lambda}}{16\pi} \int_0^{2\pi} d\sigma \left[-(\partial_\sigma n_h)^2 + \partial_\sigma n_e \partial_\sigma n_f \right] + \dots$$

COHERENT STATES:

$$n_h \sim \langle n | h | n \rangle$$

$$n_e \sim \langle n | e | n \rangle$$

$$n_f \sim \langle n | f | n \rangle$$

• TWISTED BOUNDARY CONDITIONS

$$n_e(\tau, 2\pi) = n_e(\tau, 0) (1 + \xi M_{LL}) \quad ; \quad n_h(\tau, 2\pi) = n_h(\tau, 0) \quad ; \quad n_f(\tau, 2\pi) = n_f(\tau, 0) (1 + \xi M_{LL})^{-1}$$

WITH

$$M_{LL} \sim \int_0^{2\pi} d\sigma' n_e(\tau, \sigma') \sim \Delta^{J+1}(e)$$

WE FIND TWISTED SOLUTIONS TO THE E.O.M'S WITH ENERGY

$$\xi = \frac{2\pi}{\sqrt{\lambda}} J$$

$$\tilde{\lambda} = \lambda / J^2$$

$$E_{LL} = \frac{1}{32\pi^2} \frac{(\log(1 + \xi M))^2}{J} = \frac{J^2 M^2}{8J} + O(J^{-2}) \quad \text{VS.} \quad E_{STR.} = \sqrt{J + \frac{J^2 M^2}{4}} = J + \frac{J^2 M^2}{8J} + O(J^{-1})$$

SPECTRAL PROBLEM

[SD, MOLINES '25]

① THERMODYN. LIMIT ($J \rightarrow \infty$): [STEFANSKI, TSEYTLIN '04]

• UNDEFORMED INTERACTIONS

$$H_{QM} \xrightarrow[\text{FIELDS}]{J \rightarrow \infty} H_{LL} = \frac{J\tilde{\Lambda}}{16\pi} \int_0^{2\pi} d\sigma \left[-(\partial_\sigma n_h)^2 + \partial_\sigma n_e \partial_\sigma n_f \right] + \dots$$

COHERENT STATES:

$$\begin{aligned} n_h &\sim \langle n|h|n \rangle \\ n_e &\sim \langle n|e|n \rangle \\ n_f &\sim \langle n|f|n \rangle \end{aligned}$$

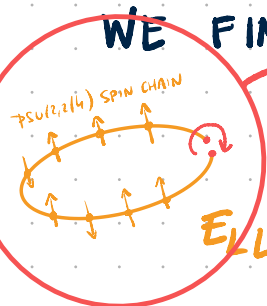
• TWISTED BOUNDARY CONDITIONS

$$n_e(\tau, 2\pi) = n_e(\tau, 0) (1 + \xi M_{LL}) \quad ; \quad n_h(\tau, 2\pi) = n_h(\tau, 0) \quad ; \quad n_f(\tau, 2\pi) = n_f(\tau, 0) (1 + \xi M_{LL})^{-1}$$

WITH

$$M_{LL} \sim \int_0^{2\pi} d\sigma' n_e(\tau, \sigma') \sim \Delta^{J+1}(e)$$

WE FIND TWISTED SOLUTIONS TO THE E.O.M'S WITH ENERGY



$$E_{LL} = \frac{1}{32\pi^2} \frac{(\log(1+\xi M))^2}{J} = \frac{J^2 M^2}{8J} + O(J^{-2})$$

VS.

$$E_{STR.} = \sqrt{J + \frac{J^2 M^2}{4}} = J + \frac{J^2 M^2}{8J} + O(J^{-1})$$



SPECTRAL PROBLEM

② LENGTH $J=2$:

[SD, MOLINES '25]
[SD, LEVKOVICH-MASLYUK, MOLINES '25]

SPECTRAL PROBLEM

[SD, MOLINER '25]

② LENGTH $J=2$:

KEY : PUT CHAIN IN $\hat{M} = \Delta(e)$ -EIGENBASIS $\leftrightarrow$ DISCRETE FOCK SPACE IN $\Delta(h)$ -EIGENBASIS
(USED FOR BETHE ANSATZ)

$$\hat{M}\Psi_M = M\Psi_M \quad \Rightarrow \quad \Psi_M(z_1, z_2) = e^{M/2(z_1+z_2)} g(z_1-z_2)$$

FIND SPECTRUM OF :

$$\hat{\tau}_\xi(u) \sim \text{TR} \left[\begin{pmatrix} 1 & \xi \Delta(h) \\ & 1 \end{pmatrix} \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1+\xi \hat{M}}} & \\ & \sqrt{1+\xi \hat{M}} \end{pmatrix} \right] \quad \text{ON } \Psi_M \rightarrow \text{NO ABA}$$

LIKE FROBENIUS : 3RD ORDER ODE IN z

$\rightarrow$ to $\mathcal{O}(\xi^6 M^6)$ [DM '25]

$\rightarrow$ WITH SPIN IN [DLHM '25]

$$g(z) = 1 + \frac{M^2 z^2}{24} + \frac{\xi M^2 z}{24} + \dots \quad \& \quad \tau_\xi(u) = u^2 \left(\frac{2 + \xi M}{\sqrt{1 + \xi M}} \right) + u^0 \left(-\frac{1}{2} - \frac{\xi^2 M^2}{48} + \mathcal{O}(\xi^3 M^3) \right)$$

$\Rightarrow$ DEFORMED IN $M \neq 0$ - SECTORS $\leftrightarrow$ UNDEFORMED IN $\Delta(h)$ STATES [KULISH, STOLIN '97, BORSATO, FERNÁNDEZ '25]
(JORDAN BLOCKS & $M=0$ STATES)

SPECTRAL PROBLEM

[SD, MOLINES '25]

[SD, LEVKOVICH-MASLYUK, MOLINES '25]

② LENGTH 1=2 :

- USE EIGENFUNCTIONS Ψ_M OF $\hat{T}_3(u)$ TO FIND SPECTRUM OF HAMILTONIAN IN INTEGRAL REP.

$$H_{12} \Psi_M(z_1, z_2) = \frac{1}{2} \int_0^1 \frac{dv}{v} \left(\Psi_M(z_1, z_2) - \Psi_M(z_1 + v(z_2 - z_1), z_2) - \Psi_M(z_1, z_2 - v(z_2 - z_1)) \right)$$

$$H^* = H_{12} + F_{12}^{-1} F_{21} H_{12} F_{21}^{-1} F_{12}$$

WE FIND: $E = \frac{\lambda}{\pi^2} \left(\frac{g^2 M^2}{96} - \frac{g^3 M^3}{96} + \frac{131}{13824} g^4 M^4 + \dots \right)$ *

- USE EIGENVALUES FOR BAXTER TQ RELATION

$$T_3(u) Q_3(u) = (u - i/2)^2 Q_3(u - i) + (u + i/2)^2 Q_3(u + i)$$

FROM YANGIAN REP,
BA, ...

EXPECTED TO BE
THE SAME FOR ALL
RATIONAL MODELS

FOR ENERGY: $E = \frac{i\lambda}{8\pi^2} \partial_u \log \frac{Q_3(u + i/2)}{Q_3(u - i/2)} \Big|_{u=0}$

→ MATCHES WITH * USING $Q_3(u) = \sum_{k=0}^{\infty} \sum_{n=0}^k c_{n,k} u^n (gM)^k$ ANSATZ

Spectral Problem

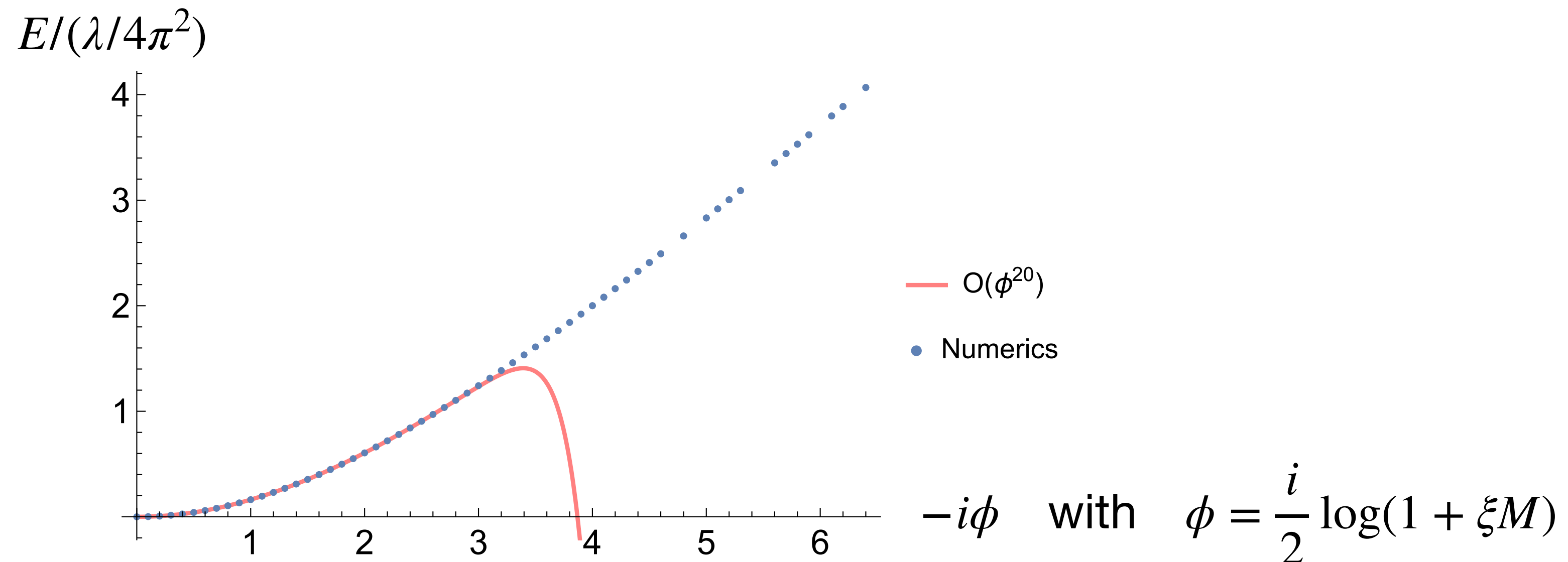
[SD, Levkovich-Maslyuk, Molines '25]

2) Length $J = 2$

In fact: the **Baxter eq.** reproduces all data [E and $\tau_\xi(u)$] by taking the ansatze

$$\tau_\xi(u) = \left(\frac{2 + \xi M}{\sqrt{1 + \xi M}} \right) u^2 + \left(-\frac{1}{2} + \sum_{k=1}^{\infty} \tau_k (\xi M)^k \right) u^0 \quad \text{and} \quad Q_\xi(u) = \sum_{k=0}^{\infty} \sum_{n=0}^k c_{n,k} u^n (\xi M)^k$$

Numerics:



Also works for higher-spin states: $E_S/(\lambda/4\pi^2) = 2h(S) + \frac{1}{8} \left(\frac{S(S+1) - 1}{4S(S+1) - 4} \right) \xi^2 M^2 + \dots$ for $S = 1, \dots, 20$

Spectral Problem

3) All length J

[SD, Molines '25]

[SD, Levkovich-Maslyuk, Molines '25]

Spectral Problem

[SD, Molines '25]

3) All length J

[SD, Levkovich-Maslyuk, Molines '25]

Large u -asymptotics:

$$\hat{\tau}_\xi(u) = 2 \cos \hat{\phi} u^J + 0 \times u^{J-1} + \dots$$

$$\tau_\xi(u) = 2 \cos \phi u^J + 0 \times u^{J-1} + \dots$$

- no $\Delta^{J-1}(h)$ operator
- trivial subleading terms indicates 'nice' Q -asymptotics

$$Q \sim \frac{1}{u} (e^{u\phi} - e^{-u\phi})$$

Solving Baxter and computing energy for $J = 2, \dots, 8$ we find

$$E = -\frac{\lambda\phi^2}{8\pi^2(J+1)} - \frac{\lambda\phi^4}{96\pi^2(J+1)^2} - \frac{(J+3)\lambda\phi^6}{2880\pi^2(J+1)^3} - \frac{(10(J-1)^3 + 100(J-1)^2 + 345(J-1) + 408)\lambda\phi^8}{483840\pi^2(J+1)^4(J+3)} + O(\phi^{10})$$

$$= \left(\frac{\eta^2 M^2}{8J} \right) \left(-\frac{\eta^2 M^2}{8J^2} \right) + \mathcal{O}(\lambda^{-1/2}, J^{-3})$$

$$\xi = \frac{2\pi}{\sqrt{\lambda}} \eta$$

Spectral Problem

[SD, Molines '25]

3) All length J

[SD, Levkovich-Maslyuk, Molines '25]

Large u -asymptotics:

$$\hat{\tau}_\xi(u) = 2 \cos \hat{\phi} u^J + 0 \times u^{J-1} + \dots$$

$$\tau_\xi(u) = 2 \cos \phi u^J + 0 \times u^{J-1} + \dots$$

- no $\Delta^{J-1}(h)$ operator
- trivial subleading terms indicates 'nice' Q-asymptotics

$$Q \sim \frac{1}{u} (e^{u\phi} - e^{-u\phi})$$

Solving Baxter and computing energy for $J = 2, \dots, 8$ we find

$$E = -\frac{\lambda\phi^2}{8\pi^2(J+1)} - \frac{\lambda\phi^4}{96\pi^2(J+1)^2} - \frac{(J+3)\lambda\phi^6}{2880\pi^2(J+1)^3} - \frac{(10(J-1)^3 + 100(J-1)^2 + 345(J-1) + 408)\lambda\phi^8}{483840\pi^2(J+1)^4(J+3)} + O(\phi^{10})$$

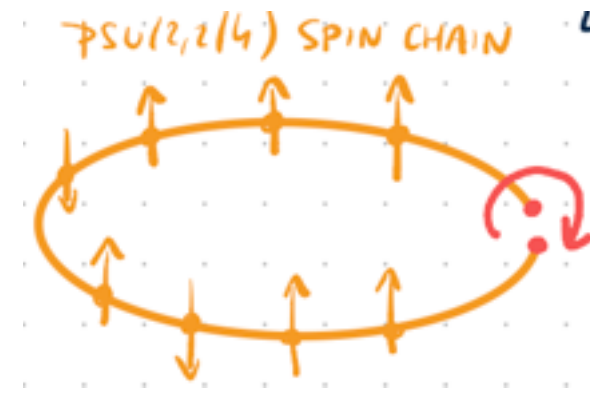
$$= \frac{\eta^2 M^2}{8J} - \frac{\eta^2 M^2}{8J^2} + \mathcal{O}(\lambda^{-1/2}, J^{-3})$$

$$\xi = \frac{2\pi}{\sqrt{\lambda}} \eta$$

matches classical string energy !

and its one-loop shift !! [Borsato, SD, Nieto-Garcia, Wyss '22]

Spectral Problem



[SD, Molines '25]

[SD, Levkovich-Maslyuk, Molines '25]

3) All length J

Large u -asymptotics:

$$\hat{\tau}_\xi(u) = 2 \cos \hat{\phi} u^J + 0 \times u^{J-1} + \dots$$

$$\tau_\xi(u) = 2 \cos \phi u^J + 0 \times u^{J-1} + \dots$$

- no $\Delta^{J-1}(h)$ operator
- trivial subleading terms indicates 'nice' Q-asymptotics

$$Q \sim \frac{1}{u} (e^{u\phi} - e^{-u\phi})$$

Solving Baxter and computing energy for $J = 2, \dots, 8$ we find

$$E = -\frac{\lambda\phi^2}{8\pi^2(J+1)} - \frac{\lambda\phi^4}{96\pi^2(J+1)^2} - \frac{(J+3)\lambda\phi^6}{2880\pi^2(J+1)^3} - \frac{(10(J-1)^3 + 100(J-1)^2 + 345(J-1) + 408)\lambda\phi^8}{483840\pi^2(J+1)^4(J+3)} + O(\phi^{10})$$

$$= \frac{\eta^2 M^2}{8J} - \frac{\eta^2 M^2}{8J^2} + \mathcal{O}(\lambda^{-1/2}, J^{-3})$$

$$\xi = \frac{2\pi}{\sqrt{\lambda}} \eta$$



matches classical string energy !

and its one-loop shift !! [Borsato, SD, Nieto-Garcia, Wyss '22]

Conclusions

Jordanian string:

- is Lax integrable: **two formulations** (deformed and twisted) **lead to spectral curve results** ['22,'24]
- S-matrix integrable? **particle production with massive asymptotic states** ['24]

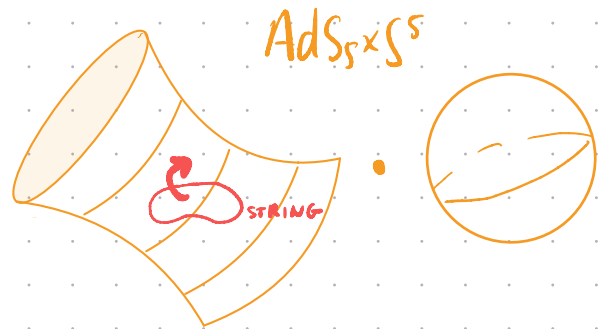
other notions of integrable scattering? $S_\eta = F S F^{-1}$? TBA? ...?

Drinfel'd twisted spin chains:

- general formulas for **twisted monodromy matrix** (and boundary conditions) ['25 - part 1]
- essential to use residual symmetry states :: **Jordanian** $\rightarrow$ **only a root generator** $\hat{M} = \Delta(e)$. **No Cartan.**
- still: it allowed to study the spectrum using the **Baxter equation** ! ['25 - part 2]
 - J=2 : numerics and perturbative results in ξ (equiv. ϕ) *analytic solutions at all ξ/ϕ ?*
 - all J : **match** with ground state solution + one-loop shift of the Jordanian **string** *excitations?*

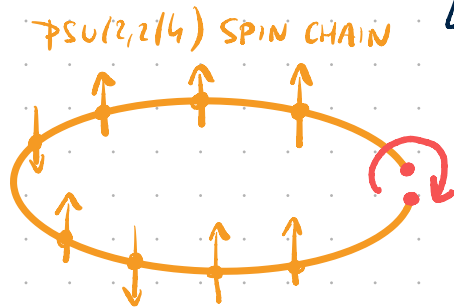
... particle number is not always a good organisation for integrability-studies

... we shouldn't discard QFTs with particle production — integrability may still be hiding there



$AdS_5 \times S^5$

"R-MATRIX" DEFORMATIONS



$PSU(2,2|4)$ SPIN CHAIN

DRINFELD-TWIST



$\mathcal{N}=4$ SYM
 $\text{Tr}(F_{\mu\nu} * F^{\mu\nu})$



TWIST - NON-COMMUTATIVE
 FIELD PRODUCTS

26xx.xxxxx
 you et. al.??

THANK YOU!

