Simple harmonic oscillators from non-semisimple Walled Brauer algebras.

S. Ramgoolam

Queen Mary, University of London

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Based on:
"Simple harmonic oscillators from non-semisimple walled Brauer algebras,"
S. Ramgoolam, M. Studzinski , https://arxiv.org/abs/2509.04234

Introduction : Representation theory of $U(N)$

- Motivating applications : AdS/CFT correspondence, quantum information theory.
- Gauge symmetry group in gauge theory, group of unitary transformations preserving inner product in a Hilbert space.
- $V_N^{\otimes n}$, $N \geq n$: Schur-Weyl duality

$$V_N^{\otimes n} = \bigoplus_{Y \vdash n} V_Y^{U(N)} \otimes V_Y^{S_n}$$

$\mathbb{C}(S_n) \in \mathrm{End}(V_N^{\otimes n})$ is the commutant of $U(N)$.

- Change of basis

$$|i_1, \cdots, i_n\rangle \to |Y, M_Y, m_Y\rangle$$

Introduction : Multiplicities from Young diagrams

$$d_Y = \frac{n!}{\prod_{a \in Y} \text{hook lengths}(a)} = \frac{n!}{H(Y)}$$

Example :

$$V_N^{\otimes 3} = V_{\yng(3)} \oplus 2\,V_{\yng(2,1)} \oplus V_{\yng(1,1,1)}$$

$$\young(31,1) \quad ; \quad \frac{3!}{3 \times 1 \times 1} = 2$$

Introduction : $N < n$ - simple cut-off on Young diagrams

$$V_N^{\otimes n} = \bigoplus_{ht(Y) \leq N} V_Y^{U(N)} \otimes V_Y^{S_n}$$

Introduction : The mixed tensor case

$N \geq (m+n)$:

$$V_N^{\otimes m} \otimes \overline{V}_N^{\otimes n} = \bigoplus_{\gamma} V_{\gamma}^{U(N)} \otimes V_{\gamma}^{B_N(m,n)}$$

Brauer Representation triples, $\mathrm{BRT}(m,n)$:

$\gamma = (k, \gamma_+, \gamma_-)$

$0 \leq k \leq \min(m,n)$

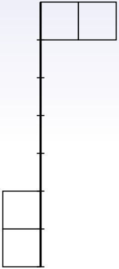
γ_+ is a Young diagram with $(m-k)$ boxes

γ_- is a Young diagram with $(n-k)$ boxes

$\Gamma(\gamma, N)$ - mixed Young diagram ; highest weight of $V_{\gamma}^{U(N)}$

$$R_i(\gamma) = r_i(\gamma_+) \quad \text{for} \quad 1 \leq i \leq c_1(\gamma_+)$$
$$R_{N-i+1}(\gamma) = -r_i(\gamma_-) \quad \text{for} \quad 1 \leq i \leq c_1(\gamma_-)$$

For $N = 7$, $\left(k = 0, \Box\!\Box, \begin{smallmatrix}\Box\\\Box\end{smallmatrix}\right)$.



Introduction : The mixed tensor case

In this range $N \geq (m+n)$, $\dim(V_\gamma^{B_N(m,n)})$ is independent of N :

$$\dim(V_\gamma^{B_N(m,n)}) = d_{m,n}(\gamma) = \frac{m!n!}{k!H(\gamma_+)H(\gamma_-)}$$

Large N regime, or stable regime, where the algebra $B_N(m,n)$ is semi-simple, i.e. has a non-degenerate trace form.

Introduction : The mixed tensor case - $B_N(m, n)$ is a diagram algebra.

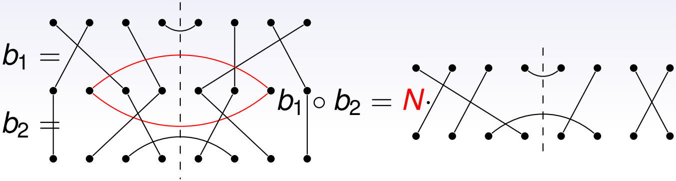


Figure: Example of graphical composition of two diagrams $b_1, b_2 \in B_N(4, 4)$. Identifying a closed loop (in red) results in multiplying the diagram by a scalar $N \in \mathbb{C}$. We see that the composition $b_1 \circ b_2$ remains within $B_N(4, 4)$.

Introduction : The mixed tensor case for $N < (m+n)$

- $B_N(m,n)$ is no longer a semi-simple algebra.
- The map representing the diagrams as operators in tensor space, which associates the lines to contractions δ_{ij}, has a non-trivial kernel.
- The quotient is a semi-simple algebra $\widehat{B}_{m,n}(N)$.
- There is SW duality

$$V_N^{\otimes m} \otimes \overline{V}_N^{\otimes n} = \bigoplus_{\substack{\gamma \in \mathrm{BRT}(m,n) \\ \mathrm{height}(\gamma) = c_1(\gamma_+) + c_1(\gamma_-) \leq N}} V_\gamma^{U(N)} \otimes V_\gamma^{\widehat{B}_N(m,n)},$$

But in general

$$d_{m,n,N} = \dim(V_\gamma^{\widehat{B}_N(m,n)}) = d_{m,n} - \delta_{m,n,N}$$

where $\delta_{m,n,N} \geq 0$ and has a description in terms of Bratteli diagrams (Stoll and Werth, 2016 + earlier math papers).

Introduction : The mixed tensor case for $N < (m+n)$.

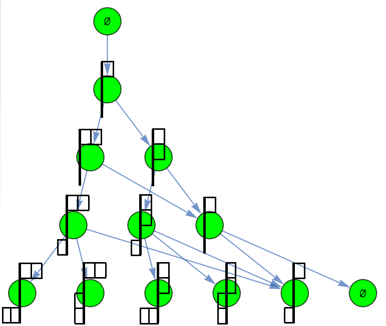
- General formulae for $\delta_{m,n,N}$ not available.
- We introduced a simplification of Bratteli diagrams, restricted Bratteli diagrams (RBD), which uses information from the stable regime to calculate rep. theory data in the non-semi-simple regime.
- Find that for $N = m + n - l$, these only depend on l for $m, n \geq (2l - 3)$.
- For $l \in \{1, 2, 3, 4\}$, and general m, n we calculate the dimension modifications.
- Found surprising connections to a partition function for an infinite tower of harmonic oscillators

$$\mathcal{Z}_{\text{univ}}(x) = \frac{x}{(1-x)(1-x^2)} \prod_{i=1}^{\infty} \frac{1}{(1-x^i)^2}$$

Outline

- Explain the Bratteli diagrams for $B_N(m, n)$ and the known algorithm.
- Define the RBD
- Relations between RBD and oscillator partition function.
- Motivations and future directions.

Bratteli diagrams in the large N regime: Example $B_{N\geq4}(2,2)$



The set of mixed Young diagrams in the bottom row corresponds to the set $\mathrm{BRT}(2,2)$. The set at intermediate levels:

$$\mathrm{BRT}(0,0) \to \mathrm{BRT}(1,0) \to \mathrm{BRT}(2,0) \to \mathrm{BRT}(2,1) \to \mathrm{BRT}(2,2)$$

Bratteli paths and $d_{m,n}(\gamma)$

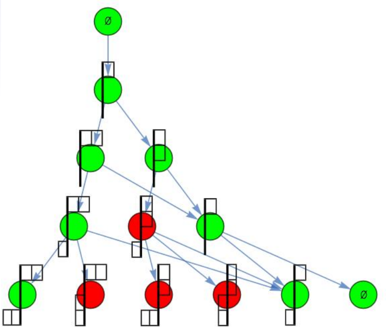
For each $\gamma \in \mathrm{BRT}(m, n)$:

$$\text{number of paths starting from the level 0 diagram and ending at } \gamma = d_{m,n}(\gamma)$$

In this example 1 path for $(k = 0, \gamma_+ = \square\square, \gamma_- = \;, \begin{array}{c}\square\\\square\end{array})$, and 4 paths for $(k = 1, \gamma_+ = \square, \gamma_- = \square)$.

Colored Bratteli diagrams for $N < (m+n)$: Excluded triples $\mathrm{height}(\gamma) > N$ are coloured red.

Example $B_{N=2}(2,2)$. $\gamma \in \mathrm{BRT}(2,2)$ with $\mathrm{height}(\gamma) > 2$ are coloured red.



Irreps of $\widehat{B}_N(m,n)$ correspond to green nodes in the last layer.

Colored Bratteli diagrams for $N < (m+n)$: Admissible paths

Some green nodes in the last layer have a subset of paths going through red nodes in earlier layers.

$\widehat{d}_{m,n}(\gamma)$ is equal to the number admissible paths, i.e. paths going through green nodes only, i.e. not passing through any red nodes.
In this case, $(k = 1, \gamma_+ = \square, \gamma_- = \square)$ has one path going through a red diagram,

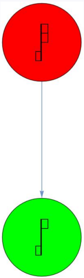
$$\widehat{d}_{m,n,N}(\gamma) = 4 - 1$$

Restricted Bratteli diagrams : simplified versions of the CBD

- Only contain at the last layer, the green nodes with a modified dimension.
- Only contain red nodes in the earlier layers which admit paths to the above green nodes, i.e. nodes relevant to the dimension modification

$$d_{m,n}(\gamma) \rightarrow d_{m,n,N}(\gamma) = d_{m,n}(\gamma) - \delta_{m,n,N}(\gamma)$$

- Only contain green nodes in earlier layers which appear in paths from the red nodes to the green nodes in the final layer.
- In the example :



- The dimension modification is 1 because for the red

$$\gamma = (k = 0, \gamma_+ = \begin{smallmatrix}\square\\\square\end{smallmatrix}, \gamma_- = \square \in \mathrm{BRT}(2,1))$$

$$d_{2,1}(\gamma) = 1$$

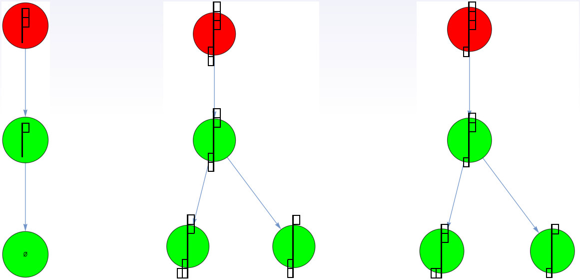
Restricted Bratteli diagrams : code and stability.

Based on the above characterisation, we wrote the mathematica-code for RBD. By modifying - with the help of ChatGPT- code written by Quantum information theorists for the standard Bratteli diagrams. (Dmitry Grinko, Adam Burchardt, and Maris Ozols, 2023)

Study $N = (m + n - l)$, with l small compared to m, n. We find from the code, and prove, stability of the RBD for $m, n \geq (2l - 3)$.

Restricted Bratteli diagrams : Stability example $l = 3$ ..

$B_{N=1}(2,2)$ $B_{N=3}(3,3)$ $B_{N=4}(3,4)$



Reds in the RBD : excess Δ and depth d

The red diagrams have $c_1(\gamma_+) + c_1(\gamma_-) > (m+n-l)$. Let

$$c_1(\gamma_+) + c_1(\gamma_-) = (m+n-l) + \Delta$$

where Δ is defined as the excess total height of γ : $\Delta \geq 1$.
Consider diagrams at depth d, irreps of $B_N(m, n-d)$, with labels (k, γ_+, γ_-).
Let $|\gamma_+|$ and $|\gamma_-|$ be the number of boxes in γ_+, γ_- respectively.

$$|\gamma_+| = m - k = c_1(\gamma_+) + |\gamma_+ \setminus c_1|$$
$$|\gamma_-| = n - k - d = c_1(\gamma_-) + |\gamma_- \setminus c_1|$$

Can use to derive

$$\boxed{d + 2k + |\gamma_+ \setminus c_1| + |\gamma_- \setminus c_1| + \Delta = l}$$

Immediately get, for red node,

$$d \leq (l - 1)$$

RBD have maximal depth $(l - 1)$.

Reds in the RBD : Counting formula

$$\boxed{d + 2k + |\gamma_+ \setminus c_1| + |\gamma_- \setminus c_1| + \Delta = l}$$

This boxed equation along with structure of Bratteli moves :

$$\Delta(d) \leq \min(l - d, d)$$

Counting of red nodes as a function of l and d - sum over partitions $|\gamma_+ \setminus c_1|$ and $|\gamma_- \setminus c_1|$ with specified number of nodes.

$$\mathcal{R}(l, d) = \sum_{\Delta=1}^{\min(l-d,d)} \sum_{k=0}^{\lfloor \frac{(l-d-\Delta)}{2} \rfloor} \sum_{l_1=0}^{l-d-2k-\Delta} p(l_1) p(l - d - 2k - \Delta - l_1)$$

Reds in the RBD : two simplidications and oscillators

$$\mathcal{Z}_{\rm univ}(x) = \sum_{s=0}^{\infty} x^s \, \mathcal{Z}(s) = \frac{x}{(1-x)(1-x^2)} \prod_{i=1}^{\infty} \frac{1}{(1-x^i)^2}$$

For d near l and d small we have two simplifications of the inequality $\Delta \leq \min(l-d, d)$.

$$\mathcal{R}(l,d) = \begin{cases} \mathcal{Z}_{\rm univ}(l-d) - \mathcal{Z}_{\rm univ}(l-2d) & \text{for } 1 \leq d \leq \lfloor \frac{l}{2} \rfloor \\ \mathcal{Z}_{\rm univ}(l-d) & \text{for } \lceil \frac{l}{2} \rceil \leq d \leq (l-1) \end{cases}$$

Green nodes : Bijection and oscillator partition function.

Lemma There is a bijection between green nodes at $d = 0$ and reds with $\Delta = 1$ at $d \geq 1$.

$$\boxed{\mathcal{G}(l, d = 0) = \mathcal{Z}_{\text{univ}}(l - 1)}$$

Physics motivations and future directions : AdS/CFT

- Orthogonal basis of polynomial matrix invariant functions of $Z, Z^\dagger$ in free-QFT-inner product is obtained from the matrix-basis (Artin-Wedderburn basis) for the $S_m \times S_n$ invariant subpspace of $\widehat{B}_N(m, n)$, labelled by γ

$$Q^{\gamma}_{r_1, r_2, \mu\nu}$$

Kimura, Ramgoolam, 2007

Physics motivations and future directions : AdS/CFT

Holomorphic invariants – are mapped to giant gravitons using the orthogonal basis in free QFT labelled by Young diagrams (Corley,Jevicki,Ramgoolam, 2001).

The non-holomorphic invariants should contain the phyiscs of something like brane-anti-brane systems (continued from the geometrical strong yang-Mills coupling limit to the weak coupling limit).

Explicit construction of the Artin-Wedderburn matrix basis for $\widehat{B}_N(m,n)$ and the $S_m \times S_n$ invariant subpsace, in the region $m+n-l=N$ with l small - should have an interpretation in terms of fluctuations of brane/anti-branes).

Physics motivations and future directions : Quantum Information

In the quantum teleportation context, N is the dimension of a Hilbert space H; Alice and Bob share m entangled pairs in $(H \otimes H)^{\otimes m}$ and they use this as a resource to teleport states in $H^{\otimes n}$. The fidelity of quantum teleportation is expressed in terms of an appropriate trace of an element in $B_N(m, n)$.

Marek Mozrzymas, Michal Studzinski, and Piotr Kopszak. Optimal multi-port-based teleportation schemes. Quantum, 5:477, 2021.

Artin-Wedderburn basis also important in the QI calculations.

Addtional technical slide

- Matrix invariants and walled-Brauer algebra elements :

$$trZZ^\dagger = Z^{i_1}_{i_2}(Z^\dagger)^{i_2}_{i_1} = \mathrm{tr}_{V_N^{\otimes 2}}(Z \otimes Z^\dagger \sigma)$$

Matrices as linear operators, σ is the permutation operator on tensor space :

$$Z|e_i\rangle = Z^{j_1}_{i}|e_j\rangle$$
$$\sigma(|e_{i_1}\rangle \otimes |e_{i_2}\rangle) = |e_{i_2}\rangle \otimes |e_{i_1}\rangle$$

$$trZZ^\dagger = Z^{i_1}_{i_2}(\overline{Z})^{i_1}_{i_2} = \mathrm{tr}_{V_N \otimes \bar{V}_N}(Z \otimes \overline{Z})b$$

$$b|e_{i_1}\rangle \otimes |e_{i_2}\rangle = \delta_{i_1 i_2}|e_k\rangle \otimes |e_k\rangle$$