Non-invertible defects from the Conway SCFT to K3 sigma models

Roberta Angius
IFT UAM/CSIC (Madrid)

Based on:

[hep-th:2402.08719] with S. Giaccari and R. Volpato

[hep-th/math-qa:2504.18619]
[hep-th/math-qa:250x.xxxxx] with S. Giaccari, S. Harrison and R. Volpato

[hep-th:2508.03612] with S. Giaccari

Κέρκυρα, September 11, 2025

The context

Standard Symmetries

| |

Topological operators (→ spacetime defects) supported on codimension 1 submanifolds of spacetime, satisfying **group-like** and **invertible** fusion rules and acting on **Local operators.**

GENERALIZATION

Higher symmetry groups

[Gaiotto Kapustin,Seiberg,Willet — 2014]

Symmetries acting on extended operators

Non-invertible symmetries

[Fuchs,Runkel,Schweigert, Bhardway,Tachikawa, Froehlich …]

Topological operators satisfying a non-invertible fusion algebra

In recent years, extensive work has been done to study the categories of topological defects in **many** physical domains: **worldsheet String Theory**, String Theory [Apruzzi et al -2022,…], QFT, condensed matter [Freed et al - 2022,…], lattice models …

State of the art in the classification of TDs in 2d CFTs:

- Invertible defects associated with standard and higher-form symmetries
- Verlinde lines in RCFTs
  [Verlinde - 1988]
- Duality defects in self-orbifold constructions
  [Tambara, Yamagami - 1988; Bhardwaj, Tachikawa - 2017]

Main purposes:

Study **topological defect lines (TDLs)** in 2-dimensional SCFTs arising as supersymmetric **non-linear σ-models** with target space a **K3** surface:

- Preserving the full $\mathcal{N} = (4{,}4)$ superconformal algebra with central charge $(c, \bar{c}) = (6{,}6)$
- Invariant under the spectral flow transformations

(My talk at "Quantum Gravity, Strings and the Swampland", Corfù 2024)

Study **topological defect lines (TDLs)** in the 2-dimensional **Conway SCFT** $V^{f\natural}$ (the unique holomorphic SCFT in 2d with $c = 12$ and no field of conformal weight $1/2$):

- Commuting with the $\mathcal{N} = 1$ superconformal algebra and $(-1)^F$;
- Satisfying some additional technical constraints

A mysterious relationship

A **mysterious relationship** between $V^{f\natural}$ and K3 NLSMs has been observed in **[Duncan, Mack-Crane - 2015]**

	$V^{f\natural}$	K3 NLSMs
SYMMETRIES [Gaberdiel, Hohenegger, Volpato - 2012]	Co_o	$G_{K3} \subset Co_0$
SYMMETRY ACTION [Duncan, Mack-Crane - 2015]	R sector ground states	RR ground states
TWINING GENUS [Duncan, Mack-Crane - 2015] [Cheng, Harrison, Volpato, Zimet - 2018]	$\phi^g(V^{f\natural}, \tau, z) = Tr_{V^{f\natural}_{tw}}\left[g(-1)^F q^{L_0-1/2} y^{J_0^3}\right]$	$\phi^g(\mathscr{C}_\Pi, \tau, z) = Tr_{RR}\left[g(-1)^{F+\bar{F}} q^{L_0-1/4} \bar{q}^{\bar{L}_0-1/4} y^{J_0^3}\right]$

Special "modified" partition functions

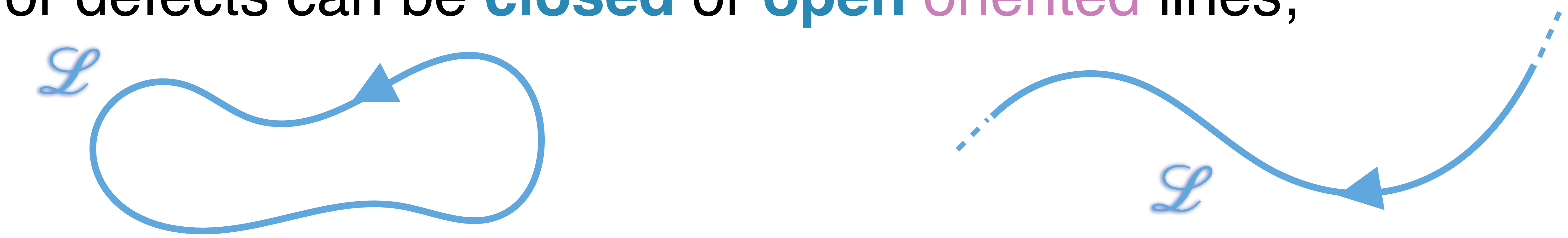
Does this connection between symmetries admits a generalization at the level of generalized symmetries?

Generalities on Topological Defects in QFT

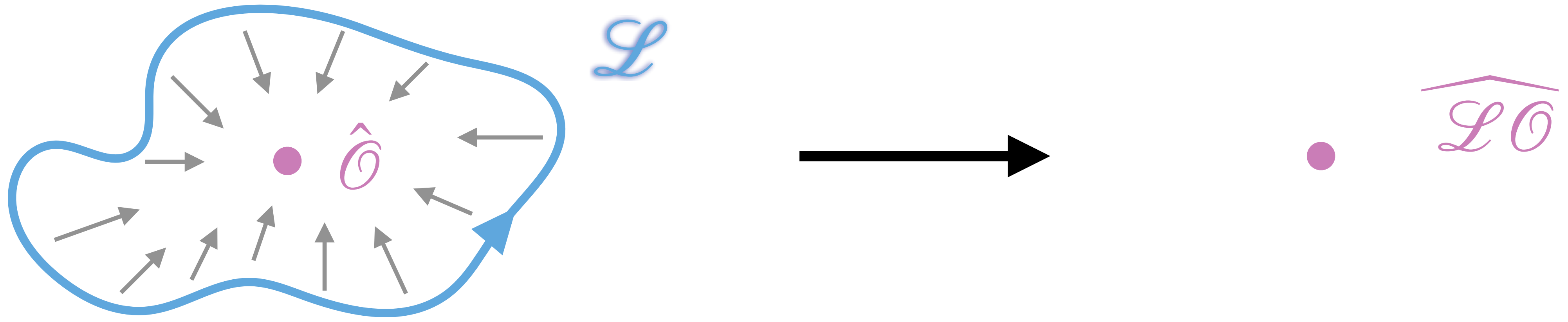
Topological Defect Lines (TDLs) in 2d QFTs

Defect Lines in a generic 2d QFT $\mathscr{Q}$ are extended operators $\hat{\mathscr{L}}(\gamma)$ supported on one-dimensional submanifolds γ (lines) of the two-dimensional spacetime $\mathscr{M}_2$.
Topological = Correlators invariant under small deformations of the line γ that do not cross other insertions.

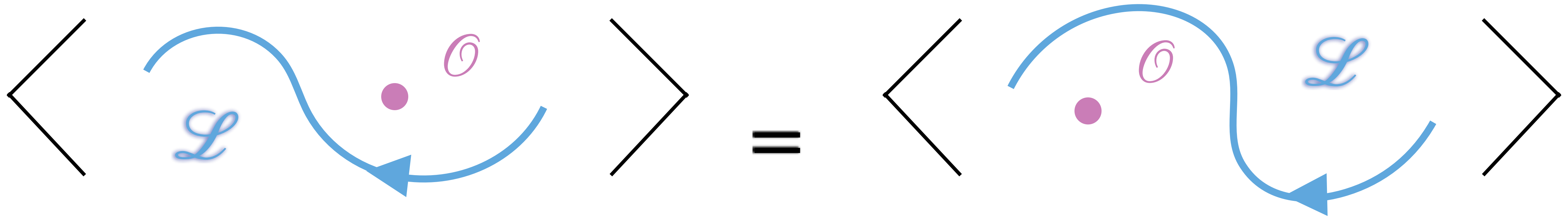
- The support manifolds for defects can be **closed** or **open** oriented lines;



- TDLs **act** on local operators of the theory producing **new** local operators.



- When the moving of $\mathscr{L}$ across a local operator $\mathscr{O}(z)$ leaves all the correlators invariant, we say that they are **transparent** one to the other and that $\mathscr{L}$ preserves the corresponding algebra.



Fusion Ring

Given the set $S = \{\mathscr{L}_i\}$ of TDLs in the 2d QFT $\mathscr{Q}$, we can define two different operations that endow S with the structure of a **ring**:

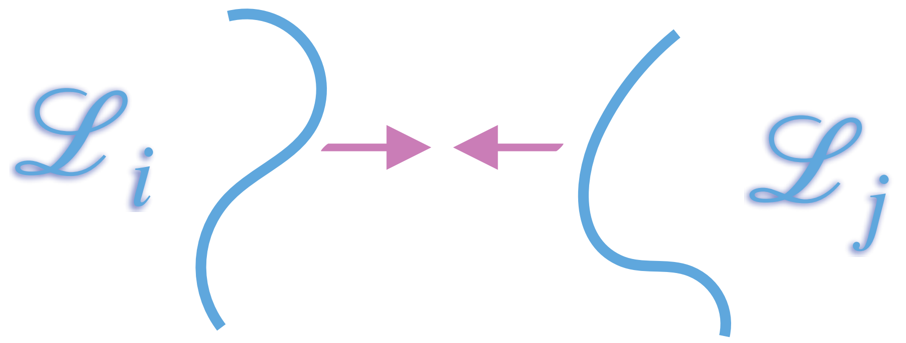
- **Superposition**:

  $\mathscr{L}_i$ and $\mathscr{L}_j$ ------> $\mathscr{L}_i + \mathscr{L}_j$, such that $\widehat{\mathscr{L}_i + \mathscr{L}_j} = \hat{\mathscr{L}}_i \oplus \hat{\mathscr{L}}_j$

  Associative and commutative

- **Fusion**:

  In the limit in which the support line of two defects $\mathscr{L}_i$ and $\mathscr{L}_j$ overlap, one can define a third defect which corresponding operator is given by the **fusion algebra** of the form:



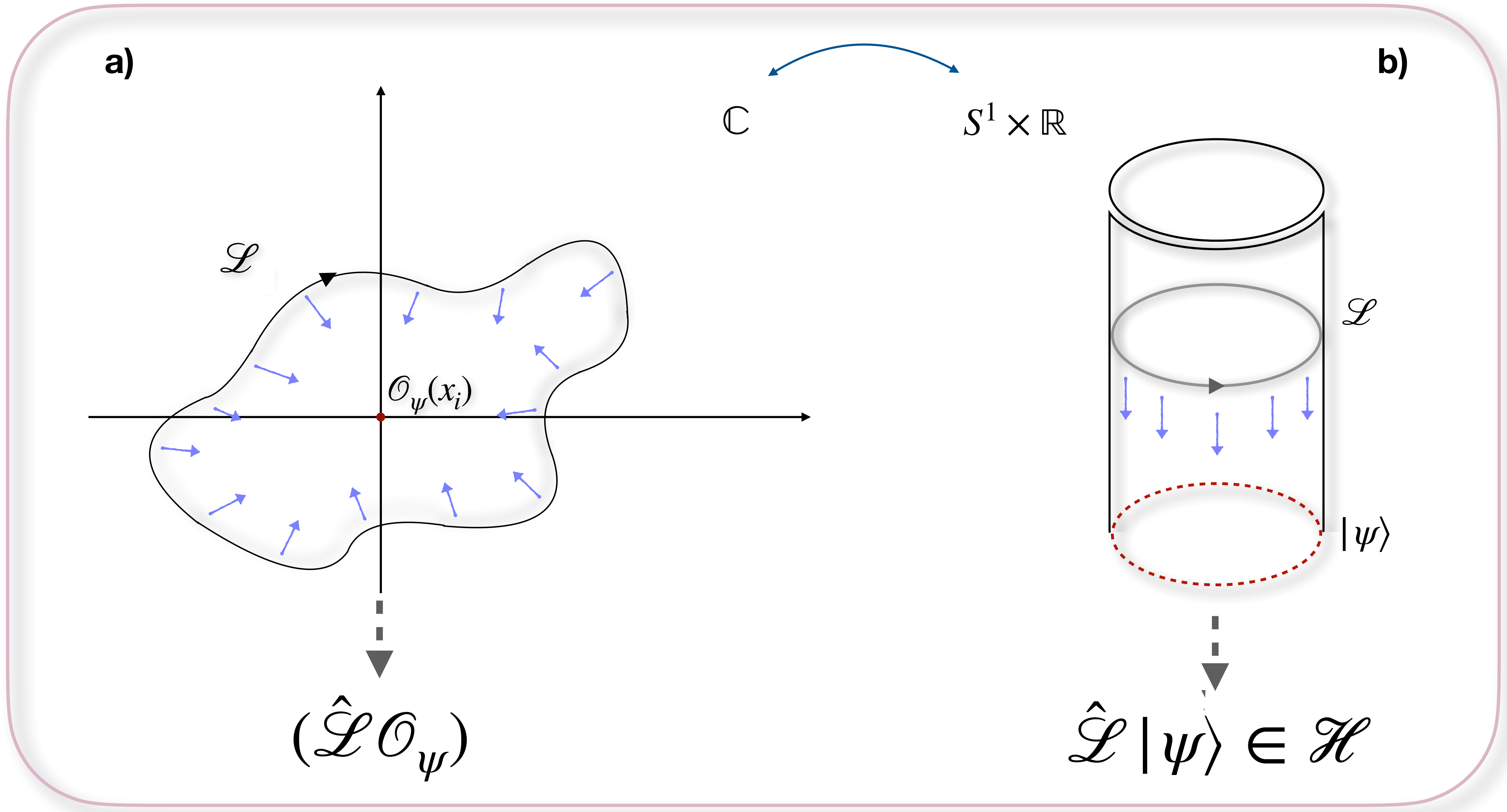
$$= \mathscr{L}_i \mathscr{L}_j = \sum_c N_{ij}^k \mathscr{L}_k$$

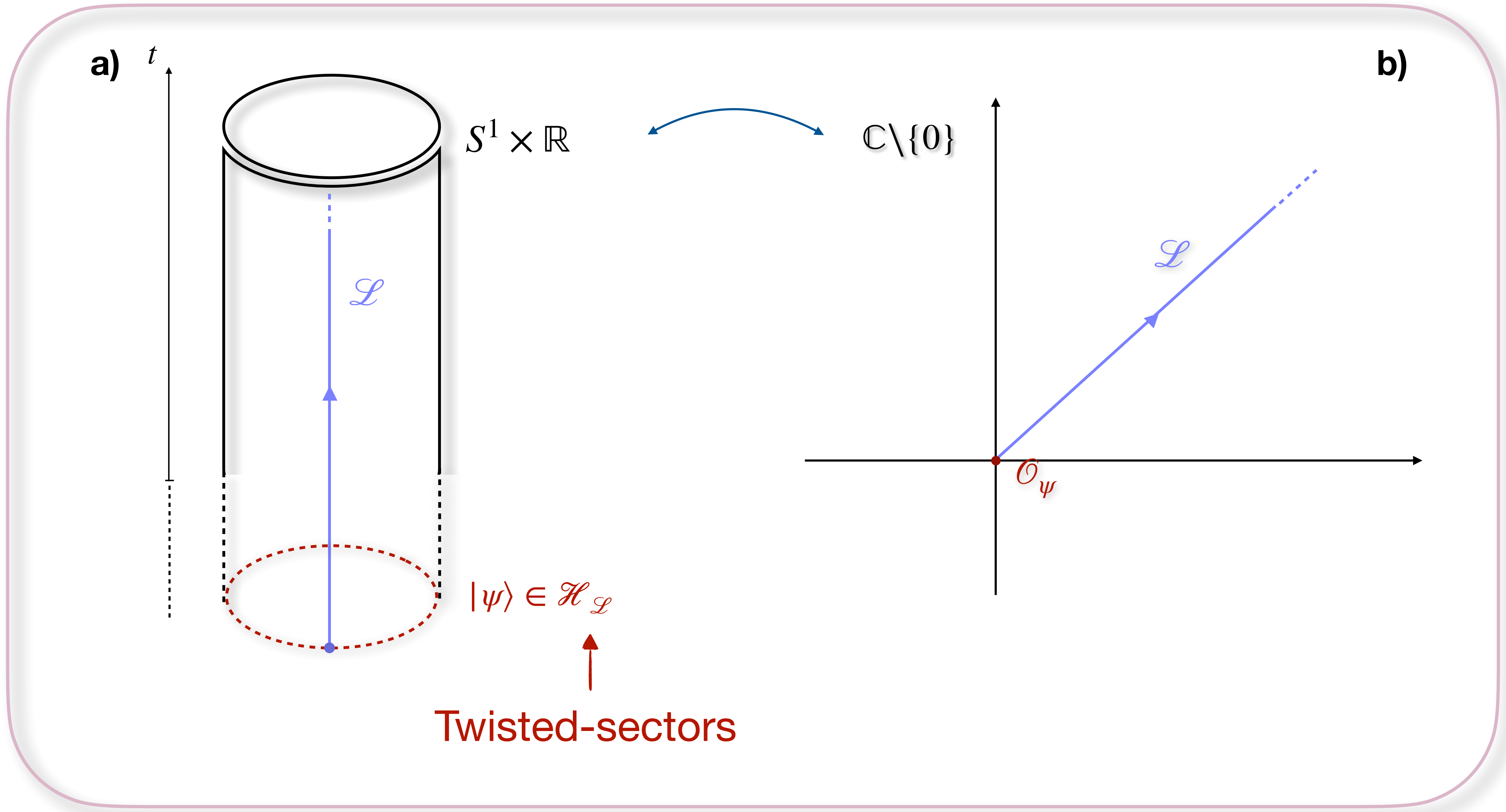
fusion coefficients

In the special case in which this fusion algebra reduces to a **group multiplication low**, the corresponding defects are called **invertible** and describe the standard global symmetries of the theory. Otherwise we talk about **non-invertible** generalized symmetries.

State/Operator correspondence

$$\mathcal{O}_\psi(z) \mapsto |\psi\rangle \in \mathcal{H}$$





We can **associate** to each top. defect $\mathscr{L}$:

- A **linear operator** $\hat{\mathscr{L}} : \mathcal{H} \mapsto \mathcal{H}$

  The vacuum is necessary an eigenstate of $\hat{\mathscr{L}}$, we call its eigenvalue "**Quantum dimension**": $\hat{\mathscr{L}}\,|0\rangle = \langle \mathscr{L} \rangle\,|0\rangle$

- A $\mathscr{L}$**-twisted space** $\mathcal{H}_{\mathscr{L}}$

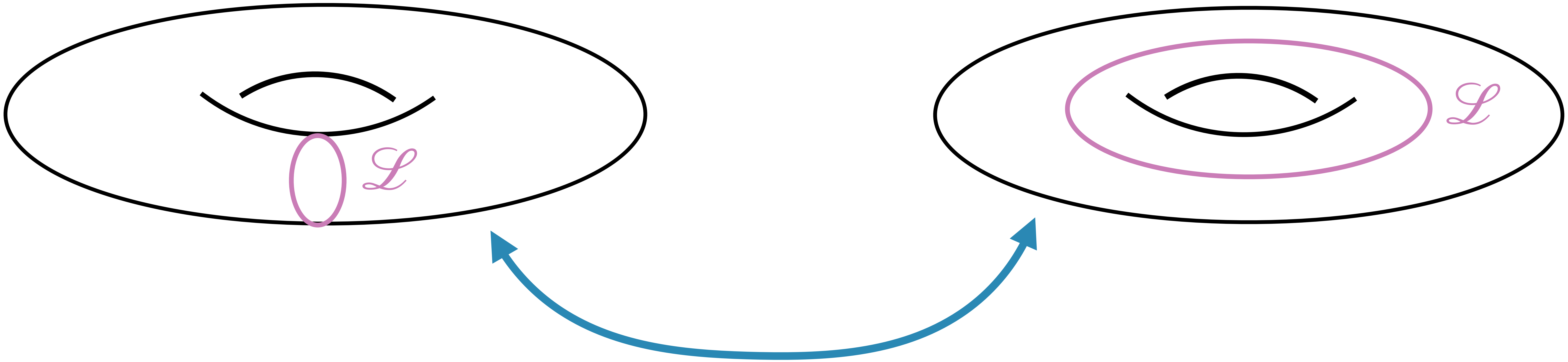
  Still providing a representation of the preserved chiral algebra

Related by **Modularity**

Modular transformation

If we put the theory on the torus:

$\mathscr{L}$-twined partition function $\mathscr{Z}^{\mathscr{L}}$ **$\mathscr{L}$-twisted partition function $\mathscr{Z}_{\mathscr{L}}$**



Modular S-transformation:

$$\tau \mapsto -1/\tau$$

Topological Defects on K3 models

Generalities on K3 models

NLSMs on K3 are ***2-dimensional*** $\mathcal{N} = (4,4)$ **superconformal fields theories** with central charge $c = \bar{c} = 6$. They arise as the worldsheet description of perturbative type II string theory on a K3 surface.

- They contain a holomorphic and an anti-holomorphic copy of the small $\mathcal{N} = 4$ superconformal algebra at central charge $c = 6$.
- We can pack the irreducible representations of $\mathcal{N} = (4,4)$ in four different sectors:

R-R **R-NS** **NS-R** **NS-NS**

For every K3-model the massless representations in this sector are $=$ **1** *4-dim* representation $\mathbf{\Pi}$ **(Spectral flow generators)** $\oplus$ **20** 1-*dim* representations $\mathbf{\Pi}^{\perp}$ $=$ **RR ground states** $\mathbf{V} \simeq \mathbb{R}^{\mathbf{4,20}}$

Moduli Space

The *80-dimensional* moduli space of **K3** models is the quotient:

$$\mathscr{M}_{K3} = O\,(4{,}20{,}\mathbb{Z})\backslash\big(O(4{,}20{,}\mathbb{R})/(O(4)\times O(20))\big)$$

T-duality group

Automorphisms of the lattice $\Gamma^{4,20}$ of RR D-brane charges with respect to the 24 U(1) RR fields

Grassmannian

Parametrising positive 4-dim subspaces Π in $V = \Gamma^{4,20} \otimes \mathbb{R} = \mathbb{R}^{4,20}$

→ The choice of a specific K3 model $\mathscr{C}_{\Pi}$ corresponds to the choice of $\Pi \subset V$.

The minimal requirements

Let Top_Π be the category of topological defects in the K3 model $\mathscr{C}_\Pi$ such that:

Implications:

1. The defects $\mathscr{L} \in Top_\Pi$ commute with the full $\mathscr{N} = (4,4)$ algebra
2. The defects $\mathscr{L} \in Top_\Pi$ commute with the spectral flow generators

- We can associate to each $\mathscr{L} \in Top_\Pi$ a linear operator:

$$\hat{\mathscr{L}} : \mathscr{H} \mapsto \mathscr{H}$$

(1) → $\mathscr{L}$ doesn’t change the representation of the states;

(2) → If we know its action on a specific sector, we can recover the action also in all the other sectors.

Action on D-branes

The defects $\mathscr{L} \in Top_{\Pi}$ map $\frac{1}{2}$ BPS D-branes into $\frac{1}{2}$ BPS D-branes then we can define the homomorphism:

$$\begin{aligned} \rho : Top_{\Pi} &\longmapsto End\left(\Gamma^{4,20}\right) \\ \mathscr{L} &\longmapsto \hat{L} : \Gamma^{4,20} \mapsto \Gamma^{4,20} \end{aligned}$$

1. $\mathscr{L}$ commute with the $\mathscr{N} = (4,4)$ algebra → L cannot mix R-R ground states in different reps.
2. $\mathscr{L}$ is transparent to spectral flow → L acts on Π them in the same way as in the vacuum.

Chosing an orthonormal basis for V compatible with the decomposition $V = \Pi \oplus \Pi^{\perp}$, the linear operators L are represented in this basis:

(1) → As 24×24 **block-diagonal** matrices;

(2) → The 4×4 block is a multiple of the **identity**, and d is the **quantum dimension** of $\mathscr{L}$;

(ρ) → **Integrality** conditions.

$$\hat{L} = \begin{pmatrix} d \cdot \mathbf{1}_{4\times4} & 0 \\ 0 & b_{20\times20} \end{pmatrix}$$

General Results

Claim:

The quantum dimension $\langle \mathscr{L} \rangle$ of a defect $\mathscr{L} \in Top_\Pi$ is an algebraic integer of degree at most 6.

If $\Pi \cap \Gamma^{4,20} \neq 0$, then $\langle \mathscr{L} \rangle$ is an integer for all the $\mathscr{L} \in Top_\Pi$.

→ Attractor points of the moduli space with respect to some brane configurations.

Claim:

For a generic K3 sigma-model $\mathscr{C}_\Pi$, the only defects in Top_Π are integral multiple of the identity.

Elliptic genus

The **elliptic genus** is a special function in K3 models that is independent on the B-field and the metric of the target manifold, but it is dependent on his topology.

$$\phi(\tau, z) = Tr_{RR}\left[q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}(-1)^{F+\bar{F}}y^{J_0^3}\right]$$

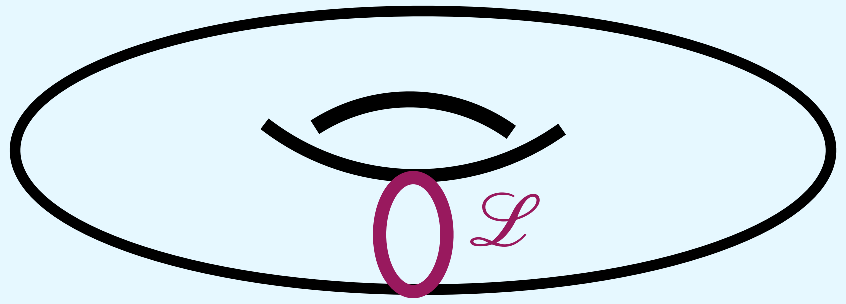
It counts states that are R ground states on the right-moving sector and unconstrained on the left-moving one.

with $q = e^{2\pi i\tau}$ and $y = e^{2\pi i z}$.

For each topological defect $\mathscr{L}$ we can construct two new genera:

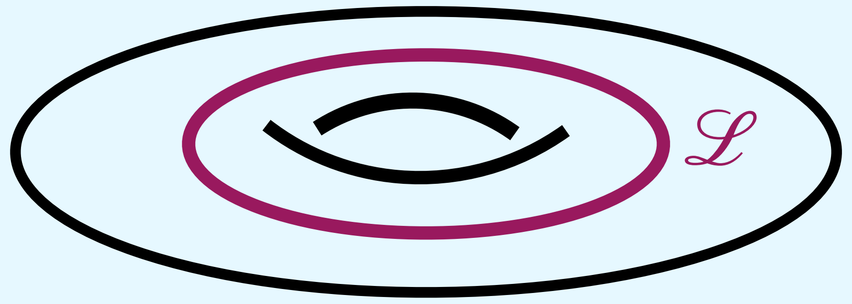
- The **twining genus**

$$\phi^{\mathscr{L}}(\tau, z) = Tr_{RR}\left[\hat{\mathscr{L}}\, q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}(-1)^{F+\bar{F}}y^{J_0^3}\right] =$$



- The **twisted genus**

$$\phi_{\mathscr{L}}(\tau, z) = Tr_{RR,\hat{\mathscr{L}}}\left[q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}(-1)^{F+\bar{F}}y^{J_0^3}\right] =$$



$$\tau \mapsto -\frac{1}{\tau}$$

Topological Defects on $V^{f\natural}$

The Conway SCFT $V^{f\natural}$

[Frenkel, Lepowsky, Meurman - 1988]
[Duncan - 2007]

- **12** holomorphic free bosons (**$\mathbf{u(1)}$ currents**) of weight 1 and standard OPE → $i\partial X^i(z) = \sum \alpha_n^i z^{-n-1}, \qquad i = 1,\ldots,12$
- The holomorphic vertex operators with momenta in the odd unimodular lattice → $\mathscr{V}_k(z) = c_k : e^{ik\cdot X(z)} : , \qquad k = (k^1, k^2, \ldots, k^{12})$

$$D_{12}^{+} = \underbrace{D_{12}}_{\text{root lattice of } so(24)_1} \cup \underbrace{(D_{12}+s)}_{\text{spinorial rep of } so(24)_1}$$

- The stress-energy tensor is $T(z) = \dfrac{1}{2}\sum : \partial X^i \partial X^i :$ at central charge $c = 12$;

In terms of $so(24)_1$ reppresentations:

NS Sector

$V^{f\natural} =$	V_0	$\oplus$	V_s
$(-1)^F$	$+1$		-1

R Sector → **$V^{f\natural}_{tw}(1/2)$ = 24 ground states of conformal weight 1/2.**

$V^{f\natural}_{tw} =$	V_v	$\oplus$	V_c
$(-1)^F$	$+1$		-1

The twisted and the twined partition functions in $V^{f\natural}$

$$Z_R^- = Tr_{V^{f\natural}_{tw}}\left[q^{L_0-1/2}(-1)^F\right]$$

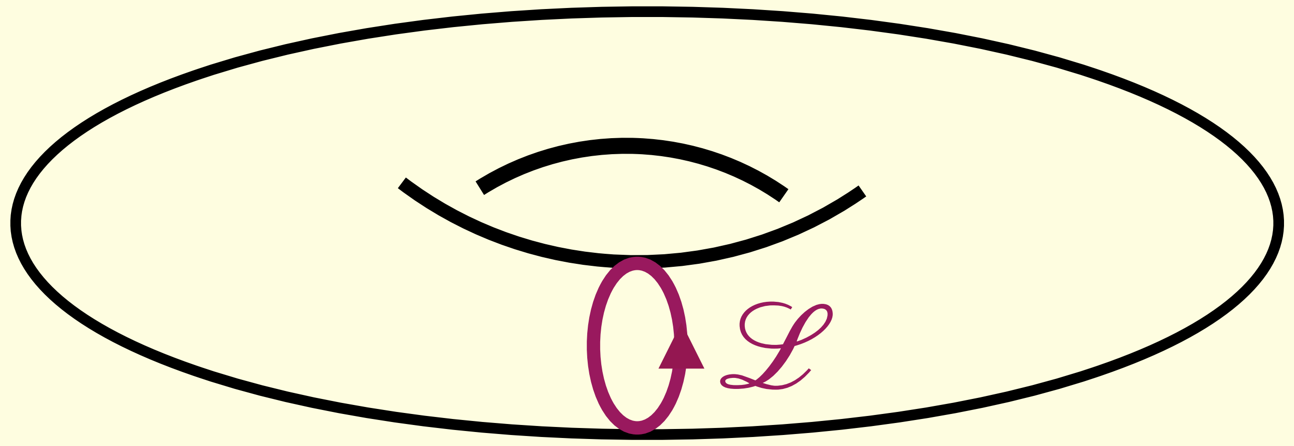
Witten index

Bosons - Fermions
in the R ground states sector

Main objects

$\mathscr{L}$-twined partition function

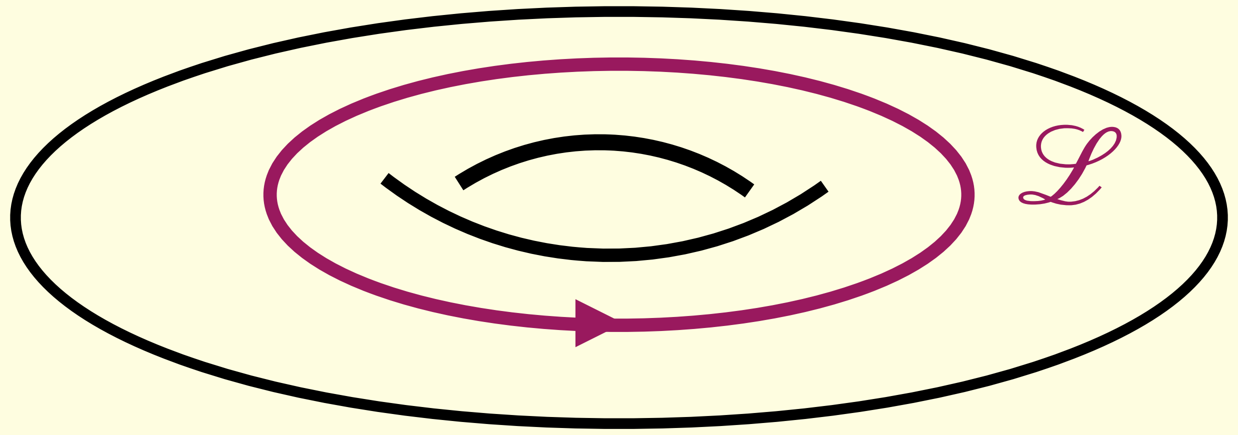
$$Z_R^{\mathscr{L},-} =$$



$$= Tr_{V^{f\natural}_{tw}}\left[q^{L_0-1/2}(-1)^F\hat{\mathscr{L}}\right]$$

$\mathscr{L}$-twisted partition function

$$Z_{\mathscr{L},R}^- =$$



$$= Tr_{V^{f\natural}_{tw,\mathscr{L}}}\left[q^{L_0-1/2}(-1)^F\right]$$

The minimal requirements

Let Top be the category of topological defects in $V^{f\natural}$ such that:

1. Each defect in Top commute with the $\mathcal{N}=1\ sVir_{c=12}$ algebra
2. Each defect in Top well behave with respect to $(-1)^F$
3. For every defect $\mathscr{L}$ we can define an $\mathscr{L}$-twined $\mathscr{Z}^{\mathscr{L}}$ and an $\mathscr{L}$-twisted $\mathscr{Z}_{\mathscr{L}}$ partition functions related by a modular S-transformation:

$$\mathscr{Z}_{\mathscr{L}}(-1/\tau)=\rho(S)\mathscr{Z}^{\mathscr{L}}(\tau)$$

Main implication:

All the $\mathscr{L}$-twined partition function on the torus with fully periodic conditions for the fermions are equal to the $\mathscr{L}$-twisted partition functions and they are integer numbers

$$Tr_{V^{f\natural}_{tw}(1/2)}\left[(-1)^F\hat{\mathscr{L}}_g\hat{\mathscr{L}}\right]\in\mathbb{Z},\forall g\in Co_0$$

The main Theorem

Theorem

Let Top be a tensor category of topological defects $\mathscr{L}$ of $V^{f\natural}$ containing all the invertible defects $\mathscr{L}_g$, with $g \in Co_0$, and such that all the objects $\mathscr{L}$ satisfy properties (1), (2) and (3). There is an embedding of the Leech lattice Λ_{Leech} in the 24 dimensional space of R-ground states $V^{f\natural}_{tw}(1/2)$, such that for every $\mathscr{L} \in Top$, we can construct a $\mathbb{C}$-linear map

$$\hat{\mathscr{L}}|_{V^{f\natural}_{tw}(1/2)} : V^{f\natural}_{tw}\left(\frac{1}{2}\right) \mapsto V^{f\natural}_{tw}\left(\frac{1}{2}\right)$$

that it is contained in $End\left(\Lambda_{Leech}\right)$:

$$\begin{aligned} \rho : Top &\longmapsto End\left(\Lambda_{Leech}\right) \\ \mathscr{L} &\longmapsto \rho(\mathscr{L}) = \hat{\mathscr{L}}|_{V^{f\natural}_{tw}(1/2)} \end{aligned}$$

This defines a surjective, non-injective ring-homomorphism.

Leech lattice Λ_{Leech}

Unique 24 dimensional unimodular lattice without roots,

***Automorphism group** = Co_0*

Topological defects satisfying properties (1), (2), (3) act on Λ_{Leech} via endomorphism

Corollaries

Corollary 1

If $\mathscr{L}$ *preserves any field in* $V_{tw}^{f\natural}(1/2)$ *then the quantum dimension* $\langle \mathscr{L} \rangle$ *must be an algebraic integer of degree at most 24.*

If $\mathscr{L} \in Top$ *preserves a field* $\lambda \in \Lambda_{Leech} \subset {}^{\mathbb{R}}V_{tw}^{f\natural}(1/2)$*,namely* $\mathscr{L}$ *acts by* $\hat{\mathscr{L}}\,|\,\lambda\rangle = \langle \mathscr{L} \rangle\,|\,\lambda\rangle$*, then necessarily*$\langle \mathscr{L} \rangle \in \mathbb{Z}_{\geq 1}$.

Corollary 2

Suppose $\mathscr{L}$ *acts trivially on some* $\psi \in V_{tw}^{f\natural}(1/2)$ *(i.e.* $\hat{\mathscr{L}}\psi = \langle \mathscr{L} \rangle \psi$*) such that* $\psi^{\perp} \cap \left(\Lambda_{Leech} \otimes \overline{\mathbb{Q}}\right) = \emptyset$*. Then,* $\mathscr{L}$ *is multiple of the identity defect, i.e.*$\mathscr{L} = dI$, $d \in \mathbb{N}$ *(and in particular* $\langle \mathscr{L} \rangle = d$ *is integral).*

The comparison

Conclusions

The connection between K3 models and $V^{f\natural}$ extend at the level of generalized symmetries

Conjecture

Each defect $\mathscr{L} \in Top$ *fixing a 4-plane* $\Pi^\natural \subset {}^{\mathbb{R}}V^{f\natural}_{tw}(1/2)$

(i) Corresponds to a defect $\mathscr{L} \in Top_{\Pi^\natural}$ *in the K3 model* $\mathscr{C}_{\Pi^\natural}$*;*

(ii) The $\mathscr{L}$*-twining partition function* $\phi^{\mathscr{L}}(V^{f\natural}, \tau, z)$ *computed in* $V^{f\natural}$ *coincides with the corresponding* $\mathscr{L}$*-twining elliptic genus* $\phi^{\mathscr{L}}(\mathscr{C}, \tau, z)$ *in the K3 model*

Tested in many examples:

- $\mathbb{Z}_k$ duality defects
- $\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2}$ duality defects
- Fibonacci defects

Thank you for your attention!