

Cartan geometries with model the future lightlike cone of Lorentz-Minkowski spacetime

Author: Rodrigo Morón Sanz (University of León)

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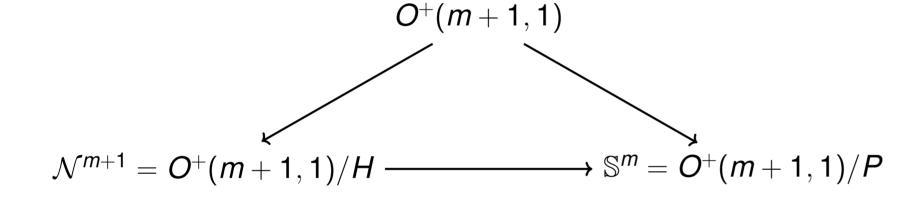


Motivation

Lightlike hypersurfaces (e.g. Killing horizons, conformal boundaries) inherit from the ambient Lorentzian metric a lightlike metric, that is, a degenerate form whose radical is a one-dimensional distribution. This degeneracy prevents the use of the Koszul formula and breaks the standard semi-Riemannian framework. Moreover, the normal and tangent bundles intersect, so the usual decomposition and theory of non-degenerate submanifolds fail. Various approaches have been developed to deal with this situation, both intrinsic and extrinsic. The most established one is the extrinsic method of Bejancu–Duggal [1], while here [5] we focus on an intrinsic perspective and present the main results shown in this poster.

Model for the considered Cartan geometries

The future lightlike cone \mathcal{N}^{m+1} in Lorentz-Minkowski spacetime \mathbb{L}^{m+2} is chosen as model due to the rigidity result [2, Thm. 1.1], which shows that its isometry group (with the induced lightlike metric) is the orthochronous Lorentz group $O^+(m+1,1)$. In contrast, lightlike hyperplanes have infinite dimensional isometry groups and cannot serve as Cartan models [6, Ex. 4.13]. The role of $O^+(m+1,1)$ in this setting parallels its central role in Riemannian conformal geometry, as summarized in the diagram below, where H corresponds to Euclidean rigid motions and P to the conformal Poincaré group:



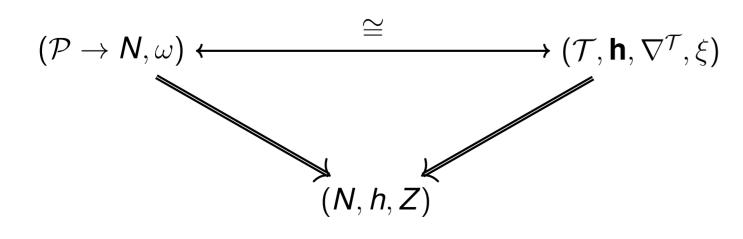
The main difficulty lies in the fact that this model is neither parabolic nor reductive [6, Lem. 2.5].

Lightlike Cartan geometries

A Cartan geometry $(\mathcal{P} \to N, \omega)$ modeled on the future lightlike cone induces a lightlike metric h and a vector field Z spanning its radical [6, Thm. 4.4]. Its automorphism group $\operatorname{Aut}(\mathcal{P}, \omega)$ is always a Lie group [4, Thm. 1.5.11] and sits as a proper subgroup of $\operatorname{Iso}(N, h, Z)$, which can be infinite dimensional. This raises the question of what additional geometric structures can be canonically derived from such a Cartan geometry.

Standard tractor bundles

The standard tractor bundle $\mathcal{T} = \mathcal{P} \times_H \mathbb{L}^{m+2} \to N$ of a lightlike Cartan geometry carries a canonical Lorentzian metric \mathbf{h} , a compatible linear tractor connection $\nabla^{\mathcal{T}}$, and a distinguished lightlike section $\xi \in \Gamma(\mathcal{T})$. There also exists a bundle morphism $\Phi := \nabla^{\mathcal{T}} \xi : TN \hookrightarrow \mathcal{T}$ that is a monomorphism and an isometry, showing that \mathcal{T} contains an isometric copy of the tangent bundle. The point of view adopted here, in the spirit of the tractor calculus for parabolic geometries [3], is to describe such Cartan geometries equivalently in terms of these tractor bundles just introduced.



Tractor calculus

Each $\tau \in S(N, h, Z) := \{\tau \in \Omega^1(N, \mathbb{R}) : \tau(Z) = 1\}$ determines a distribution An $(\tau) := \{W \in TN : \tau(W) = 0\}$, together with an associated field of endomorphisms $P^{\tau}(W) := W - \tau(W)Z$. The standard tractor bundle admits a τ -dependent splitting

$$\mathcal{T}\cong\underline{\mathbb{R}}\oplus\operatorname{An}\left(au
ight)\oplus\underline{\mathbb{R}}.$$

In this splitting, the tractor connection $\nabla^{\mathcal{T}}$ can be expressed as

$$\nabla_{W}^{\tau} \begin{pmatrix} \alpha \\ X \\ \beta \end{pmatrix} \stackrel{\tau}{=} \begin{pmatrix} W(\alpha) + \alpha \tau(W) - h(X, D^{\tau}(W)) \\ \alpha P^{\tau}(W) + \beta D^{\tau}(W) + \nabla_{W}^{\tau} X \\ W(\beta) - \beta \tau(W) - h(X, W) \end{pmatrix}$$

where ∇^{τ} is a metric linear connection on $(\operatorname{An}(\tau) \to N, h)$ and D^{τ} is a vector bundle morphism from TN to $\operatorname{An}(\tau)$.

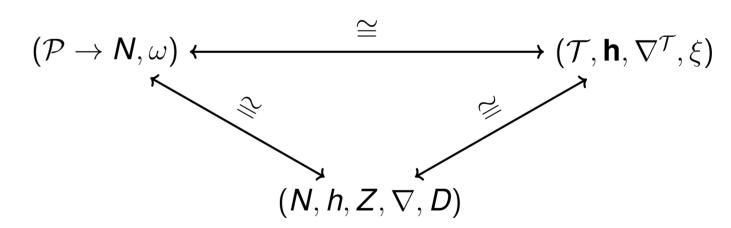
Let $(\mathcal{P} \to \mathcal{N}, \omega)$ be a lightlike Cartan geometry. Then the manifold \mathcal{N} carries:

- A lightlike metric *h* and a vector field *Z* spanning its radical.
- A map ∇ assigning to each $\tau \in S(N, h, Z)$ a metric linear connection ∇^{τ} on $(An(\tau) \to N, h)$.
- A map D assigning to each $\tau \in S(N, h, Z)$ a vector bundle morphism $D^{\tau}: TN \to An(\tau)$.

The maps ∇ and D satisfy explicit transformation laws under changes of τ .

Main result [5, Thm. 6.17]

The pair (∇, D) is exactly the additional geometric content carried by the Cartan connection beyond (h, Z).



Future work

First, we aim to provide a canonical choice of the pair (∇, D) by imposing appropriate normalization conditions on lightlike Cartan geometries; this problem has been partially addressed in [5, Thm. 7.9]. Second, we plan to study local invariants arising in the intrinsic geometry of (N, h, Z) to better understand their geometric and physical properties.

References

- [1] A. Bejancu and K.L. Duggal, Lightlike submanifolds of semi-Riemannian manifolds and applications, *Mathematics and its Applications*, **364**, Kluwer Academic Publishers Group, Dordrecht, 1996.
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