Cartan geometries with model the future lightlike cone of Lorentz-Minkowski spacetime

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Motivation

Lightlike hypersurfaces (e.g. Killing horizons, conformal boundaries) inherit from the ambient Lorentzian metric a lightlike metric, that is, a degenerate form whose radical is a one-dimensional distribution. This degeneracy prevents the use of the Koszul formula and breaks the standard semi-Riemannian framework. Moreover, the normal and tangent bundles intersect, so the usual decomposition and theory of non-degenerate submanifolds fail. Various approaches have been developed to deal with this situation, both intrinsic and extrinsic. The most established one is the extrinsic method of Bejancu–Duggal [1], while here [5] we focus on an intrinsic perspective and present the main results shown in this poster.

Model for the considered Cartan geometries

The future lightlike cone $\mathcal{N}^{m+1}$ in Lorentz-Minkowski spacetime $\mathbb{L}^{m+2}$ is chosen as model due to the rigidity result [2, Thm. 1.1], which shows that its isometry group (with the induced lightlike metric) is the orthochronous Lorentz group $O^{+}(m+1,1)$. In contrast, lightlike hyperplanes have infinite dimensional isometry groups and cannot serve as Cartan models [6, Ex. 4.13]. The role of $O^{+}(m+1,1)$ in this setting parallels its central role in Riemannian conformal geometry, as summarized in the diagram below, where H corresponds to Euclidean rigid motions and P to the conformal Poincaré group:

$$\begin{array}{ccc} & O^{+}(m+1,1) & \\ \swarrow & & \searrow \\ \mathcal{N}^{m+1}=O^{+}(m+1,1)/H & \longrightarrow & \mathbb{S}^{m}=O^{+}(m+1,1)/P \end{array}$$

The main difficulty lies in the fact that this model is neither parabolic nor reductive [6, Lem. 2.5].

Lightlike Cartan geometries

A Cartan geometry $(\mathcal{P}\to N,\omega)$ modeled on the future lightlike cone induces a lightlike metric h and a vector field Z spanning its radical [6, Thm. 4.4]. Its automorphism group $\mathrm{Aut}(\mathcal{P},\omega)$ is always a Lie group [4, Thm. 1.5.11] and sits as a proper subgroup of $\mathrm{Iso}(N,h,Z)$, which can be infinite dimensional. This raises the question of what additional geometric structures can be canonically derived from such a Cartan geometry.

Standard tractor bundles

The standard tractor bundle $\mathcal{T}=\mathcal{P}\times_{H}\mathbb{L}^{m+2}\to N$ of a lightlike Cartan geometry carries a canonical Lorentzian metric $\mathbf{h}$, a compatible linear tractor connection $\nabla^{\mathcal{T}}$, and a distinguished lightlike section $\xi\in\Gamma(\mathcal{T})$. There also exists a bundle morphism $\Phi:=\nabla^{\mathcal{T}}\xi: TN\hookrightarrow\mathcal{T}$ that is a monomorphism and an isometry, showing that $\mathcal{T}$ contains an isometric copy of the tangent bundle. The point of view adopted here, in the spirit of the tractor calculus for parabolic geometries [3], is to describe such Cartan geometries equivalently in terms of these tractor bundles just introduced.

$$\begin{array}{ccc} (\mathcal{P}\to N,\omega) & \overset{\cong}{\longleftrightarrow} & (\mathcal{T},\mathbf{h},\nabla^{\mathcal{T}},\xi) \\ & \searrow\!\!\searrow \quad \swarrow\!\!\swarrow & \\ & (N,h,Z) & \end{array}$$

Tractor calculus

Each $\tau\in S(N,h,Z):=\{\tau\in\Omega^{1}(N,\mathbb{R}):\tau(Z)=1\}$ determines a distribution $\mathrm{An}\,(\tau):=\{W\in TN:\tau(W)=0\}$, together with an associated field of endomorphisms $P^{\tau}(W):=W-\tau(W)Z$. The standard tractor bundle admits a τ-dependent splitting

$$\mathcal{T}\cong\underline{\mathbb{R}}\oplus\mathrm{An}\,(\tau)\oplus\underline{\mathbb{R}}.$$

In this splitting, the tractor connection $\nabla^{\mathcal{T}}$ can be expressed as

$$\nabla^{\mathcal{T}}_{W}\begin{pmatrix}\alpha\\ X\\ \beta\end{pmatrix}\overset{\tau}{=}\begin{pmatrix}W(\alpha)+\alpha\,\tau(W)-h(X,D^{\tau}(W))\\ \alpha\,P^{\tau}(W)+\beta\,D^{\tau}(W)+\nabla^{\tau}_{W}X\\ W(\beta)-\beta\,\tau(W)-h(X,W)\end{pmatrix},$$

where ∇^{τ} is a metric linear connection on $(\mathrm{An}\,(\tau)\to N,h)$ and D^{τ} is a vector bundle morphism from TN to $\mathrm{An}\,(\tau)$.

Let $(\mathcal{P}\to N,\omega)$ be a lightlike Cartan geometry. Then the manifold N carries:

- A lightlike metric h and a vector field Z spanning its radical.
- A map ∇ assigning to each $\tau\in S(N,h,Z)$ a metric linear connection ∇^{τ} on $(\mathrm{An}\,(\tau)\to N,h)$.
- A map D assigning to each $\tau\in S(N,h,Z)$ a vector bundle morphism $D^{\tau}:TN\to\mathrm{An}\,(\tau)$.

The maps ∇ and D satisfy explicit transformation laws under changes of τ.

Main result [5, Thm. 6.17]

The pair (∇,D) is exactly the additional geometric content carried by the Cartan connection beyond (h,Z).

$$\begin{array}{ccc} (\mathcal{P}\to N,\omega) & \overset{\cong}{\longleftrightarrow} & (\mathcal{T},\mathbf{h},\nabla^{\mathcal{T}},\xi) \\ \cong\nwarrow\!\!\searrow & & \swarrow\!\!\nearrow\cong \\ & (N,h,Z,\nabla,D) & \end{array}$$

Future work

First, we aim to provide a canonical choice of the pair (∇,D) by imposing appropriate normalization conditions on lightlike Cartan geometries; this problem has been partially addressed in [5, Thm. 7.9]. Second, we plan to study local invariants arising in the intrinsic geometry of (N,h,Z) to better understand their geometric and physical properties.

References

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