Matrix Geometries with an Internal Space

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Based on joint work with John Barrett

Outline

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Spectral Triples and Physics

- Almost-Commutative Geometries a la Connes provide a natural geometric origin for the standard model
- Investigate a non-commutative 'spacetime' with internal space e.g. Perez-Sanchez [1]
- Matrix geometries provide a nice space of toy geometries (with commutative geometries as limits e.g. fuzzy sphere).
- Functional Integrals are well defined for matrix geometries

Matrix Geometries (Real Finite Spectral Triples)

Matrix Geometries (Barrett [2]) are real finite spectral triples, $(\mathcal{A}, \mathcal{H}, \mathcal{D}; J, \Gamma)$, such that

- $\mathcal{A} = M_n(\mathbb{R})$ or $M_{n/2}(\mathbb{H})$
- $\mathcal{H} = \mathbb{C}^k \otimes M_n(\mathbb{C})$ for $\mathbb{C}^k$ a (p, q) Clifford Module
- $\mathcal{D}$ is a general Dirac operator constructed of combinations of Clifford generators (gamma matrices) and (anti-)commutators of matrices e.g for a $(0, 4)$ geometry

$$D_4 = \sum_{\mu} \gamma^{\mu} \otimes [L_{\mu}, \cdot] + \sum_{i<j<k} \gamma^i \gamma^j \gamma^k \otimes \{H_{ijk}, \cdot\}$$

Outline

2 Fluctuations of Matrix Geometries

- The Model
- Fluctuations
- The Action

Matrix Geometry with an Internal Space

Take the product of a $(0,4)$ space and a '$U(1)$' internal space. Gives a KO-2 triple:

$$\mathcal{A} = M_n(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \quad \text{or} \quad M_n(\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C}$$

$$\mathcal{H} = \mathbb{C}^4 \otimes M_n(\mathbb{C}) \otimes \mathbb{C}^2$$

$$D = D_4 \otimes \mathbb{1}_{\mathcal{H}_F} + \Gamma \otimes D_F$$

$$J = C \otimes (\cdot)^* \otimes J_F$$

$$\gamma = \Gamma \otimes \gamma_F = \gamma_5 \otimes \mathbb{1}_n \otimes \gamma_F$$

Representations of the algebra are given by

$$\pi(A \otimes \lambda) = \mathbb{1}_4 \otimes A \otimes \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbb{1}_4 \otimes \lambda A & 0 \\ 0 & \mathbb{1}_4 \otimes \bar{\lambda} A \end{pmatrix}$$

The Gauge Group

The gauge group of the spectral triple is given by

$$\mathfrak{G}(\mathcal{A}, \mathcal{H}; J) = U(n)/\mathbb{Z}_2$$

This agrees with Bhomwick et al.[3]

The gauge algebra of the spectral triple is

$$\mathfrak{g}(\mathcal{A}, \mathcal{H}; J) = \mathfrak{u}(n)$$

Fluctuations

Inner fluctuations are Connes one-forms

$$\omega = \pi(A \otimes \alpha)[D, \pi(B \otimes \beta)]$$

For almost commutative geometries inner automorphisms provide gauge- and scalar- field content.

In this case, all automorphisms are inner.

- Gauge and spacetime fluctuations are associated to the 'manifold' Dirac operator, $\pi(A \otimes \alpha)[D_4 \otimes \mathbb{1}_{\mathcal{H}_F}, \pi(B \otimes \beta)]$
- Scalar fluctuations originate from the mass Dirac operator, $\pi(A \otimes \alpha)[\Gamma \otimes D_F, \pi(B \otimes \beta)]$

Spacetime Vs Gauge I

Fluctuations associated to D_4 take the form

$$\Omega_\Lambda = \gamma^\mu \otimes \begin{pmatrix} \lambda\Lambda_\mu(\cdot) - (\cdot)\bar{\lambda}\Lambda_\mu & 0 \\ 0 & \bar{\lambda}\Lambda_\mu(\cdot) - (\cdot)\lambda\Lambda_\mu \end{pmatrix}$$
$$+ \gamma^i\gamma^j\gamma^k \otimes \begin{pmatrix} \lambda\Lambda_{ijk}(\cdot) + (\cdot)\bar{\lambda}\Lambda_{ijk} & 0 \\ 0 & \bar{\lambda}\Lambda_{ijk}(\cdot) + (\cdot)\lambda\Lambda_{ijk} \end{pmatrix}$$

for $\lambda \in \mathbb{C}$, $\Lambda_\mu, \Lambda_{ijk} \in M_n(\mathbb{C})$.

Spacetime Vs Gauge II

Upon applying the Hermiticity constraint,there are two possible types of fluctuation related to $\lambda = \nu + i\kappa$

- Real fluctuations, $\nu \neq 0, \kappa = 0$, are associated to the 'manifold algebra' - Spacetime Fluctuations
- Imaginary fluctuations, $\nu = 0, \kappa \neq 0$, are associated to both the manifold and internal algebras - Gauge Fields

General fluctuations are a sum of these two types

Spacetime Vs Gauge III

- **Real / Spacetime Fluctuations**

$$\Omega_{\mathbb{R}} = \left(\gamma^\mu \otimes \frac{1}{2}[\sigma_\mu, \cdot] + \gamma^i\gamma^j\gamma^k \otimes \frac{1}{2}\{y_{ijk}, \cdot\}\right) \otimes \mathbb{1}_{\mathcal{H}_F}$$
$$= (\Sigma + Y) \otimes \mathbb{1}_{\mathcal{H}_F}$$

- **Imaginary / Gauge fluctuations**

$$\Omega_{i\mathbb{R}} = \left(\gamma^\mu \otimes \frac{1}{2}\{\theta_\mu, \cdot\} + \gamma^i\gamma^j\gamma^k \otimes \frac{1}{2}\left[ix_{ijk}, \cdot\right]\right) \otimes \gamma_F$$
$$= (\Theta + X) \otimes \gamma_F$$

Higgs terms

The mass fluctuations are generated by the internal Dirac operator.

These fluctuations are determined by a complex, symmetric matrix ϕ

$$\begin{aligned}\Omega_\Phi &= \gamma_5 \otimes \begin{pmatrix} 0 & \bar{\mu}\{\phi, \cdot\} \\ \mu\{\bar{\phi}, \cdot\} & 0 \end{pmatrix} \\ &= \gamma_5 \otimes \begin{pmatrix} 0 & \bar{\mu}\Phi \\ \mu\bar{\Phi} & 0 \end{pmatrix}\end{aligned}$$

The Total Dirac Operator

$$D = \begin{pmatrix} (D_4 + \Sigma + Y) + (\Theta + X) & \bar{\mu}\Gamma(1 + \Phi) \\ \mu\Gamma(1 + \bar{\Phi}) & (D_4 + \Sigma + Y) - (\Theta + X) \end{pmatrix}$$
$$= \begin{pmatrix} D_4' + (\Theta + X) & \bar{\mu}\Gamma(1 + \Phi) \\ \mu\Gamma(1 + \bar{\Phi}) & D_4' - (\Theta + X) \end{pmatrix}$$

The spacetime fluctuations can be absorbed via a redefinition of D_4

The Fermionic Action

The fermionic action is defined as $S = \frac{1}{2}\langle J\Psi, D\Psi\rangle = \frac{1}{2}(\Psi, D\Psi)$ [4]. The action for a fermion, $\Psi = \chi \otimes e + \xi \otimes \bar{e}$, in this model is

$$
\begin{aligned}
(\Psi, D\Psi) = &(\xi, (D_4' + (\Theta + X))\chi) \\
&+ (\chi, (D_4' - (\Theta + X))\xi) \\
&+ \mu(\xi, \Gamma(1 + \Phi)\xi) + \bar{\mu}(\chi, \Gamma(1 + \Phi)\chi)
\end{aligned}
$$

Outline

3 Fermion Integrals for Matrix Geometries with Internal Space

- The Integral of Interest
- Extending the Real Integral

The Real Fermionic Integral

The real fermionic integral is given by

$$Z_{\mathbb{R}} = \int_{\mathcal{H}} \mathcal{D}\psi e^{\frac{i}{2}\langle J\Psi, D\Psi\rangle}$$

The conjugation operator ensures this integration is only over the fermion field.

For a KO-2 triple, and if there is a mass to regulate zero modes (or there are no zero modes), this results in [5]

$$Z_{\mathbb{R}} = \mathrm{pf}[\langle Je_i, iDe_j\rangle] = \sqrt{\det(D)}$$

Real Integral with Spacetime Fluctuations

Consider the Dirac operator with only real fluctuations, $D_{ST} = D + \Omega_{\mathbb{R}}$

- Spacetime fluctuations are global, thus a basis can be found in terms of a modified D_4'
- Barrett's method can then be used [5], giving an identical form just with a modified eigenvalue spectrum
- $\int_{\mathcal{H}} \mathcal{D}\psi e^{\frac{i}{2}\langle J\Psi, D_{ST}\Psi\rangle} = \sqrt{\det(D_{ST})} = \sqrt{\det\left(D_4'^2 + |\mu|^2\right)}$

Real Integral with a Gauge Field I

Now 'turn on' the imaginary fluctuations to give the operator

$$D_G = D_{ST} + D_{i\mathbb{R}}$$
$$= \begin{pmatrix} D_4' + (\Theta + X) & \bar{\mu}\Gamma \\ \mu\Gamma & D_4' - (\Theta + X) \end{pmatrix}$$

In this case a basis of D_4' cannot be used, instead one is constructed directly for D_G

Real Integral with a Gauge Field II

Hence, $\text{pf}(M_G) = \sqrt{\det(D_G)}$, with $\det(D_G)$ equal to,

$$\det\left(D_4'^2 + |\mu|^2 - \underbrace{\left((\Theta + X)^2 + \left[D_4', \Theta + X\right]\right)}_{F} \right)$$

These induced terms take the form

$$\begin{aligned} F &= (\Theta + X)^2 + \left[D_4', \Theta + X\right] \\ &= \Theta^2 + \left[D_4', \Theta\right] + X^2 + \left[D_4', X\right] + \{\Theta, X\} \end{aligned}$$

Outline

Conclusion/Recap

- Investigation of a matrix geometry with a $U(1)$ internal space
- Formed a spectral fermionic action functional that provided gauge and scalar field interaction terms
- In addition shown the fluctuations generate a new charge dependent derivative-like term from the spin connection
- Symbolically computed fermionic integrals - these contribute 'Field Strength'-like correction terms to the action
- Outlook - Higgs effects, Chiral fermion integral, Spectral action, Geometries with commutative limit

The End

Thank you for your time! :)

References

[1] C. I. Perez-Sanchez, "On multimatrix models motivated by random noncommutative geometry II: A Yang-Mills-Higgs matrix model," *Annales Henri Poincare*, vol. 23, no. 6, pp. 1979–2023, May 2021.

[2] J. W. Barrett, "Matrix geometries and fuzzy spaces as finite spectral triples," *Journal of Mathematical Physics*, vol. 56, no. 8, p. 82 301, Aug. 2015.

[3] J. Bhowmick, F. D'Andrea, B. Das, and L. Dabrowski, "Quantum gauge symmetries in Noncommutative Geometry," *Journal of Noncommutative Geometry*, vol. 8, no. 2, pp. 433–471, Dec. 2011.

[4] W. D. van Suijlekom, *Noncommutative Geometry and Particle Physics* (Mathematical Physics Studies). Cham: Springer Nature Switzerland, 2025.

[5] J. W. Barrett, "Fermion integrals for finite spectral triples," *Journal*