Entropy on the Bloch sphere from the spectral action

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work in progress with

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Introduction

Dialogue between entropy and geometry via noncommutative geometry, following some considerations of v. Suijlekom (CIRM Luminy, last april) concerning his paper "Entropy and the spectral action" with Chamseddine and Connes (2018)

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Connes's characterisation of riemannian geometry in terms of spectral data:

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In particular, identifying a point z of $\mathcal{M}$ with the pure state of $C^\infty(\mathcal{M})$

$$\delta_z : f \mapsto f(z),$$

the geodesic distance is retrieved as

$$d(\delta_x,\delta_y) = \sup_{f\in C^\infty(\mathcal{M})} \left\{ \delta_x(f) - \delta_y(f), \quad ||[\partial\!\!\!/, f]|| \leq 1 \right\}.$$

Connes, Lott (1992)

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with a suitably chosen finite dimensional algebra $\mathcal{A}_F$ and matrix D_F, one retrieves the bosonic part of the lagrangian of the Standard Model as the asymptotic expansion $\Lambda \to \infty$ of the spectral action

$$\mathrm{Tr}\left(f\left(\frac{D_A^2}{\Lambda^2}\right)\right),$$

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$$\mathrm{Tr}\left(f\left(\frac{D_A^2}{\Lambda^2}\right)\right),$$

where $D_A = D + A + JAJ^{-1}$ is the covariant Dirac operator defined by a generalised 1-form

$$A = A^* \in \left\{\sum_i a_i[D, b_i],\ a_i, b_i \in \mathcal{A}\right\} \quad \text{with } J \text{ the "real structure"},$$

and the test function f is a smooth approximation of the characteristic function of the interval $[0, 1]$.

Chamseddine, Connes, Marcolli, Suijlekom (1996-2015)

By studying a specific C^*-dynamical system associated to a spectral triple, one singles out a unique state, whose entropy turns out to be related to the spectral action for a specific test function.

Chamseddine, Connes, van Suijlekom (2018)

What does this entropy says about the geometry, in particular about the distance between the (pure) states ?

1. **Entropy from the spectral action**
   - C^*-dynamical system
   - KMS state for the Clifford algebra
   - The result of Chamseddine, Connes and v. Suijlekom
2. **The two point space**
   - Relation between distance and entropy
3. **Entropy on the Bloch sphere**

I. Entropy from the spectral action

C^*-dynamical system: a (unital) C^*-algebra $\mathcal{C}$ together with a 1-parameter group of automorphism $\sigma_t \in \mathrm{Aut}(\mathcal{C})$, $t \in \mathbb{R}$.

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A state φ on $\mathcal{C}$ (i.e. linear, positive map $\mathcal{C} \to \mathbb{R}$ of norm 1) then satisfies the *KMS* condition at inverse temperature $\beta > 0$ if

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Finite dimensional case: for $\mathcal{C} = M_n(\mathbb{C})$, any σ_t is associated with a self-adjoint $H = H^* \in M_n(\mathbb{C})$:

$$\sigma_t(A) = e^{itH}\, A\, e^{-itH} \quad \forall A \in M_n(\mathbb{C}),\ t \in \mathbb{R}.$$

For any $\beta > 0$, there exists a unique *KMS* state φ_β on $(\mathcal{C}, \sigma_t)$:

$$\varphi_\beta(\cdot) = \mathrm{Tr}(\rho_\beta\, \cdot) \quad \text{for } \rho_\beta = Z e^{-\beta H} \text{ with } Z = \frac{1}{\mathrm{Tr}(e^{-\beta H})}.$$

Any density matrix ρ on an Hilbert space $\mathcal{H}$ (i.e. positive operator with trace 1) defines a state

$$\varphi(.) := \mathrm{Tr}(\rho\cdot)$$

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$$S(\varphi) := -\mathrm{Tr}\left(\rho \log \rho\right).$$

Example — Chamseddine, Connes, V. Suijlekom

For $x > 0$, the partition of the unit interval by two intervals α, β of ratio x is

$$\alpha = \frac{1}{x+1}, \quad \beta = \frac{x}{x+1}.$$

The corresponding density matrix

$$\rho = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

defines a state with entropy

$$\mathcal{E}(x) := \log(x+1) - \frac{x \log x}{x+1}.$$

Proposition Chamseddine, Connes, V. Suijlekom (2018)

Let $\mathcal{H}$ be a complex Hilbert space, D a selfadjoint operator on $\mathcal{H}$ with compact resolvent. Let

- $\mathcal{C} = \mathbb{C}l(\mathcal{H}_{\mathbb{R}})$ be the complex Clifford algebra of the underlying real Hilbert space $\mathcal{H}_{\mathbb{R}}$,
- σ_t the 1-parameter group of automorphisms associated to $e^{itD} \in \mathrm{Aut}(\mathcal{H}_R)$,

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Moreover, if the operator $e^{-\beta|D|}$ is trace-class, the entropy of φ_β is equal to the spectral action,

$$S(\varphi_\beta) = \mathrm{Tr}\left(h(\beta D)\right)$$

for the test function

$$h(x) = \mathcal{E}(e^{-x}).$$

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- The function h is linked to Riemann ζ function.
- Given any spectral triple $(\mathcal{A}, \mathcal{H}, D)$ such that $e^{-\beta|D|}$ is trace-class, the spectral action for the test function h measures a "intrinsic" entropy associated with the noncommutative geometry.

What does this entropy say about the geometry ?

II. Entropy of a two point space

Suijlekom (2025)

$$\mathcal{A} = \mathbb{C}^2, \quad \mathcal{H} = \mathbb{C}^2, \quad D = \begin{pmatrix} 0 & m \\ \bar{m} & 0 \end{pmatrix}$$

with $m \in \mathbb{C}^*$ and $\mathcal{A}$ acts on $\mathcal{H}$ as

$$\pi(z_1, z_2) = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \quad \text{for} \quad (z_1, z_2) \in \mathbb{C}^2.$$

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The two pure states of $\mathcal{A}$ are

$$\delta_1(z_1, z_2) = z_1, \quad \delta_1(z_1, z_2) = z_2,$$

at distance

$$d(\delta_1, \delta_2) = \frac{1}{|m|} =: r$$

from one another.

The entropy depends only on the distance

$$S(r) = \mathrm{Tr}\left(h\left(\beta D\right)\right) = 2\left(\log(1 + e^{-\frac{\beta}{r}}) + \frac{\beta}{r}\frac{1}{1+e^{\frac{\beta}{r}}}\right).$$

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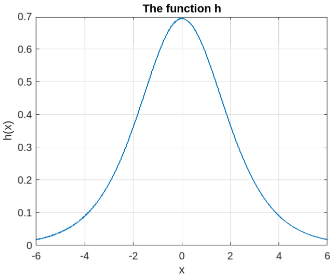

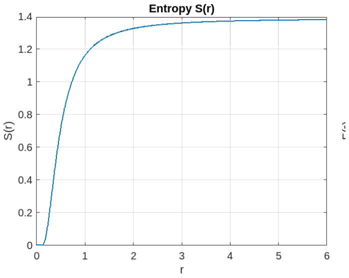

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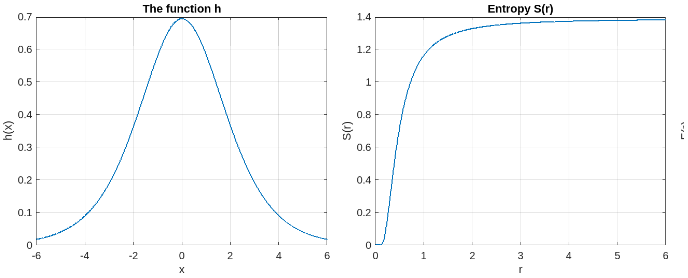

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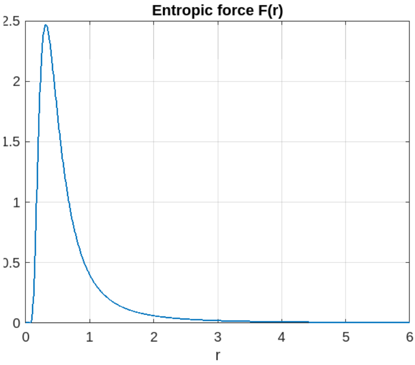
- Maximal (yet finite) disorder when the points are far from one another. Minimal disorder when the two points coincide.

The entropic force

$$F(r) := \frac{1}{\beta}\partial_r S$$

is

$$F(r) = \frac{\beta}{2r^3}\frac{1}{\cosh^2(\frac{\beta}{2r})}.$$



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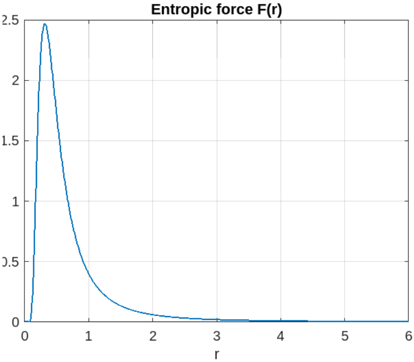

- The maximum of $F(r)$, for $r \simeq 0.308\beta$, singles out one particular geometrical configuration (i.e. distance) of the two points.

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$$D = \begin{pmatrix} 0 & m \\ \bar{m} & 0 \end{pmatrix} \quad \text{with} \quad D' = \begin{pmatrix} \lambda_1 & m \\ \bar{m} & \lambda_2 \end{pmatrix},$$

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the distance is unchanged but the entropy

$$S(r', t') = h\left(\frac{\beta}{r'} + \frac{\beta}{t'}\right) + h\left(\frac{\beta}{r'} - \frac{\beta}{t'}\right)$$

now depends on two parameters

$$r' = \frac{2}{|D_1 - D_2|}, \quad t' = \frac{2}{|D_1 + D_2|}.$$

(Martinetti, Priou, 2025)

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- When the diagonal is zero, then $t' \to \infty$ and $r' = \frac{1}{|m|} = r$, so that one finds back $S(r', t') = S(r)$.
- When the diagonal is non-zero, $d(\delta_1, \delta_2) = r \neq r'$.

Possible to express the entropy as $S(r, \tilde{r})$ for some parameter $\tilde{r}$?

III. Entropy of the Bloch sphere

$\mathcal{A} = M_2(\mathbb{C}), \quad \mathcal{H} = \mathbb{C}^2, \quad D \in M_2(\mathbb{C})$ with eigenvalues D_1, D_2.

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The space of pure states of $\mathcal{A}$ is $\mathbb{C}P^1$:

$$\xi = \left(\xi_1, \xi_2\right) \in \mathbb{C}^2,\ |\xi_1|^2 + |\xi_2|^2 = 1 \quad \text{such that} \quad \xi \simeq \zeta \Longleftrightarrow \xi = e^{i\theta}\zeta.$$

Any class of equivalence defines a pure state

$$\omega_\xi(m) = \langle \xi, \xi m \rangle \quad \forall m \in M_2(\mathbb{C})$$

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and any pure state comes in this way. This space is isomorphic to the sphere,

$$\xi \to \begin{pmatrix} x_\xi \\ y_\xi \\ z_\xi \end{pmatrix} \in S^2 \text{ where } x_\xi = 2\mathcal{R}e(\bar{\xi}_1\xi_2), \quad y_\xi = 2\mathcal{I}m(\bar{\xi}_1\xi_2,), \quad z_\xi = |xi_1|^2 - |\xi_2|^2,$$

called the Bloch sphere in quantum computing (describing a q-bit).

The distance coincides with the euclidean distance on a disk

$$d(\omega_\xi, \omega_\zeta) = \begin{cases} \frac{2}{|D_1 - D_2|}\sqrt{(x_\xi - x_\zeta)^2 + (y_\xi - y_\zeta)^2} & \text{if } z_\xi = z_\zeta, \\ = \infty & \text{if } z_\xi \neq z_\zeta. \end{cases}$$

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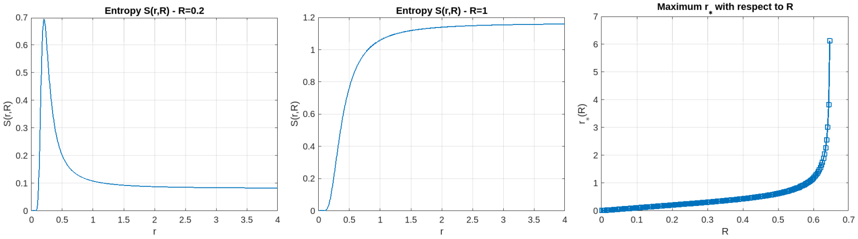
- The parameter r fixes the radius of the circle.

Assume the value of t is fixed. Then

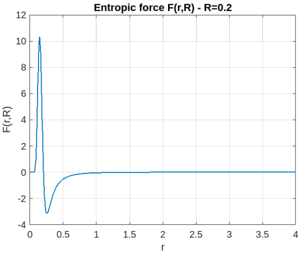
$$S(r,t) \to 0 \quad \text{when } r \to 0,$$

$$S(r,t) \to 2h(\frac{\beta}{R}) \quad \text{when } r \to \infty.$$

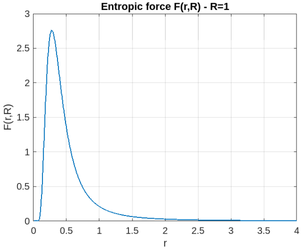
- For $t < 0,647\beta$, the maximum of the entropy is reached for a finite r



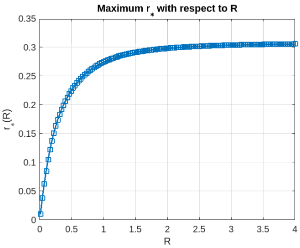
The entropic force tends to 0 as r tends to 0 or ∞.



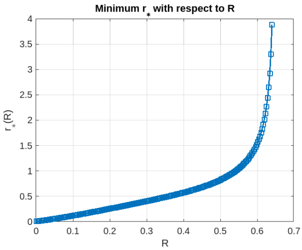
(a) $R = 0.2$



(b) $R = 1$



(c) Maximum



(d) Minimum

It has negative minimum for $t < 0,647\beta$.

Conclusion

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- Test function f related to the coupling constants. What happens for h ?
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