Towards loop space quantum gravity

Niccolò Cribiori

KU LEUVEN fwo

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Work in progress with C. Kneissl, A. Makridou

M-theory is the non-perturbative completion of string theory conjectured in [Townsend '95, Witten '95].

Remarkably its low-energy limit is 11D SUGRA, constructed in [Cremmer, Julia, Scherk '78]. It contains graviton, gravitino and a **3-form "gauge field"**, henceforth **C-field**

Its fundamental formulation and mathematical structure has remained elusive since 1995. It is the missing corner in our theoretical understanding of quantum gravity.

For recent progress, see talks by R. Blumenhagen, A. Paraskevopoulou and J. Van Muiden.

In this talk I will argue that

$$\Omega_n^{\mathrm{Spin}}(BLE_8) = \text{spin-bordism of the loop space of } E_8$$

is **relevant to quantum gravity**: it contains info on **M-theory**.

I will follow an **empirical** and **bottom-up** approach and discuss only topological properties.

I will show preliminary results that may serve as proof of principles.

Motivating bordism in M-theory

Some puzzles (1/2)

Dimensional reduction of M-theory gives type IIA string theory

- Distinction between RR and NS sector is perturbative.
  M-theory should mix them. How?

- For $H_3 = 0$, RR fields G_p classified by K-theory [Moore, Minasian '97] all on the same footing (Bott periodicity).
  Unclear from M-theory: e.g., G_2 requires non-trivial spacetime circle bundle, while G_4 not.

- For $H_3 \neq 0$, RR fields G_p classified by twisted K-theory. Not satisfactory as it threats NS and RR sectors differently.

Some puzzles (2/2)

Describing the C-field of M-theory is a long-standing open problem.

- [Diaconescu, Freed, Moore, '03]: C-field is shifted differential character *possibly* modelled by triplet (P, A, c), where P is a **principal** E_8 **bundle** with connection A and c is a three-form

- [Sati '13; Fiorenza, Sati, Schreiber '19], *hypothesis H*: C-field quantized in certain twisted cohomotopy cohomology. It is "dual" version of **homotopy** and twist involves an S^4.

Notice that $\pi_3(E_8) = \pi_4(S^4)$, but otherwise not clear (to me) precise relation between the two proposals.

Is there a third way in the between?

Homotopy groups and G-bundles

Topologically non-trivial G-bundles can be detected by homotopy

$$\pi_{n-1}(G) \neq 0 \qquad \Rightarrow \qquad \exists \quad \text{non-trivial } G\text{-bundle on } S^n$$

Idea: G-bundles can be constructed from configurations on the northern and southern hemisphere by gluing along the equator S^{n-1}. Then, count different gauge transformations on S^{n-1}.

Example: Magnetic monopoles in 4D are topologically non-trivial U(1) bundles detected by $\pi_1(U(1)) = \mathbb{Z}$ and with charge $\int_{S^2} F_2$.

Codimension $n+1$ branes detected by $\pi_{n-1}(G)$. New examples in heterotic string [Kaidi, Ohmori, Tachikawa, Yonekura '23, Kneissl '24].

More general homotopy groups (1/2)

Generalization: homotopy groups of more complicated objects:

$$\pi_n(G) \qquad \longrightarrow \qquad \pi_n(X)$$

To guess X, recall that 11D SUGRA has fermions, hence at least

$$X = MSpin$$

$MSpin$ = Thom spin spectrum: collects spaces with spin-structure but any possible topology. It goes beyond spheres.

To **fix some topological information** Y, we can refine

$$X \to MSpin \wedge Y$$

More general homotopy groups (2/2)

Indeed, we want a C-field. Drawing from [Witten '96; Diaconescu, Moore, Witten '00] we model the **C-field as principal E_8-bundle**:

$$Y = BE_8 = K(Z, 4) = \text{classifying space of 4-forms}$$

(the functor B is called classifying space)

Hence, we arrive at (Thom-Pontryagin)

$$\pi_n(MSpin \wedge BE_8) = \Omega_n^{spin}(BE_8)$$

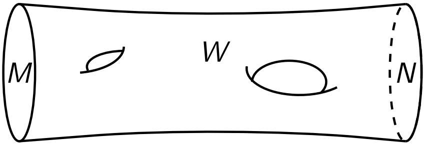
This captures non-trivial topology of spacetime and C-field. **Unification** expected in quantum gravity.

Now, the point is that bordism groups can be computed!

Spin bordism of a E_8 gauge theory

What is bordism?

Bordism is tool to classify closed n-dimensional manifold. Two manifolds are equivalent if bound $(n+1)$-dimensional manifold



$\Rightarrow \quad M \sim N$

equivalence relation

The **group of equivalence classes** can be further refined with

- ξ-tangential structure: Ω_n^ξ
- G-bundle information: $\Omega_n^\xi(BG)$

Claim [McNamara,Vafa '19]: bordism classifies backgrounds in quantum gravity.

Some bordism groups of interest

The first non-trivial fact is [Stong '86]

$$\Omega_4^{Spin}(BE_8)\Big|_{gauge\ part} = \mathbb{Z}$$

implying $\exists$ 4-manifolds M_4 on which G_4 can be integrated.

Recall 4D magnetic monopole detected by $\int_{S^2} F_2$.
Now, we are seeing a charge $\sim \int_{M_4} G_4$.

However [Stong '86]

$$\Omega_n^{Spin}(BE_8)\Big|_{gauge\ part} = 0 \qquad \text{for} \qquad n = 2, 3$$

Hence, **no room** for G_2 and H_3 $\Rightarrow$ **something else is needed**.

What is missing and why?

Absence of G_2 and H_3 because at no point we introduced info on the **dimensional reduction** M-theory $\to$ IIA.

How to implement dimensional reduction in bordism explained in [Blumenhagen, NC, Kneissl, Makridou '22].
In brief (independently of E_8 part)

$$\Omega_n^{\xi}(Z) \quad \to \quad \text{dimensional reduction on } Z$$

it reproduces expected pattern of charges. Checked explicitly for $Z = S^m, T^m, K3, CY_3$.

One can see that $BE_8 \times S^1$ will not do the job. We need a more sophisticated version. This is where the **loop space** enters.

Loop space bordism and M-theory

Loop space construction

Find way to combine circle bundle with E_8 bundle of the C-field.

A proposal made by [Bergman, Varadarajan '04]. Two steps:

- First construct **free loop space** $LE_8 = E_8 \times \omega E_8$
  (ωX = space of based loops on X)
  However, this will force the circle bundle to be trivial. Hence, we cannot see G_2.
- Then, correct with U(1) bundle to get $\widehat{LE_8} = LE_8 \rtimes U(1)$

Our idea [NC, Kneissl, Makridou, wip]: bordism groups

$$\Omega_n^{Spin}(\widehat{LE_8})$$

see all G_2, H_3, G_4 arising from dimensional reduction of C-field.

We can compute those groups and check!

Some loop space bordism groups

By explicit computation, we find [NC, Kneissl, Makridou, wip]

$$\Omega_n^{Spin}(B\widehat{LE_8})\Big|_{gauge\ part} = \mathbb{Z} \qquad \text{for} \qquad n = 2, 3, 4$$

corresponding to non-trivial charges

$$\int_{M_2} G_2, \qquad \int_{M_3} H_3, \qquad \int_{M_4} G_4 \qquad \neq 0$$

Comments:

- non-trivial consistency check
- G_2 and G_4 on same footing
- NS and RR sector on same footing

What next?

1. Complete computation of all bordism groups for M-theory:

$$\Omega_n^{Spin}(B\widehat{LE_8}) \qquad \text{for} \quad n = 0, 1, \ldots, 12$$

2. Then, complete matching with known M-theory quantities.

3. We find not just a "gauge" part but also other structures
   - purely gravitational (it survives turning off E_8 bundle)
   - torsion $\mathbb{Z}_k$ for $k = 2, 3$

4. Once step 2 completed, we can propose physical interpretation for new structures in step 3:

New non-SUSY branes and new anomalies?

Thank you!

Extra slides

RR fields and K-theory

RR charges in string theory are not really described by cohomology but by K-theory [Moore, Minasian '97]. The same applies to RR fields [Moore, Witten '99]. Given spacetime X and element $x \in K(X)$, we have

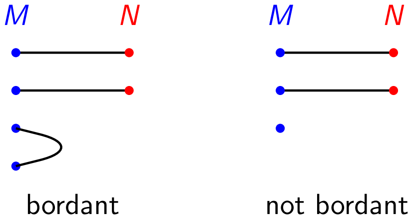
$$\frac{G_{2p}}{2\pi} = \sqrt{\hat{A}(X)}\, ch(x)|_{2p}$$

For example

$$\frac{G_2}{2\pi} = c_1(x), \qquad \frac{G_4}{2\pi} = \frac{1}{2}c_1(x)^2 - c_2(x)$$

Some simple examples

- $\Omega_0 = \mathbb{Z}_2$.
  For example $M = \sqcup_m \mathrm{pt}$, $N = \sqcup_n \mathrm{pt}$ bordant iff $m + n$ is even



- $\Omega_1 = 0$. Indeed the circle is null-bordant