Coloring of some simple Lie (super)algebras

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Color algebras and color superalgebras

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- color algebras and color superalgebras: graded by some abelian grading group Γ
- the simplest case not coinciding with a Lie algebra or Lie superalgebra is for $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$.
- for an algebra graded by $\mathbb{Z}_2 \times \mathbb{Z}_2 = \mathbb{Z}_2^2$, there are already two distinct choices for the Lie bracket: referred to as $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras and $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras

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- graded (quantum) mechanics and quantization [Bruce 2020; Aizawa, Kuznetsova, Toppan 2020, 2021; Quesne 2021]
- $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded two-dimensional models [Bruce 2021, Toppan 2021]
- parastatistics [Tolstoy 2014, Stoilova and Van der Jeugt 2018-2023]
- alternative descriptions of parabosons and parafermions [Toppan 2021-2025]
- algebraic structute and representation theory [Aizawa 2018-2025, Issac 2019, 2024, Rui Lu 2023], Stoilova and Van der Jeugt 2018-2025

The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras and Lie superalgebras

V. Rittenberg and D. Wyler (1978)

- $\mathfrak{g} = \bigoplus_{\boldsymbol{a}} \mathfrak{g}_{\boldsymbol{a}} = \mathfrak{g}_{(0,0)} \oplus \mathfrak{g}_{(0,1)} \oplus \mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(1,1)}$
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- homogeneous elements of $\mathfrak{g}_{\boldsymbol{a}}$: $x_{\boldsymbol{a}}$ with degree $deg\ x_{\boldsymbol{a}} = \boldsymbol{a}$
- $\mathfrak{g}$ with bracket $[\![.,.]\!]$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra, resp.
  $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra:

$$[\![x_{\boldsymbol{a}}, y_{\boldsymbol{b}}]\!] \in \mathfrak{g}_{\boldsymbol{a}+\boldsymbol{b}}, \quad \textit{grading}$$

$$[\![x_{\boldsymbol{a}}, y_{\boldsymbol{b}}]\!] = -(-1)^{\boldsymbol{a}\cdot\boldsymbol{b}}[\![y_{\boldsymbol{b}}, x_{\boldsymbol{a}}]\!], \quad \textit{symmetry}$$

$$[\![x_{\boldsymbol{a}}, [\![y_{\boldsymbol{b}}, z_{\boldsymbol{c}}]\!]]\!] = [\![[\![x_{\boldsymbol{a}}, y_{\boldsymbol{b}}]\!], z_{\boldsymbol{c}}]\!] + (-1)^{\boldsymbol{a}\cdot\boldsymbol{b}}[\![y_{\boldsymbol{b}}, [\![x_{\boldsymbol{a}}, z_{\boldsymbol{c}}]\!]]\!], \textit{Jacobi identities}$$

where

$\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1, a_2 + b_2) \in \mathbb{Z}_2 \times \mathbb{Z}_2$,

$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_2 - a_2 b_1$ - $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra

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General remarks

- Note: in general, a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra is NOT a Lie algebra, nor a Lie superalgebra.
- (Similarly: a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra is NOT a Lie superalgebra.)

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- $[\mathfrak{g}_{(0,0)}, \mathfrak{g}_{\boldsymbol{a}}] \subset \mathfrak{g}_{\boldsymbol{a}}, \quad [\![\mathfrak{g}_{\boldsymbol{a}}, \mathfrak{g}_{\boldsymbol{a}}]\!] \subset \mathfrak{g}_{(0,0)}, \qquad \boldsymbol{a} \in \mathbb{Z}_2 \times \mathbb{Z}_2$
- Let $\mathfrak{g}$ be an associative $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebra, with a product denoted by $x \cdot y$:

$$\mathfrak{g}_{\boldsymbol{a}} \cdot \mathfrak{g}_{\boldsymbol{b}} \subset \mathfrak{g}_{\boldsymbol{a}+\boldsymbol{b}}$$

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  then $(\mathfrak{g}, [\![\cdot, \cdot]\!])$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra, resp. a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra, by defining

$$[\![x_{\boldsymbol{a}}, y_{\boldsymbol{b}}]\!] = x_{\boldsymbol{a}} \cdot y_{\boldsymbol{b}} - (-1)^{\boldsymbol{a}\cdot\boldsymbol{b}} y_{\boldsymbol{b}} \cdot x_{\boldsymbol{a}},$$

  with $\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_2 - a_2 b_1$, resp. with $\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$.

Construction of classical $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras (J.Math.Phys.64 (2023) 061702; Springer proceedings in mathematics and statistics 473 (2025) 123)

- Now consider: $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras
- Assume at least two nontrivial subspaces in $\mathfrak{g}_{(0,1)} \oplus \mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(1,1)}$
- $\{\mathfrak{g}_{\boldsymbol{a}}, \mathfrak{g}_{\boldsymbol{b}}\} \subset \mathfrak{g}_{\boldsymbol{c}}$ if $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are mutually distinct elements of $\{(1,0),(0,1),(1,1)\}$.
- Classes of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras analogues of the classical Lie algebras (denining matrices)
- Natural to assume that $\mathfrak{g}$ is generated by $\mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(0,1)}$.
- Then one can deduce

$$\mathfrak{g}_{(0,0)} = [\![\mathfrak{g}_{(1,0)}, \mathfrak{g}_{(1,0)}]\!] + [\![\mathfrak{g}_{(0,1)}, \mathfrak{g}_{(0,1)}]\!]$$
$$\mathfrak{g}_{(1,1)} = [\![\mathfrak{g}_{(1,0)}, \mathfrak{g}_{(0,1)}]\!].$$

Construction of classical $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras

Let V be a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded linear space of dimension n:
$V = V_{(0,0)} \oplus V_{(0,1)} \oplus V_{(1,0)} \oplus V_{(1,1)}$, subspaces of dimension $p + q + r + s = n$.
$\mathrm{End}(V)$ is then a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded associative algebra, and turned into a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra by the bracket $[\![\cdot,\cdot]\!]$. Denoted by $\mathfrak{gl}_{p,q,r,s}(n)$. In matrix form:

$$\begin{array}{cc} \begin{array}{cccc} p & q & r & s \end{array} & \\ \begin{pmatrix} a_{(0,0)} & a_{(0,1)} & a_{(1,0)} & a_{(1,1)} \\ b_{(0,1)} & b_{(0,0)} & b_{(1,1)} & b_{(1,0)} \\ c_{(1,0)} & c_{(1,1)} & c_{(0,0)} & c_{(0,1)} \\ d_{(1,1)} & d_{(1,0)} & d_{(0,1)} & d_{(0,0)} \end{pmatrix} & \begin{array}{c} p \\ q \\ r \\ s \end{array} \end{array}$$

The indices of the matrix blocks refer to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading.

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The indices of the matrix blocks refer to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading.
One can check: $\mathrm{Tr}[\![A, B]\!] = 0$, hence $\mathfrak{g} = \mathfrak{sl}_{p,q,r,s}(n)$ is subalgebra of traceless elements.

Graded transpose

If $A \in \mathfrak{sl}_{p,q,r,s}(n) \subset \mathrm{End}(V)$, then $A^* \in \mathrm{End}(V^*)$ by requirement:

$$\langle A^* y_{\boldsymbol{b}}, x\rangle = (-1)^{\boldsymbol{a}\cdot\boldsymbol{b}} \langle y_{\boldsymbol{b}}, Ax\rangle$$

where $\langle \cdot, \cdot \rangle$ is natural pairing of V and V^*.

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In matrix form, this leads to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded transpose A^T of A:

$$A = \begin{pmatrix} a_{(0,0)} & a_{(0,1)} & a_{(1,0)} & a_{(1,1)} \\ b_{(0,1)} & b_{(0,0)} & b_{(1,1)} & b_{(1,0)} \\ c_{(1,0)} & c_{(1,1)} & c_{(0,0)} & c_{(0,1)} \\ d_{(1,1)} & d_{(1,0)} & d_{(0,1)} & d_{(0,0)} \end{pmatrix}, A^T = \begin{pmatrix} a^t_{(0,0)} & b^t_{(0,1)} & c^t_{(1,0)} & d^t_{(1,1)} \\ a^t_{(0,1)} & b^t_{(0,0)} & -c^t_{(1,1)} & -d^t_{(1,0)} \\ a^t_{(1,0)} & -b^t_{(1,1)} & c^t_{(0,0)} & -d^t_{(0,1)} \\ a^t_{(1,1)} & -b^t_{(1,0)} & -c^t_{(0,1)} & d^t_{(0,0)} \end{pmatrix}$$

Property:

$$\boxed{(AB)^T = (-1)^{\boldsymbol{a}\cdot\boldsymbol{b}} B^T A^T}$$

Subalgebra $\mathfrak{g} = \mathfrak{so}_{p,q,r,s}(n) \subset \mathfrak{sl}_{p,q,r,s}(n)$

$$\mathfrak{g} = \mathfrak{so}_{p,q,r,s}(n) = \{A \in \mathfrak{sl}_{p,q,r,s}(n) \mid A^T + A = 0\}$$

If $A, B \in \mathfrak{g}$, then

$$\begin{aligned} [\![A, B]\!]^T &= (AB - (-1)^{\boldsymbol{a}\cdot\boldsymbol{b}} BA)^T \\ &= (-1)^{\boldsymbol{a}\cdot\boldsymbol{b}} B^T A^T - A^T B^T = (-1)^{\boldsymbol{a}\cdot\boldsymbol{b}} BA - AB = -[\![A, B]\!] \end{aligned}$$

Matrices of the form:

$$\begin{array}{cc} \begin{array}{cccc} p & q & r & s \end{array} & \\ \begin{pmatrix} a_{(0,0)} & a_{(0,1)} & a_{(1,0)} & a_{(1,1)} \\ -a^t_{(0,1)} & b_{(0,0)} & b_{(1,1)} & b_{(1,0)} \\ -a^t_{(1,0)} & b^t_{(1,1)} & c_{(0,0)} & c_{(0,1)} \\ -a^t_{(1,1)} & b^t_{(1,0)} & c^t_{(0,1)} & d_{(0,0)} \end{pmatrix} & \begin{array}{c} p \\ q \\ r \\ s \end{array} \end{array}$$

where $a_{(0,0)}$, $b_{(0,0)}$, $c_{(0,0)}$ and $d_{(0,0)}$ are antisymmetric matrices.

Disadvantages: Cartan subalgebra? (classical choice not abelian)

$\mathfrak{so}_{p,q,r,s}(n)_{(0,0)}$

Different approach

Analogues of classical Lie algebras of type B, C, D?

$G = \mathfrak{so}(2n+1)$ (dim $G = 2n^2 + n$)
$$\begin{array}{cc} \begin{matrix} n & n & 1 \end{matrix} & \\ \begin{pmatrix} a & b & c \\ d & -a^t & e \\ -e^t & -c^t & 0 \end{pmatrix} & \begin{matrix} n \\ n \\ 1 \end{matrix} \end{array}$$
b and d antisymmetric;

$G = \mathfrak{sp}(2n)$ (dim $G = 2n^2 + n$)
$$\begin{array}{cc} \begin{matrix} n & n \end{matrix} & \\ \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} & \begin{matrix} n \\ n \end{matrix} \end{array}$$
b and c symmetric;

$G = \mathfrak{so}(2n)$ (dim $G = 2n^2 - n$)
$$\begin{array}{cc} \begin{matrix} n & n \end{matrix} & \\ \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} & \begin{matrix} n \\ n \end{matrix} \end{array}$$
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Different approach

- start from a set of generators of the classical Lie algebra (in the defining matrix form)
- associate a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading on these generators
- compute new elements with these generators using the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded bracket, and see which matrix structures and algebras arise in this way.

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How to do this systematically?

- Let generating subspace S of the classical Lie algebra G correspond to the subspace $\mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(0,1)}$ of the associated $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra $\mathfrak{g}$, and generate $\mathfrak{g}$.
- Thus we are looking for generating subspaces S of a classical Lie algebra G such that $G = S + [S, S]$ (as vector space).
- Use all so-called 5-gradings $G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_1 \oplus G_2$ of G such that G is generated by $S = G_{-1} \oplus G_1$.

Different approach

Classification of those 5-gradings: [Stoilova and Van der Jeugt 2005]

Procedure:

- For each of the 5-gradings of G, let $S = G_{-1} \oplus G_1$ (as a subspace of the vector space of G).
- Partition S in all possible ways in two subspaces $\mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(0,1)}$.
- Construct from here the matrix elements of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra $\mathfrak{g}$ using the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded bracket.

This construction process is straightforward but very elaborate.

For $\mathfrak{sl}(n)$: same graded algebras $\mathfrak{sl}_{p,q,r,s}(n)$.
Results on following slides.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras of type B

The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra $\mathfrak{g} = \mathfrak{so}_p(2n+1)$ consists of all matrices of the following block form:

$$
\begin{array}{cc}
\begin{array}{ccccc} p & n-p & p & n-p & 1 \end{array} & \\
\left(\begin{array}{cc:cc:c}
a_{(0,0)} & a_{(1,1)} & b_{(0,0)} & b_{(1,1)} & c_{(0,1)} \\
\tilde{a}_{(1,1)} & \tilde{a}_{(0,0)} & b^t_{(1,1)} & \tilde{b}_{(0,0)} & c_{(1,0)} \\
\hdashline
d_{(0,0)} & d_{(1,1)} & -a^t_{(0,0)} & \tilde{a}^t_{(1,1)} & e_{(0,1)} \\
d^t_{(1,1)} & \tilde{d}_{(0,0)} & a^t_{(1,1)} & -\tilde{a}^t_{(0,0)} & e_{(1,0)} \\
\hdashline
-e^t_{(0,1)} & -e^t_{(1,0)} & -c^t_{(0,1)} & -c^t_{(1,0)} & 0
\end{array}\right) &
\begin{array}{c} p \\ n-p \\ p \\ n-p \\ 1 \end{array}
\end{array}
$$

where $b_{(0,0)}$, $\tilde{b}_{(0,0)}$, $d_{(0,0)}$ and $\tilde{d}_{(0,0)}$ are antisymmetric matrices.

$$
\begin{aligned}
&\dim \mathfrak{g}_{(0,0)} = 2n^2 - n - 4p(n-p)^2 \\
&\dim \mathfrak{g}_{(0,1)} = 2p, \quad \dim \mathfrak{g}_{(1,0)} = 2(n-p) \\
&\dim \mathfrak{g}_{(1,1)} = 4p(n-p).
\end{aligned}
$$

Note: $\dim \mathfrak{so}_p(2n+1) = \dim \mathfrak{so}(2n+1)$.

One can verify that $\mathfrak{g} = \mathfrak{so}_p(2n+1)$ consists of all matrices A of $\mathfrak{sl}_{2p,0,2n-2p,1}(2n)$ that satisfy

$$\boxed{A^T K + KA = 0}$$

where

$$K = \begin{array}{c} \left(\begin{array}{cc|cc|c} 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & -I & 0 \\ \hline I & 0 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right) \end{array} \begin{array}{c} p \\ n-p \\ p \\ n-p \\ 1 \end{array}$$

Note: $K^T = K$, $K^{-1} = K^t$.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras of type B

- Cartan subalgebra is straightforward, as it consists of the set of diagonal matrices
- basis for the Cartan subalgebra $\mathfrak{h}$ is given by (e_{ij}, 1 in the entry of row i, column j and 0 elsewhere)

$$h_i = e_{i,i} - e_{n+i,n+i} \qquad i = 1, \ldots, n$$

- now $\mathfrak{h} \subset \mathfrak{g}_{(0,0)}$, i.e. the Cartan subalgebra is just the Cartan subalgebra of the Lie algebra $\mathfrak{g}_{(0,0)}$, which makes further structure theory feasible

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras of type B

- In terms of the dual basis ϵ_j ($j = 1, \ldots, n$) of $\mathfrak{h}^*$ the roots and corresponding root vectors of $\mathfrak{so}_q(2n+1)$ are given by:

root	deg	root vector	
ϵ_j	$(0,1)$	$e_{j,2n+1} - e_{2n+1,j+n}$	$j = 1, \ldots, q$
ϵ_j	$(1,0)$	$e_{j,2n+1} - e_{2n+1,j+n}$	$j = q+1, \ldots, n$
$-\epsilon_j$	$(0,1)$	$e_{n+j,2n+1} - e_{2n+1,j}$	$j = 1, \ldots, q$
$-\epsilon_j$	$(1,0)$	$e_{n+j,2n+1} - e_{2n+1,j}$	$j = q+1, \ldots, n$
$\epsilon_j - \epsilon_k$	$(0,0)$	$e_{jk} - e_{k+n,j+n}$	$j \neq k = 1, .., q$ or $j \neq k = q+1, .., n$
$\epsilon_j - \epsilon_k$	$(1,1)$	$e_{jk} + e_{k+n,j+n}$	$j = 1, \ldots, q;\ k = q+1, \ldots, n$ or $j = q+1, \ldots, n;\ k = 1, \ldots, q$
$\epsilon_j + \epsilon_k$	$(0,0)$	$e_{j,k+n} - e_{k,j+n}$	$j < k = 1, .., q$ or $j < k = q+1, .., n$
$\epsilon_j + \epsilon_k$	$(1,1)$	$e_{j,k+n} + e_{k,j+n}$	$j = 1, .., q;\ k = q+1, .., n$
$-\epsilon_j - \epsilon_k$	$(0,0)$	$e_{j+n,k} - e_{k+n,j}$	$j < k = 1, .., q$ or $j < k = q+1, .., n$
$-\epsilon_j - \epsilon_k$	$(1,1)$	$e_{j+n,k} + e_{k+n,j}$	$j = 1, \ldots, q;\ k = q+1, \ldots, n$

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras of type B

- the positive roots are given by

$$\Delta^+ = \{\epsilon_j \ (j = 1, \ldots, n); \epsilon_j - \epsilon_k, \epsilon_j + \epsilon_k \ (1 \le j < k \le n)\}$$

- but note that there are four different types of roots, according to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ degree:

$$\begin{aligned}
\Delta^+_{(0,1)} &= \{\epsilon_j \ (j = 1, \ldots, q)\} \\
\Delta^+_{(1,0)} &= \{\epsilon_j \ (j = q+1, \ldots, n)\} \\
\Delta^+_{(0,0)} &= \{\epsilon_j - \epsilon_k, \epsilon_j + \epsilon_k \ (j < k = 1, \ldots, q \text{ or } j < k = q+1, \ldots, n) \\
\Delta^+_{(1,1)} &= \{\epsilon_j - \epsilon_k, \epsilon_j + \epsilon_k \ (j = 1, \ldots, q; k = q+1, \ldots, n)\}
\end{aligned}$$

- a set of simple roots (with their degrees) is given by

$$\begin{array}{ccccccccc}
\epsilon_1 - \epsilon_2 & \ldots & \epsilon_{q-1} - \epsilon_q & \epsilon_q - \epsilon_{q+1} & \epsilon_{q+1} - \epsilon_{q+2} & \ldots & \epsilon_{n-1} - \epsilon_n & \epsilon_n \\
(0,0) & \ldots & (0,0) & (1,1) & (0,0) & \ldots & (0,0) & (1,0)
\end{array}$$

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras of type B

- for the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra $\mathfrak{so}_q(2n+1)$ we have the same root space decomposition as for the Lie algebra $\mathfrak{so}(2n+1)$
- the main difference being the degree of the roots
- and the fact that both commutators and anti-commutators appear among the brackets between root vectors.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras of type C

The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra $\boxed{\mathfrak{g} = \mathfrak{sp}_p(2n)}$ (matrices):

$$\begin{array}{cc} \begin{array}{cccc} p & n-p & p & n-p \end{array} & \\ \left(\begin{array}{cc|cc} a_{(0,0)} & a_{(1,0)} & b_{(1,1)} & b_{(0,1)} \\ \tilde{a}_{(1,0)} & \tilde{a}_{(0,0)} & -b^t_{(0,1)} & \tilde{b}_{(1,1)} \\ \hline c_{(1,1)} & c_{(0,1)} & -a^t_{(0,0)} & -\tilde{a}^t_{(1,0)} \\ -c^t_{(0,1)} & \tilde{c}_{(1,1)} & -a^t_{(1,0)} & -\tilde{a}^t_{(0,0)} \end{array}\right) & \begin{array}{c} p \\ n-p \\ p \\ n-p \end{array} \end{array}$$

where $b_{(1,1)}$, $\tilde{b}_{(1,1)}$, $c_{(1,1)}$ and $\tilde{c}_{(1,1)}$ are symmetric matrices

$$\boxed{A^T J + JA = 0} \qquad (*)$$

$$J = \begin{array}{cc} \left(\begin{array}{cc|cc} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ \hline -I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array}\right) & \begin{array}{c} p \\ n-p \\ p \\ n-p \end{array} \end{array}$$

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras of type D

The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra $\boxed{\mathfrak{g} = \mathfrak{so}_p(2n)}$ (matrices):

$$\begin{array}{cc} \begin{array}{cccc} p & n-p & p & n-p \end{array} & \\ \left(\begin{array}{cc|cc} a_{(0,0)} & a_{(1,0)} & b_{(1,1)} & b_{(0,1)} \\ \tilde{a}_{(1,0)} & \tilde{a}_{(0,0)} & b^t_{(0,1)} & \tilde{b}_{(1,1)} \\ \hline c_{(1,1)} & c_{(0,1)} & -a^t_{(0,0)} & -\tilde{a}^t_{(1,0)} \\ c^t_{(0,1)} & \tilde{c}_{(1,1)} & -a^t_{(1,0)} & -\tilde{a}^t_{(0,0)} \end{array}\right) & \begin{array}{c} p \\ n-p \\ p \\ n-p \end{array} \end{array}$$

where $b_{(1,1)}$, $\tilde{b}_{(1,1)}$, $c_{(1,1)}$ and $\tilde{c}_{(1,1)}$ are antisymmetric matrices

$$\boxed{A^T K + KA = 0}$$

where

$$K = \begin{array}{cc} \left(\begin{array}{cc|cc} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ \hline I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \end{array}\right) & \begin{array}{c} p \\ n-p \\ p \\ n-p \end{array} \end{array}$$

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras (J.Phys.A 57 (2024) 095202)

- Now consider: $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras
- Let $\mathfrak{g}$ be an associative $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebra, with a product denoted by $x \cdot y$:

$$\mathfrak{g}_{\boldsymbol{a}} \cdot \mathfrak{g}_{\boldsymbol{b}} \subset \mathfrak{g}_{\boldsymbol{a}+\boldsymbol{b}}$$

then $(\mathfrak{g}, [\![\cdot,\cdot]\!])$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra by defining

$$[\![x_{\boldsymbol{a}}, y_{\boldsymbol{b}}]\!] = x_{\boldsymbol{a}} \cdot y_{\boldsymbol{b}} - (-1)^{\boldsymbol{a}\cdot\boldsymbol{b}} y_{\boldsymbol{b}} \cdot x_{\boldsymbol{a}} \ ,$$

with $\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded general linear Lie superalgebra

- Let V be a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded linear space, $V = V_{(0,0)} \oplus V_{(1,1)} \oplus V_{(1,0)} \oplus V_{(0,1)}$, with subspaces of dimension m_1, m_2, n_1 and n_2 respectively. End(V) is then a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded associative algebra.
- By the previous bracket: turned into a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra. This algebra is usually denoted by $\mathfrak{gl}(m_1, m_2|n_1, n_2)$.
- In matrix form, the elements are written as:

$$A = \begin{array}{c} \begin{array}{cccc} m_1 & m_2 & n_1 & n_2 \end{array} \\ \left(\begin{array}{cccc} a_{(0,0)} & a_{(1,1)} & a_{(1,0)} & a_{(0,1)} \\ b_{(1,1)} & b_{(0,0)} & b_{(0,1)} & b_{(1,0)} \\ c_{(1,0)} & c_{(0,1)} & c_{(0,0)} & c_{(1,1)} \\ d_{(0,1)} & d_{(1,0)} & d_{(1,1)} & d_{(0,0)} \end{array}\right) \end{array} \begin{array}{l} \\ m_1 \\ m_2 \\ n_1 \\ n_2 \end{array}$$

(the indices refer to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading; the size of the blocks is indicated in the lines above and to the right of the matrix.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded special linear Lie superalgebra

$$A = \begin{array}{cc} \begin{array}{cccc} m_1 & m_2 & n_1 & n_2 \end{array} & \\ \begin{pmatrix} a_{(0,0)} & a_{(1,1)} & a_{(1,0)} & a_{(0,1)} \\ b_{(1,1)} & b_{(0,0)} & b_{(0,1)} & b_{(1,0)} \\ c_{(1,0)} & c_{(0,1)} & c_{(0,0)} & c_{(1,1)} \\ d_{(0,1)} & d_{(1,0)} & d_{(1,1)} & d_{(0,0)} \end{pmatrix} & \begin{array}{c} m_1 \\ m_2 \\ n_1 \\ n_2 \end{array} \end{array}$$

The matrices of the Lie algebra $\mathfrak{gl}(m_1 + m_2 + n_1 + n_2)$, of the Lie superalgebra $\mathfrak{gl}(m_1 + m_2 | n_1 + n_2)$ and of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{gl}(m_1, m_2 | n_1, n_2)$ are all the same, but of course the bracket is different in all of these cases.

One can check that $\mathrm{Str}[\![A, B]\!] = 0$, where $\mathrm{Str}(A) = \mathrm{tr}(a_{(0,0)}) + \mathrm{tr}(b_{(0,0)}) - \mathrm{tr}(c_{(0,0)}) - \mathrm{tr}(d_{(0,0)})$ is the graded supertrace in terms of the ordinary trace tr. Hence $\mathfrak{sl}(m_1, m_2 | n_1, n_2)$ is defined as the subalgebra of elements of $\mathfrak{gl}(m_1, m_2 | n_1, n_2)$ with graded supertrace equal to 0.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded supertranspose A^T of A

Let $A \in \mathfrak{sl}(m_1, m_2|n_1, n_2) \subset \mathrm{End}(V)$ of degree $\boldsymbol{a} \in \mathbb{Z}_2 \times \mathbb{Z}_2$; V^* - dual to V, inheriting the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading from V; $\langle \cdot, \cdot \rangle$ - the natural pairing of V and V^*. Then $A^* \in \mathrm{End}(V^*)$ is determined by:

$$\langle A^* y_{\boldsymbol{b}}, x \rangle = (-1)^{\boldsymbol{a}\cdot\boldsymbol{b}} \langle y_{\boldsymbol{b}}, Ax \rangle, \qquad \forall y_{\boldsymbol{b}} \in V^*_{\boldsymbol{b}}, \forall x \in V.$$

This is extended by linearity to all elements of $\mathfrak{sl}(m_1, m_2|n_1, n_2)$. In matrix form, this yields the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded supertranspose A^T of A:

$$A^T = \begin{pmatrix} a^t_{(0,0)} & b^t_{(1,1)} & -c^t_{(1,0)} & -d^t_{(0,1)} \\ a^t_{(1,1)} & b^t_{(0,0)} & c^t_{(0,1)} & d^t_{(1,0)} \\ a^t_{(1,0)} & -b^t_{(0,1)} & c^t_{(0,0)} & -d^t_{(1,1)} \\ a^t_{(0,1)} & -b^t_{(1,0)} & -c^t_{(1,1)} & d^t_{(0,0)} \end{pmatrix},$$

a^t - ordinary matrix transpose. One can check (case by case, according to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading) that the graded supertranspose of matrices satisfies

$$(AB)^T = (-1)^{\boldsymbol{a}\cdot\boldsymbol{b}} B^T A^T$$

The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{osp}(2m_1+1, 2m_2|2n_1, 2n_2)$ consists of the set of matrices A from $\mathfrak{sl}(2m_1+1, 2m_2|2n_1, 2n_2)$ such that

$$A^T J + JA = 0$$

where

$$J = \begin{pmatrix} 0 & I_{m_1+m_2} & 0 & 0 & 0 \\ I_{m_1+m_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_1+n_2} \\ 0 & 0 & 0 & -I_{n_1+n_2} & 0 \end{pmatrix}.$$

(Explicit blocks of the matrices can be given, but technical.)

By deleting row $2m_1 + 2m_2 + 1$ and column $2m_1 + 2m_2 + 1$ in the matrix A of $\mathfrak{osp}(2m_1 + 1, 2m_2|2n_1, 2n_2)$, one obtains the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras $\mathfrak{osp}(2m_1, 2m_2|2n_1, 2n_2)$, the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras corresponding to the Lie superalgebras of type C and D.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra of type G_2 (J.Phys.A 58 (2025) 365201)

- the applied technique to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie (super)algebras of type A, B, C, D did not lead to any results for exceptional Lie (super)algebras
- first: a basis for the ordinary Lie algebra G_2 in terms of 7×7-matrices; (E_{ij} is the 7×7-matrix with a 1 at position (i, j) and zeroes elsewhere)
- Matrix form of Chevalley basis, as 7×7-matrices.

$$h_1 = -E_{11} + 2E_{22} - E_{33} + E_{44} - 2E_{55} + E_{66}, \quad h_2 = E_{11} - E_{22} - E_{44} + E_{55}$$
$$x_1 = E_{35} - E_{26} + \sqrt{2}E_{71} - \sqrt{2}E_{47}, \quad x_2 = E_{16} - E_{34} + \sqrt{2}E_{72} - \sqrt{2}E_{57},$$
$$x_3 = -E_{15} + E_{24} + \sqrt{2}E_{73} - \sqrt{2}E_{67}, \quad y_1 = -E_{53} + E_{62} - \sqrt{2}E_{17} + \sqrt{2}E_{74},$$
$$y_2 = -E_{61} + E_{43} - \sqrt{2}E_{27} + \sqrt{2}E_{75}, \quad y_3 = E_{51} - E_{42} - \sqrt{2}E_{37} + \sqrt{2}E_{76},$$
$$a_{12} = E_{12} - E_{54}, \quad a_{23} = E_{23} - E_{65}, \quad a_{13} = E_{13} - E_{64},$$
$$a_{21} = E_{21} - E_{45}, \quad a_{32} = E_{32} - E_{56}, \quad a_{31} = E_{31} - E_{46}.$$

G_2 commutator table

$[\cdot,\cdot]$	h_1	h_2	a_{12}	a_{13}	a_{23}	a_{21}	a_{31}	a_{32}	x_1	x_2	x_3	y_1	y_2	y_3
h_1	0	0	$-3a_{12}$	0	$3a_{23}$	$3a_{21}$	0	$-3a_{32}$	x_1	$-2x_2$	x_3	$-y_1$	$2y_2$	$-y_3$
h_2		0	$2a_{12}$	a_{13}	$-a_{23}$	$-2a_{21}$	$-a_{31}$	a_{32}	$-x_1$	x_2	0	y_1	$-y_2$	0
a_{12}			0	0	a_{13}	h_2	$-a_{32}$	0	$-x_2$	0	0	0	y_1	0
a_{13}				0	0	$-a_{23}$	h_1+2h_2	a_{12}	$-x_3$	0	0	0	0	y_1
a_{23}					0	0	a_{21}	h_1+h_2	0	$-x_3$	0	0	0	y_2
a_{21}						0	0	$-a_{31}$	0	$-x_1$	0	y_2	0	0
a_{31}							0	0	0	0	$-x_1$	y_3	0	0
a_{32}								0	0	0	$-x_2$	0	y_3	0
x_1									0	$2y_3$	$-2y_2$	h_1+3h_2	$3a_{21}$	$3a_{31}$
x_2										0	$2y_1$	$3a_{12}$	h_1	$3a_{32}$
x_3											0	$3a_{13}$	$3a_{23}$	$-2h_1-3h_2$
y_1												0	$2x_3$	$-2x_2$
y_2													0	$2x_1$
y_3														0

Note that this is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded basis of G_2, with the degree of the elements given as follows:

$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
h_1, h_2	x_1, y_1, a_{23}, a_{32}	x_2, y_2, a_{13}, a_{31}	x_3, y_3, a_{12}, a_{21}

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded (color) Lie algebra of type G_2

Same principle as before: certain sign changes in defining matrices, e.g.

$$x_2 = \begin{bmatrix} 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-\sqrt{2}\\ 0&0&0&0&0&0&0\\ 0&\sqrt{2}&0&0&0&0&0 \end{bmatrix} \quad \to \quad \tilde{x}_2 = \begin{bmatrix} 0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&\sqrt{2}\\ 0&0&0&0&0&0&0\\ 0&\sqrt{2}&0&0&0&0&0 \end{bmatrix}$$

$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$\tilde{h}_1, \tilde{h}_2$	$\tilde{x}_1, \tilde{y}_1, \tilde{a}_{23}, \tilde{a}_{32}$	$\tilde{x}_2, \tilde{y}_2, \tilde{a}_{13}, \tilde{a}_{31}$	$\tilde{x}_3, \tilde{y}_3, \tilde{a}_{12}, \tilde{a}_{21}$

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra of type G_2,

$$[\![x_a, y_b]\!] = x_a \cdot y_b - (-1)^{a_1 b_2 - a_2 b_1} y_b \cdot x_a$$

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded (color) Lie algebra of type G_2: brackets

$[\![\cdot,\cdot]\!]$		$\tilde{h}_1$	$\tilde{h}_2$	$\tilde{a}_{12}$	$\tilde{a}_{13}$	$\tilde{a}_{23}$	$\tilde{a}_{21}$	$\tilde{a}_{31}$	$\tilde{a}_{32}$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{y}_1$	$\tilde{y}_2$	$\tilde{y}_3$
(0,0)	$\tilde{h}_1$	0	0	$-3\tilde{a}_{12}$	0	$3\tilde{a}_{23}$	$3\tilde{a}_{21}$	0	$-3\tilde{a}_{32}$	$\tilde{x}_1$	$-2\tilde{x}_2$	$\tilde{x}_3$	$-\tilde{y}_1$	$2\tilde{y}_2$	$-\tilde{y}_3$
(0,0)	$\tilde{h}_2$		0	$2\tilde{a}_{12}$	$\tilde{a}_{13}$	$-\tilde{a}_{23}$	$-2\tilde{a}_{21}$	$-\tilde{a}_{31}$	$\tilde{a}_{32}$	$-\tilde{x}_1$	$\tilde{x}_2$	0	$\tilde{y}_1$	$-\tilde{y}_2$	0
(1,1)	$\tilde{a}_{12}$			0	0	$\tilde{a}_{13}$	$\tilde{h}_2$	$\tilde{a}_{32}$	0	$\tilde{x}_2$	0	0	0	$\tilde{y}_1$	0
(1,0)	$\tilde{a}_{13}$				0	0	$\tilde{a}_{23}$	$\tilde{h}_1+2\tilde{h}_2$	$\tilde{a}_{12}$	$\tilde{x}_3$	0	0	0	0	$-\tilde{y}_1$
(0,1)	$\tilde{a}_{23}$					0	0	$\tilde{a}_{21}$	$\tilde{h}_1+\tilde{h}_2$	0	$\tilde{x}_3$	0	0	0	$-\tilde{y}_2$
(1,1)	$\tilde{a}_{21}$						0	0	$\tilde{a}_{31}$	0	$\tilde{x}_1$	0	$\tilde{y}_2$	0	0
(1,0)	$\tilde{a}_{31}$							0	0	0	0	$\tilde{x}_1$	$-\tilde{y}_3$	0	0
(0,1)	$\tilde{a}_{32}$								0	0	0	$\tilde{x}_2$	0	$-\tilde{y}_3$	0
(0,1)	$\tilde{x}_1$									0	$2\tilde{y}_3$	$-2\tilde{y}_2$	$\tilde{h}_1+3\tilde{h}_2$	$-3\tilde{a}_{21}$	$3\tilde{a}_{31}$
(1,0)	$\tilde{x}_2$										0	$-2\tilde{y}_1$	$-3\tilde{a}_{12}$	$\tilde{h}_1$	$3\tilde{a}_{32}$
(1,1)	$\tilde{x}_3$											0	$-3\tilde{a}_{13}$	$-3\tilde{a}_{23}$	$2\tilde{h}_1+3\tilde{h}_2$
(0,1)	$\tilde{y}_1$												0	$2\tilde{x}_3$	$-2\tilde{x}_2$
(1,0)	$\tilde{y}_2$													0	$-2\tilde{x}_1$
(1,1)	$\tilde{y}_3$														0

Lie algebra G_2

$$e_1 = A^{010}_{100} \quad \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&\bar{1}&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&\bar{1}&0 \end{bmatrix}$$

$$e_2 = A^{001}_{100} \quad \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&\bar{1}&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&\bar{1} \\ 0&0&0&0&0&0&1&0 \end{bmatrix}$$

$$e_3 = A^{100}_{010} \quad \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&\bar{1}&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&\bar{1} \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \end{bmatrix}$$

$$e_4 = A^{001}_{010} \quad \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&\bar{1}&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&\bar{1}&0&0 \end{bmatrix}$$

$$e_5 = A^{110}_{110} \quad \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&\bar{1}&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&\bar{1}&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix}$$

$$e_6 = A^{001}_{110} \quad \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&\bar{1} \\ 0&0&0&0&0&0&\bar{1}&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&1&0&0&0 \end{bmatrix}$$

$$e_7 = A^{100}_{001} \quad \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0 \\ 0&\bar{1}&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&\bar{1}&0&0&0&0 \end{bmatrix}$$

$$e_8 = A^{010}_{001} \quad \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&\bar{1}&0 \\ 0&0&0&0&0&0&0&\bar{1} \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \end{bmatrix}$$

$$e_9 = A^{101}_{101} \quad \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&\bar{1}&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&\bar{1}&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix}$$

$$e_{10} = A^{010}_{101} \quad \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&\bar{1} \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&\bar{1}&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \end{bmatrix}$$

$$e_{11} = A^{100}_{011} \quad \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&\bar{1} \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&\bar{1}&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \end{bmatrix}$$

$$e_{12} = A^{011}_{011} \quad \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&\bar{1}&0&0 \\ 0&0&\bar{1}&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix}$$

$$e_{13} = A^{101}_{111} \quad \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&\bar{1}&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&\bar{1}&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix}$$

$$e_{14} = A^{011}_{111} \quad \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&\bar{1}&0&0 \\ 0&0&0&0&\bar{1}&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix}$$

($\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \equiv \mathbb{Z}_2^3$-graded) Lie algebra G_2

Relation with Chevalley basis

One can now take as basis of a Cartan subalgebra of $\mathfrak{g}$:

$$h_1 = ie_2 - ie_1\,, \quad h_2 = ie_1$$

and consider the following twelve root vectors with respect to this Cartan subalgebra:

$$x_1 = e_3 + \frac{1}{2}e_4 + ie_5 + \frac{i}{2}e_6\,, \quad x_2 = e_{11} + \frac{1}{2}e_{12} + ie_{13} + \frac{i}{2}e_{14}\,, \quad x_3 = e_7 +$$
$$y_1 = e_3 + \frac{1}{2}e_4 - ie_5 - \frac{i}{2}e_6\,, \quad y_2 = e_{11} + \frac{1}{2}e_{12} - ie_{13} - \frac{i}{2}e_{14}\,, \quad y_3 = e_7 +$$
$$a_{12} = \frac{1}{2}e_8 - \frac{i}{2}e_{10}\,, \quad a_{13} = -\frac{1}{2}e_{12} + \frac{i}{2}e_{14}\,, \quad a_{23} = \frac{1}{2}e_4 + \frac{i}{2}e_6\,,$$
$$a_{21} = -\frac{1}{2}e_8 - \frac{i}{2}e_{10}\,, \quad a_{31} = \frac{1}{2}e_{12} + \frac{i}{2}e_{14}\,, \quad a_{32} = -\frac{1}{2}e_4 + \frac{i}{2}e_6\,.$$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-graded color Lie algebra of type G_2: case 1

- Sign factor

$$\langle \alpha, \beta \rangle = \alpha_1 \beta_2 + \alpha_2 \beta_1, \quad \alpha = (\alpha_1, \alpha_2, \alpha_3)$$

$$[\![x_\alpha, y_\beta]\!] = x_\alpha \cdot y_\beta - (-1)^{\langle \alpha, \beta \rangle} y_\beta \cdot x_\alpha$$

- New basis elements $\tilde{e}_i$ (old G_2 basis e_i), $i = 1, \ldots, 14$.
- Grading:

000	100	010	110	001	101	011	111
	e_1, e_2	e_3, e_4	e_5, e_6	e_7, e_8	e_9, e_{10}	e_{11}, e_{12}	e_{13}, e_{14}
	$\tilde{e}_1, \tilde{e}_2$	$\tilde{e}_3, \tilde{e}_4$	$\tilde{e}_5, \tilde{e}_6$	$\tilde{e}_7, \tilde{e}_8$	$\tilde{e}_9, \tilde{e}_{10}$	$\tilde{e}_{11}, \tilde{e}_{12}$	$\tilde{e}_{13}, \tilde{e}_{14}$

- Brackets: $[e_i, e_j] \sim e_k \to [\![\tilde{e}_i, \tilde{e}_j]\!] \sim \pm \tilde{e}_k$, e.g.

$$\begin{array}{llll} [e_1, e_4] = -e_6 & [e_1, e_5] = e_3 & [e_1, e_6] = e_4 & [e_1, e_7] = e_{10} \quad \to \\ \{\tilde{e}_1, \tilde{e}_4\} = \tilde{e}_6 & \{\tilde{e}_1, \tilde{e}_5\} = -\tilde{e}_3 & \{\tilde{e}_1, \tilde{e}_6\} = -\tilde{e}_4 & [\tilde{e}_1, \tilde{e}_7] = \tilde{e}_{10} \end{array}$$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times Z_2$-graded color Lie algebra of type G_2: one case

$\tilde{e}_1 \sim A^{010}_{100}$
$$\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&\bar{1}&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&\bar{1}&0 \end{bmatrix}$$

$\tilde{e}_2 \sim A^{001}_{100}$
$$\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&\bar{1}&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&\bar{1}\\ 0&0&0&0&0&0&1&0 \end{bmatrix}$$

$\tilde{e}_3 \sim A^{100}_{010}$
$$\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&\bar{1}&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&\bar{1}\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0 \end{bmatrix}$$

$\tilde{e}_4 \sim A^{001}_{010}$
$$\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&{\color{red}1}&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&{\color{red}\bar{1}}&0&0&0\\ 0&0&0&0&0&\bar{1}&0&0 \end{bmatrix}$$

$\tilde{e}_5 \sim A^{110}_{110}$
$$\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&{\color{red}1}&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&{\color{red}1}&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}$$

$\tilde{e}_6 \sim A^{001}_{110}$
$$\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&{\color{red}1}\\ 0&0&0&0&0&0&\bar{1}&0\\ 0&0&0&0&0&{\color{red}\bar{1}}&0&0\\ 0&0&0&0&1&0&0&0 \end{bmatrix}$$

$\tilde{e}_7 \sim A^{100}_{001}$
$$\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 0&\bar{1}&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&\bar{1}&0&0&0&0 \end{bmatrix}$$

$\tilde{e}_8 \sim A^{010}_{001}$
$$\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&\bar{1}&0\\ 0&0&0&0&0&0&0&\bar{1}\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0 \end{bmatrix}$$

$\tilde{e}_9 \sim A^{101}_{101}$
$$\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&\bar{1}&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&\bar{1}&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}$$

$\tilde{e}_{10} \sim A^{010}_{101}$
$$\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&\bar{1}\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&\bar{1}&0&0&0&0\\ 0&0&1&0&0&0&0&0 \end{bmatrix}$$

$\tilde{e}_{11} \sim A^{100}_{011}$
$$\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&\bar{1}\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&\bar{1}&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0 \end{bmatrix}$$

$\tilde{e}_{12} \sim A^{011}_{011}$
$$\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&{\color{red}\bar{1}}&0&0&0\\ 0&0&0&0&0&\bar{1}&0&0\\ 0&0&{\color{red}1}&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}$$

$\tilde{e}_{13} \sim A^{101}_{111}$
$$\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&\bar{1}&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&{\color{red}1}&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&{\color{red}\bar{1}}&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}$$

$\tilde{e}_{14} \sim A^{011}_{111}$
$$\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&{\color{red}1}&0&0\\ 0&0&0&0&\bar{1}&0&0&0\\ 0&0&0&{\color{red}\bar{1}}&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}$$

($\mathbb{Z}_2^3$-graded color Lie algebra of type G_2: case 1)

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-graded color Lie algebra of type G_2: other cases

Case 2: sign factor

$$\langle \alpha, \beta \rangle = \alpha_1\beta_3 + \alpha_3\beta_1$$

Case 3: sign factor

$$\langle \alpha, \beta \rangle = \alpha_2\beta_3 + \alpha_3\beta_2$$

All of type 3_2.
Note: type 3_1 is ordinary Lie algebra.
Type 3_3 ($(-1)^{\alpha_i\beta_i}$), type 3_4 ($(-1)^{\alpha_i\beta_i+\alpha_j\beta_j}$) and type 3_5 ($(-1)^{\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3}$) would correspond to Γ-graded color superalgebras, so not surprising that they are missing.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded A-statistics (J.Geom.Symmetry Phys. 71 (2025) 1)

- A - arbitrary $(m_1 + m_2 + n_1 + n_2 + 1 \times m_1 + m_2 + n_1 + n_2 + 1)$-matrix of the following block form:

$$A = \begin{array}{cc} \begin{array}{cccc} m_1+1 & m_2 & n_1 & n_2 \end{array} & \\ \begin{pmatrix} a_{(0,0)} & a_{(1,1)} & a_{(1,0)} & a_{(0,1)} \\ b_{(1,1)} & b_{(0,0)} & b_{(0,1)} & b_{(1,0)} \\ c_{(1,0)} & c_{(0,1)} & c_{(0,0)} & c_{(1,1)} \\ d_{(0,1)} & d_{(1,0)} & d_{(1,1)} & d_{(0,0)} \end{pmatrix} & \begin{array}{l} m_1+1 \\ m_2 \\ n_1 \\ n_2 \end{array} \end{array}$$

- $$[\![A_{(a_1,a_2)}, \tilde{A}_{(b_1,b_2)}]\!] = A_{(a_1,a_2)}\tilde{A}_{(b_1,b_2)} - (-1)^{a_1 b_1 + a_2 b_2}\tilde{A}_{(b_1,b_2)}A_{(a_1,a_2)}$$

- Graded supertrace
  $\mathrm{Str}(A) = tr(a_{(0,0)}) + tr(b_{(0,0)}) - tr(c_{(0,0)}) - tr(d_{(0,0)})$
- $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded $\mathfrak{sl}(m_1+1, m_2|n_1, n_2)$ is the subalgebra of $\mathfrak{gl}(m_1+1, m_2|n_1, n_2)$ with graded supertrace equal to 0.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded A-statistics

- Let

$$d_i = \begin{cases} (0,0);\, i = 0, \ldots, m_1 \\ (1,1);\, i = m_1+1, \ldots, m_1+m_2 = m \\ (1,0);\, i = m_1+m_2+1, \ldots, m_1+m_2+n_1 = m+n_1 \\ (0,1);\, i = m_1+m_2+n_1+1, \ldots, m_1+m_2+n_1+n_2 = m+ \end{cases}$$

- and let $e_{ij},\ i,j = 0,1,\ldots,m_1+m_2+n_1+n_2 = m+n$ (where $m_1+m_2 = m, n_1+n_2 = n$) be the $(m+n+1 \times m+n+1)$ matrix A with 1 in the entry of row i, column j and 0 elsewhere. These matrices are homogeneous and the grading $\deg(e_{ij})$ is as follows:

$$\deg(e_{ij}) = d_i + d_j.$$

- A set of generators of $\mathfrak{sl}(m_1+1, m_2|n_1, n_2)$:

$$a_i^+ = e_{i0},\ \ a_i^- = e_{0i},\ \ i = 1,\ldots,m+n,\ \ (\deg(a_i^\pm) = d_i)$$

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded A-statistics

- Denote these generators by

$$a_i^{\pm} \equiv b_i^{\pm} \in \mathfrak{g}_{(0,0)}, \quad i = 1, \dots, m_1,$$
$$a_i^{\pm} \equiv \tilde{b}_{i-m_1}^{\pm} \in \mathfrak{g}_{(1,1)}, \quad i = m_1 + 1, \dots, m,$$
$$a_i^{\pm} \equiv f_{i-m}^{\pm} \in \mathfrak{g}_{(1,0)}, \quad i = m + 1, \dots, m + n_1,$$
$$a_i^{\pm} \equiv \tilde{f}_{i-m-n_1}^{\pm} \in \mathfrak{g}_{(0,1)}, \quad i = m + n_1 + 1, \dots, m + n.$$

- $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded $\mathfrak{sl}(m_1 + 1, m_2 | n_1, n_2)$ can be defined in terms of the generators $a_i^{\pm}, \ i = 1, \dots, m + n$ and the following relations:

$$[\![a_i^{\xi}, a_j^{\xi}]\!] = 0, \quad \xi = \pm, \ i, j = 1, \dots, m + n,$$
$$[\![[\![a_i^{+}, a_j^{-}]\!], a_k^{+}]\!] = \delta_{jk} a_i^{+} + (-1)^{d_i \cdot d_i} \delta_{ij} a_k^{+},$$
$$[\![[\![a_i^{+}, a_j^{-}]\!], a_k^{-}]\!] = -(-1)^{(d_i + d_j) \cdot d_k} \delta_{ik} a_j^{-} - (-1)^{d_i \cdot d_i} \delta_{ij} a_k^{-}, \quad i, j, k = 1, .$$

Fock representations

- assume that the corresponding representation space W_p contains (up to a multiple) a cyclic vector $|0\rangle$, such that

$$a_i^-|0\rangle = 0, \quad i = 1, 2, \ldots, n+m,$$
$$[\![a_i^-, a_j^+]\!]|0\rangle = \delta_{ij}p|0\rangle, \; p \in \mathbb{N}, \; i, j = 1, 2, \ldots, n+m.$$

- basis in the Fock space

$$|p; r_1, .., r_{m_1}, l_1, \ldots, l_{m_2}, \theta_1, \ldots, \theta_{n_1}, \lambda_1, \ldots, \lambda_{n_2}) = \sqrt{\frac{(p-R)!}{p! r_1! \ldots \lambda_{n_2}!}} \times$$

$$(b_1^+)^{r_1} \ldots (b_{m_1}^+)^{r_{m_1}} (\tilde{b}_1^+)^{l_1} \ldots (\tilde{b}_{m_2}^+)^{l_{m_2}} (f_1^+)^{\theta_1} \ldots (f_{n_1}^+)^{\theta_{n_1}} (\tilde{f}_1^+)^{\lambda_1} .. (\tilde{f}_{n_2}^+)^{\lambda_{n_2}}$$

$$r_i, l_i \in \mathbb{Z}_+, \; \theta_i, \lambda_i \in \{0, 1\}, \; R = \sum_{i=1}^{m_1} r_i + \sum_{i=1}^{m_2} l_i + \sum_{i=1}^{n_1} \theta_i + \sum_{i=1}^{n_2} \lambda_i \leq p.$$

Fock representations

The transformation of the basis:

$$b_i^+|p;..,r_{i-1},r_i,r_{i+1},\ldots) = \sqrt{(r_i+1)(p-R)}|p;\ldots,r_{i-1},r_i+1,r_{i+1},\ldots)$$

$$\tilde{b}_i^+|p;\ldots,l_{i-1},l_i,l_{i+1},\ldots) = \sqrt{(l_i+1)(p-R)}|p;\ldots,l_{i-1},l_i+1,l_{i+1},\ldots)$$

$$f_i^+|p;\ldots,\theta_{i-1},\theta_i,\theta_{i+1},\ldots) = (1-\theta_i)(-1)^{l_1+\ldots+l_{m_2}}(-1)^{\theta_1+\ldots+\theta_{i-1}}$$
$$\times\sqrt{p-R}|p;\ldots,\theta_{i-1},\theta_i+1,\theta_{i+1},\ldots),$$

$$\tilde{f}_i^+|p;\ldots,\lambda_{i-1},\lambda_i,\lambda_{i+1},\ldots) = (1-\lambda_i)(-1)^{l_1+\ldots+l_{m_2}}(-1)^{\lambda_1+\ldots+\lambda_{i-1}}$$
$$\times\sqrt{p-R}|p;\ldots,\theta_{i-1},\theta_i+1,\theta_{i+1},\ldots),$$

Fock representations

The transformation of the basis:

$$b_i^-|p;\ldots,r_{i-1},r_i,r_{i+1},\ldots) = \sqrt{r_i(p-R+1)}|p;\ldots,r_{i-1},r_i-1,r_{i+1},\ldots)$$

$$\tilde{b}_i^-|p;\ldots,l_{i-1},l_i,l_{i+1},\ldots) = \sqrt{l_i(p-R+1)}|p;\ldots,l_{i-1},l_i-1,l_{i+1},\ldots),$$

$$f_i^-|p;\ldots,\theta_{i-1},\theta_i,\theta_{i+1},\ldots) = \theta_i(-1)^{l_1+\ldots+l_{m_2}}(-1)^{\theta_1+\ldots+\theta_{i-1}}$$
$$\times\sqrt{p-R+1}|p;\ldots,\theta_{i-1},\theta_i-1,\theta_{i+1},\ldots),$$

$$\tilde{f}_i^-|p;\ldots,\lambda_{i-1},\lambda_i,\lambda_{i+1},\ldots) = \lambda_i(-1)^{l_1+\ldots+l_{m_2}}(-1)^{\lambda_1+\ldots+\lambda_{i-1}}$$
$$\times\sqrt{p-R+1}|p;\ldots,\theta_{i-1},\theta_i+1,\theta_{i+1},\ldots).$$

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$$H = \sum_{i=1}^{m} \varepsilon_i (\llbracket a_i^+, a_i^- \rrbracket + \llbracket a_i^+, a_i^- \rrbracket)$$

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$$H = \sum_{i=1}^{m} \varepsilon_i ([\![a_i^+, a_i^-]\!] + [\![a_i^+, a_i^-]\!])$$

- then

$$[H, a_i^\pm] = \pm \varepsilon_i a_i^\pm .$$

Microscopic properties of the underlying statistics

interpretation:

- $r_i,\ l_i,\ \theta_i,\ \lambda_i,$ - the number of particles on the corresponding orbital;
- then the operator a_i^+ increases this number by one, it "creates" a particle in the one-particle state ($=$ orbital) i
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- these are, Fermi like (resp. Bose like) properties
- **essential difference** - if the order of the statistics is p then no more than p "particles" can be accommodated in the system, $\sum_{i=1}^{m_1} r_i + \sum_{i=1}^{m_2} l_i + \sum_{i=1}^{n_1} \theta_i + \sum_{i=1}^{n_2} \lambda_i \leq p$

Microscopic properties of the underlying statistics

consider some configurations for $m = n = 6$, assume $p = 5$; denote by $\bullet$ a $b-$particle and by $\circ$ an $f-$particle, and represent the six orbibals by six boxes.

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- the state $|\bullet\circ\bullet|\circ\circ| \quad | \quad | \quad | \quad |$ is forbidden, because it contains two $f-$particles in the second box

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- the state $|\bullet\bullet\bullet|\bullet\circ\bullet|\quad|\quad|\quad|\quad|$ is forbidden; it is not possible to accommodate more than $p = 5$ particles
- the state $|\bullet\circ\bullet|\circ\bullet|\quad|\quad|\quad|\quad|$ is completely filled. It contains 5 particles; no more particles can be "loaded" even in the empty "boxes" (orbitals)
- the state $|\bullet\circ\bullet|\circ\circ|\quad|\quad|\quad|\quad|$ is forbidden, because it contains two $f-$particles in the second box
- consider the state $|\bullet\bullet\circ|\ \bullet\ |\quad|\quad|\quad|\quad|$; a new $b-$particle can be accommodated in any box, whereas the first box is "locked" for an $f-$particle; an $f-$particle can however be accommodated in any other box

Microscopic properties of the underlying statistics

consider some configurations for $m = n = 6$, assume $p = 5$; denote by $\bullet$ a $b-$particle and by $\circ$ an $f-$particle, and represent the six orbibals by six boxes.

- the state $|\bullet\bullet\bullet|\bullet\circ\bullet| \quad | \quad | \quad | \quad |$ is forbidden; it is not possible to accommodate more than $p = 5$ particles
- the state $|\bullet\circ\bullet|\circ\bullet| \quad | \quad | \quad | \quad |$ is completely filled. It contains 5 particles; no more particles can be "loaded" even in the empty "boxes" (orbitals)
- the state $|\bullet\circ\bullet|\circ\circ| \quad | \quad | \quad | \quad |$ is forbidden, because it contains two $f-$particles in the second box
- consider the state $|\bullet\bullet\circ| \ \bullet \ | \quad | \quad | \quad | \quad |$; a new $b-$particle can be accommodated in any box, whereas the first box is "locked" for an $f-$particle; an $f-$particle can however be accommodated in any other box
- this statistics - belongs to the class of the so-called *(fractional) exclusion statistics*

(Fractional) exclusion statistics (FES)

- first introduced by Haldane (Phys.Rev.Lett. **67** (1991) 937)
- it has emerged as a unifying framework for describing quantum systems whose quasiparticle excitations interpolate between bosonic and fermionic behavior
- prominent applications of FES arises in the study of low-dimensional electron systems, particularly the fractional quantum Hall effect, spin chains and integrable models such as the Haldane–Shastry and Calogero–Sutherland systems
- beyond condensed matter, FES has been employed in black hole thermodynamics and quantum gravity, and to thermodynamics of strongly correlated electron gases, Luttinger liquids, and quantum wires (A. Khare, *Fractional Statistics and Quantum Theory*, 2nd edn. (World Scientific, Singapore, 2005))

Conclusions / summary

- $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded classical Lie algebras and basic classical Lie superalgebras - definition, structure and representation theory;
- $\mathbb{Z}_2^2$-grading of G_2, graded Chevalley basis
- Can also be lifted to $\mathbb{Z}_2^2$-graded color Lie algebras of type G_2
- Graded basis labeled by points and lines of the Fano plane
- Colorings: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-graded color Lie algebras of type G_2
- Example: A-statistics: microscopic properties
- Outlook: $G(3)$? F_4 or $F(4)$?

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