On 1-loop graviton 4-point amplitudes with reduced supersymmetry

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Overview

- Motivation and earlier results
- Some details of our calculation
- Conclusion and outlook

Motivation

- [Brink, Green, Schwarz 1982]:
  1-loop 4-point graviton amplitude for $\mathcal{N}=8$ SUGRA from low energy limit of type II string theory on T^6

- [Dunbar, Norridge 1994]:
  Contribution to 1-loop 4-point graviton amplitude from arbitrary matter field in loop (i.e. arbitrary $\mathcal{N}$) via string based worldline methods [Bern, Dunbar, Shimada 1993]

However: Used spinor helicity methods

Spinor helicity formalism

- Only physical on-shell states used
- Much more compact formulas

However:

- Lose contact to local Lagrangian (i.e. low energy EFT)
- Dimension dependent

Motivation cont'd.

- [Tourkine, Vanhove 2013]:

  1-loop 4-point graviton amplitude for $\mathcal{N}=4$ SUGRA from low energy limit of type II orbifolds

  However: Result still given in spinor helicity formalism

- [Bern, Edison, Kosower, Parra-Martinez 2017]:

  Covariant result for *pure* $\mathcal{N}=4$ SUGRA using double copy $\mathcal{N}=4\otimes\mathcal{N}=0$

IR divergence from graviton running in the loop

L_i on next slide

$$M^{1-\text{loop}}_{\mathcal{N}=4,\text{SG}} = c_\Gamma \Big[M^{\text{tree}}_{\mathcal{N}=4,\text{SG}} \Big(-\frac{2}{\epsilon^2} \sum_{i<j}^{3} s_{ij} \Big(\frac{-s_{ij}}{\mu^2} \Big)^{-\epsilon} + L_1(s,t,u) \Big)$$

$$+ \Big(\frac{4}{3} t_8 \text{tr}(R_1 R_2 R_3 R_4) \Big(\frac{1}{st} + L_2(s,t,u) \Big)$$

$$+ t_8 \text{tr}(R_1 R_2) \text{tr}(R_3 R_4) \Big(\frac{1}{s^2} + L_3(s,t,u) \Big) + \text{cyclic}(2,3,4) \Big) \Big]$$

$$\frac{2}{\alpha'} R_{i\mu\nu\rho\sigma} = (k_{i\mu}\epsilon_{i\nu} - k_{i\nu}\epsilon_{i\mu})(k_{i\rho}\epsilon_{i\sigma} - k_{i\sigma}\epsilon_{i\rho})$$

2 independent R^4-terms in D=4: [Fulling, King, Wybourne, Cummins 1992]

$t_8 \text{tr} R^4$ & $t_8 (\text{tr} R^2)^2$

$$L_1(s,t,u) = -s\ln\left(\frac{-s}{\mu^2}\right) - \frac{2s^2+st+2t^2}{2u}\left(\ln^2\left(\frac{-s}{-t}\right)+\pi^2\right) + \text{cyclic}(s,t,u)$$

$$L_2(s,t,u) = \Big[-\frac{2s}{t^2u}\ln\left(\frac{-s}{-u}\right) + \frac{1}{4u^2}\left(\ln^2\left(\frac{-s}{-t}\right)+\pi^2\right) + \frac{s-2t}{t^3}\left(\ln^2\left(\frac{-s}{-u}\right)+\pi^2\right)\Big] + (s\leftrightarrow t)$$

$$L_3(s,t,u) = \frac{1}{stu}\left(-s\ln\left(\frac{-s}{\mu^2}\right) - t\ln\left(\frac{-t}{\mu^2}\right) - u\ln\left(\frac{-u}{\mu^2}\right)\right) + \frac{t-u}{s^3}\ln\left(\frac{-t}{-u}\right) + \frac{2s^2-tu}{s^4}\left(\ln^2\left(\frac{-t}{-u}\right)+\pi^2\right)$$

(Euclidean region $s,t,u<0$)

Here:

- Covariant result from low energy limit of type II on $T^4/\mathbb{Z}_N \times T^2$
  - ★ Related to symmetric double copy $\mathcal{N}=2 \otimes \mathcal{N}=2$
  - ★ $\mathcal{N}=4$ SUGRA coupled to vector
- Develop methods which might be applicable more widely

Starting point

String integrand for type II on $T^4/\mathbb{Z}_N \times T^2$: [Berg, Buchberger, Schlotterer 2016]

z, τ kinematics (Berends-Giele currents)

$$\Big[\Big| X_{23,4} C_{1|234} + X_{24,3} C_{1|243} + [s_{12}(f_{12}^{(2)} + f_{34}^{(2)}) P_{1|2|3,4} + (2 \leftrightarrow 3,4)] - 2F_{1/2}^{(2)}(\gamma) t_8(1,2,3,4) \Big|^2$$

$$+ \frac{\pi}{\mathrm{Im}\tau} (X_{23} C^m_{1|23,4} + X_{24} C^m_{1|24,3} + X_{34} C^m_{1|34,2})(\bar{X}_{23} \tilde{C}^m_{1|23,4} + \bar{X}_{24} \tilde{C}^m_{1|24,3} + \bar{X}_{34} \tilde{C}^m_{1|34,2})$$

$$+ \Big(\frac{\pi}{\mathrm{Im}\tau}\Big)^2 (\tfrac{1}{2} C^{mn}_{1|2,3,4} \tilde{C}^{mn}_{1|2,3,4} - P_{1|2|3,4} \tilde{P}_{1|2|3,4} - P_{1|3|2,4} \tilde{P}_{1|3|2,4} - P_{1|4|2,3} \tilde{P}_{1|4|2,3}) \Big] \, KN$$

$z_i - z_j$

$$X_{12,3} = s_{12} f_{12}^{(1)} (s_{13} f_{13}^{(1)} + s_{23} f_{23}^{(1)}) \ , \qquad f_{ij}^{(1)} = \partial \ln \vartheta_1(z_{ij}, \tau) + 2\pi i \frac{\mathrm{Im}(z_{ij})}{\mathrm{Im}(\tau)}$$

$$KN = \exp\Big(\sum_{1 \le i < j}^{4} s_{ij} G(z_{ij}) \Big) \ , \qquad s_{ij} = \tfrac{1}{2}(k_i + k_j)^2 = k_i \cdot k_j$$

- Take field theory limit
- Regulate (in particular IR divergences)
- Map to integrals known from field theory

Field theory limit

- $\alpha' \to 0\ ,\ \ \tau_2 \to \infty$ **with** $\mathfrak{t} = \alpha' \tau_2$ **finite**

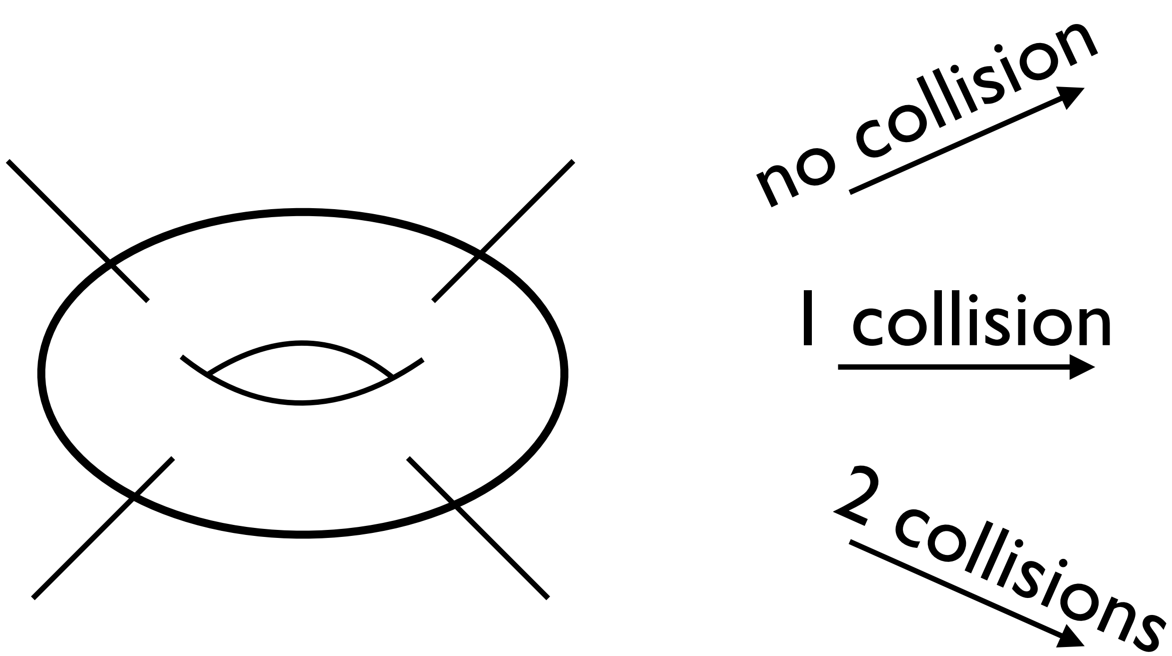
- $z = \mathrm{Re}(z) + i\tau_2 \nu$ **with** $\nu \in [0,1]$

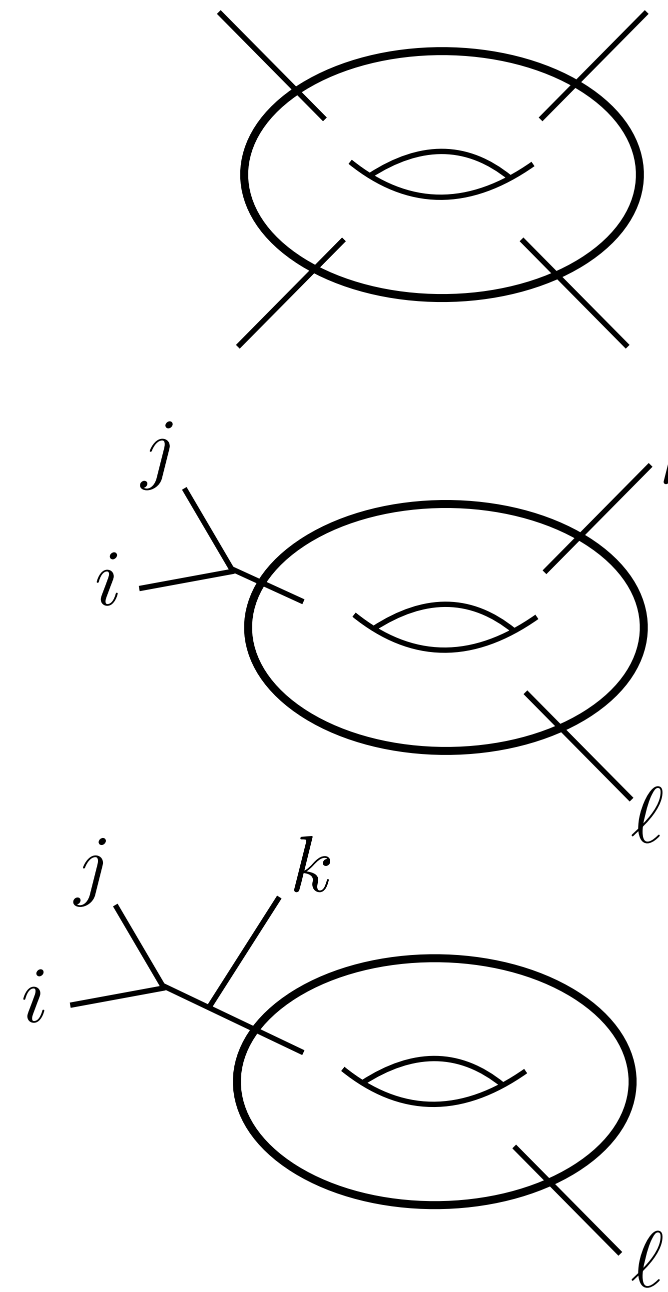
- E.g. $$G(z;\tau) = \frac{\alpha'}{2}\left(\ln \left| \frac{\vartheta_1(z;\tau)}{\vartheta_1'(0;\tau)} \right|^2 - \frac{2\pi}{\mathrm{Im}\tau} \mathrm{Im}^2(z) \right)$$
$$\xrightarrow{\text{f.t.}} -\pi \mathfrak{t}(\nu^2 - |\nu|)$$

$\Longrightarrow$ ★ integral over $\mathrm{Re}(z)$ becomes trivial

★ singularity at collision points $z \to 0$ lost

$\Longrightarrow$ Treat collisions separately using $\int d^2z|z|^{s-2}g(z) = \frac{\pi}{s}g(0)$ before field theory limit





$\sim \frac{1}{s_{ij}}$

$\sim \frac{1}{s_{ij}s_{ijk}}$

$s_{ijk} = \frac{1}{2}(k_i + k_j + k_k)^2 = 0$

on-shell, momentum conservation

cancelled by similar term in numerator

$\Longrightarrow$ Need regularization

Regularization

- Add 5th momentum $$\sum_{i=1}^{4} k_i = \kappa$$ [Minahan 1987]

$$\kappa^2 = 0 \implies \text{Modular invariance}$$

- Possible interpretation via addition of low momentum dilaton vertex operator in the amplitude?

[Dine, Ichinose, Seiberg 1987]

- $$s_{ijk} = \frac{1}{2}(k_i + k_j + k_k)^2 = \frac{1}{2}(-k_\ell - \kappa)^2 = \underbrace{k_\ell \cdot \kappa}_{m_s^2 \theta_\ell} \neq 0$$ but small

- Also regularizing IR divergences in field theory integrals (see below)

Integration contour

- $$KN = \exp\Big(-\sum_{1\le i<j}^{4} \pi t\, s_{ij}(\nu_{ij}^2 - |\nu_{ij}|)\Big)$$

  integrated from 0 to ∞

  $\nu_i - \nu_j$ $\qquad$ $\nu_i \in [0,1]$

- No choice of external kinematics with $m_s^2 \ll |s_{ij}|$ for which KN damped for all orders of ν_i

  [e.g. D'Hoker, Phong 1993]

- E.g. $(\nu_1 \equiv 0)$

$$\frac{1}{2}\sum_{1\le i<j}^{4} s_{ij}(\nu_{ij}^2 - |\nu_{ij}|) = \begin{cases} s_{12}(\nu_3-\nu_2)(1-\nu_4) + s_{14}\nu_2(\nu_4-\nu_3) + \mathcal{O}(m_s^2) \\ \qquad (0 \le \nu_2 \le \nu_3 \le \nu_4 \le 1) \\ s_{13}(\nu_2-\nu_3)(1-\nu_4) + s_{14}\nu_3(\nu_4-\nu_2) + \mathcal{O}(m_s^2) \\ \qquad (0 \le \nu_3 \le \nu_2 \le \nu_4 \le 1) \end{cases}$$

$\implies$ s_{12}, s_{13} and s_{14} all need to be positive, but

$$s_{12} + s_{13} + s_{14} = -(s+t+u) = 0 + \mathcal{O}(m_s^2)$$

Solution in analogy with QFT [Witten 2013]

Euclidean propagator:

Euclidean Schwinger parameter $\leftrightarrow \mathfrak{t}$ in ST

$$\frac{1}{p^2+m^2} = \int_0^\infty d\mathfrak{t}_E \exp\Big(-\mathfrak{t}_E(p^2+m^2)\Big)$$

only convergent for $p^2+m^2>0$

Lorentzian propagator:

Lorentzian proper time

$$\frac{-i}{p^2+m^2-i\epsilon} = \int_0^\infty d\mathfrak{t}_L \exp\Big(-i\mathfrak{t}_L(p^2+m^2-i\epsilon)\Big)$$

converges for arbitrary values of p^2+m^2, due to factor $e^{-\mathfrak{t}_L\epsilon}$

$$\Longrightarrow \quad KN = \exp\Big(-\mathfrak{t}(\dots)\Big) \longrightarrow \exp\Big(-i\mathfrak{t}_L(\dots - i\epsilon)\Big)$$

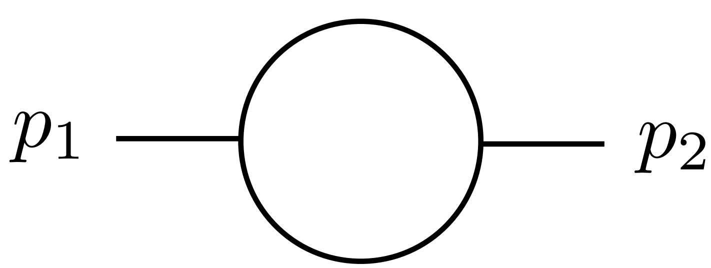
[cf. also Eberhardt, Mizera 2022; Baccianti, Eberhardt, Mizera 2025]

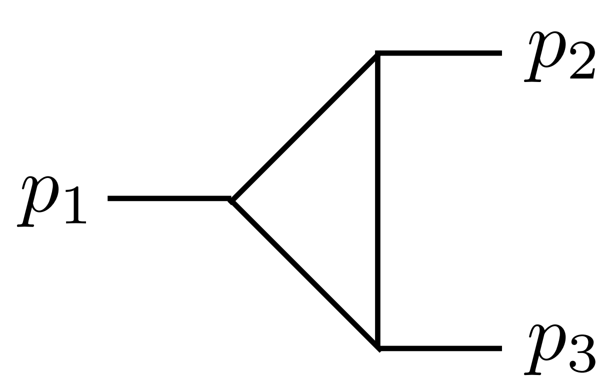
Mapping to field theory integrals

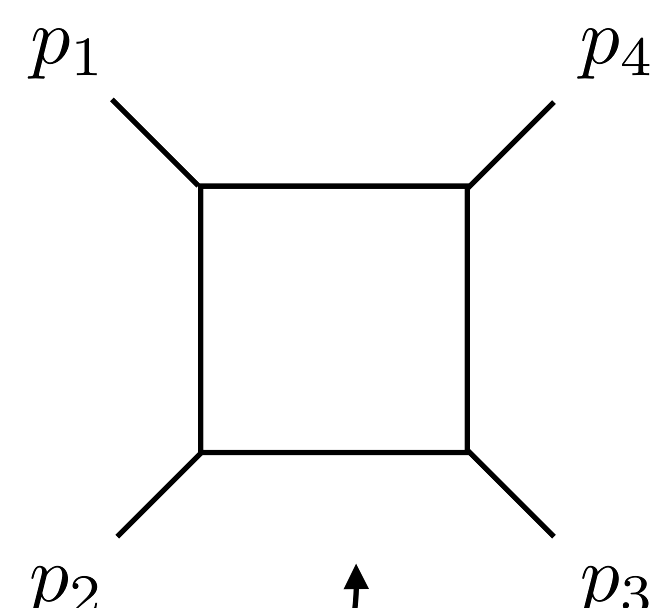
- t_L integration
- $\nu_2 = x_1 \; , \; \nu_3 = x_1 + x_2 \; , \; \nu_4 = x_1 + x_2 + x_3$

[Montag, Weisberger 1991]

$\Longrightarrow$ Field theory integrals (with external masses but massless internal lines)







$p_i^2 = -m_i^2 \neq 0$

elementary (needs dim. reg.)

can be evaluated in $D = 4$

FT triangle and box integrals can all be obtained from

- $$I_3[1](m_1^2, m_2^2, m_3^2) = \left(\prod_{i=1}^{3} \int_0^1 dx_i\right) \frac{\delta(x_1 + x_2 + x_3 - 1)}{\mathcal{F}_3 - i\epsilon}$$

  with $\mathcal{F}_3 = -m_1^2 x_1 x_2 - m_2^2 x_2 x_3 - m_3^2 x_1 x_3$

- $$I_4[1](\hat{s}, \hat{t}, m_1^2, m_2^2, m_3^2, m_4^2) = \left(\prod_{i=1}^{4} \int_0^1 dx_i\right) \frac{\delta(x_1 + x_2 + x_3 + x_4 - 1)}{(\mathcal{F}_4 - i\epsilon)^2}$$

  with $\mathcal{F}_4 = -\hat{s} x_1 x_3 - \hat{t} x_2 x_4 - m_1^2 x_1 x_2 - m_2^2 x_2 x_3 - m_3^2 x_3 x_4 - m_4^2 x_1 x_4$

- Field theory parameters $\hat{s}, \hat{t}, m_i^2$ are functions of string theory kinematic variables $s_{ij} \longleftarrow$ only 5 independent $(\kappa^2 = 0)$

  More convenient basis of string theory kinematics, e.g.:

$$s_{23},\ s_{24},\ s_{123} = m_s^2 \theta_4,\ s_{124} = m_s^2 \theta_3,\ s_{234} = m_s^2 \theta_1$$

Example: $\int_0^\infty d\mathfrak{t}_L \, \mathfrak{t}_L \exp\Big(- i\mathfrak{t}_L (\ldots - i\epsilon)\Big) \sim \frac{1}{\big(\ldots - i\epsilon\big)^2}$

$$0 \leq \nu_2 \leq \nu_3 \leq \nu_4 \leq 1 \Longrightarrow$$

$$\begin{aligned} \ldots = -(-2s_{23} + s_{123} + s_{124} + s_{234})x_1x_3 - (2s_{23} + 2s_{24})x_2x_4 \\ -(s_{123} + s_{124} + s_{234})x_1x_2 - (-s_{123} - 2s_{234})x_2x_3 \\ -(-s_{124})x_3x_4 - (s_{123} - s_{234})x_1x_4 \end{aligned}$$

Comparison with $\mathcal{F}_4$:

$$\hat{s} = -2s_{23} + m_s^2(\theta_4 + \theta_3 + \theta_1) \qquad \hat{t} = 2(s_{23} + s_{24})$$

$$m_1^2 = m_s^2(\theta_4 + \theta_3 + \theta_1) \qquad m_2^2 = m_s^2(-\theta_4 - 2\theta_1)$$

$$m_3^2 = m_s^2(-\theta_3) \qquad m_4^2 = m_s^2(\theta_4 - \theta_1)$$

Note: All $m_i^2 \longrightarrow 0$ for $m_s^2 \longrightarrow 0$

Integrals

(for FT Euclidean region, i.e. $i\epsilon \to 0$)

$$I_3[1](m_1^2, m_2^2, m_3^2) = \Big(\prod_{i=1}^{3}\int_0^1 dx_i\Big)\frac{\delta(x_1+x_2+x_3-1)}{-m_1^2 x_1 x_2 - m_2^2 x_2 x_3 - m_3^2 x_1 x_3}$$

- Factor out $m_3^2 \implies$ nontrivial fct. of $u = \dfrac{m_1^2}{m_3^2}$ & $v = \dfrac{m_2^2}{m_3^2}$
- Change variables $z\bar{z} = u$, $(1-z)(1-\bar{z}) = v$

$$I_3[1](u,v) = -\frac{1}{m_3^2}\frac{P_2(z,\bar{z})}{z-\bar{z}}$$

[Duplancic, Nizic 2002;
Chavez-Duhr 2012;
Bourjaily, Hannesdottir,
McLeod, Schwartz, Vergu 2020;
Weinzierl's book 2022; ...]

$$P_2(z,\bar{z}) = 2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) - \ln(z\bar{z})\big(\mathrm{Li}_1(z) - \mathrm{Li}_1(\bar{z})\big)$$

$$\int_0^z \frac{dx}{x}\mathrm{Li}_1(x)$$

$$\int_0^z \frac{dx}{1-x} = -\ln(1-z)$$

$$I_4[1](\hat{s},\hat{t},m_1^2,m_2^2,m_3^2,m_4^2) = I_3[1](\hat{s}\hat{t},m_1^2m_3^2,m_2^2m_4^2)$$

- Reason for simplicity: Dual conformal symmetry

- Relation can get (known) corrections away from Euclidean region [Duplancic, Nizic 2002; Corcoran, Loebbert, Miczajka, Staudacher 2020]

- Back to example $0 \leq \nu_2 \leq \nu_3 \leq \nu_4 \leq 1$

m_s IR regulator

$$I_4[1](\hat{s},\hat{t},m_1^2,m_2^2,m_3^2,m_4^2)$$

$$\xrightarrow{m_s\to 0} -\frac{1}{4s_{23}(s_{23}+s_{24})}\Big[\Big(\ln\big(\frac{m_s^2(\theta_3-\theta_1)}{2(s_{23}+s_{24})}\big)+\ln\big(\frac{m_s^2(2\theta_1+\theta_4)}{2s_{23}}\big)\Big)\times$$
$$\Big(\ln\big(-\frac{m_s^2\theta_4}{2(s_{23}+s_{24})}\big)+\ln\big(-\frac{m_s^2(\theta_1+\theta_3+\theta_4)}{2s_{23}}\big)\Big)+\frac{\pi^2}{3}\Big]$$

- Expect all IR divergences to cancel between orderings and diagrams, except for IR div. from graviton in loop

Tensor triangles

$$I_3[x_1^\ell x_2^m x_3^n](m_1^2, m_2^2, m_3^2) = \Big(\prod_{i=1}^{3} \int_0^1 dx_i\Big) \frac{\boxed{x_1^\ell x_2^m x_3^n}\,\delta(x_1 + x_2 + x_3 - 1)}{-m_1^2 x_1 x_2 - m_2^2 x_2 x_3 - m_3^2 x_1 x_3}$$

and tensor boxes $I_4[x_1^k x_2^\ell x_3^m x_4^n](\hat{s}, \hat{t}, m_1^2, m_2^2, m_3^2, m_4^2)$

can all be obtained from scalar integrals $I_3[1]$ & $I_4[1]$ via

Bern-Dixon-Kosower differentiation method

[Bern, Dixon, Kosower 1993]

Preliminary results

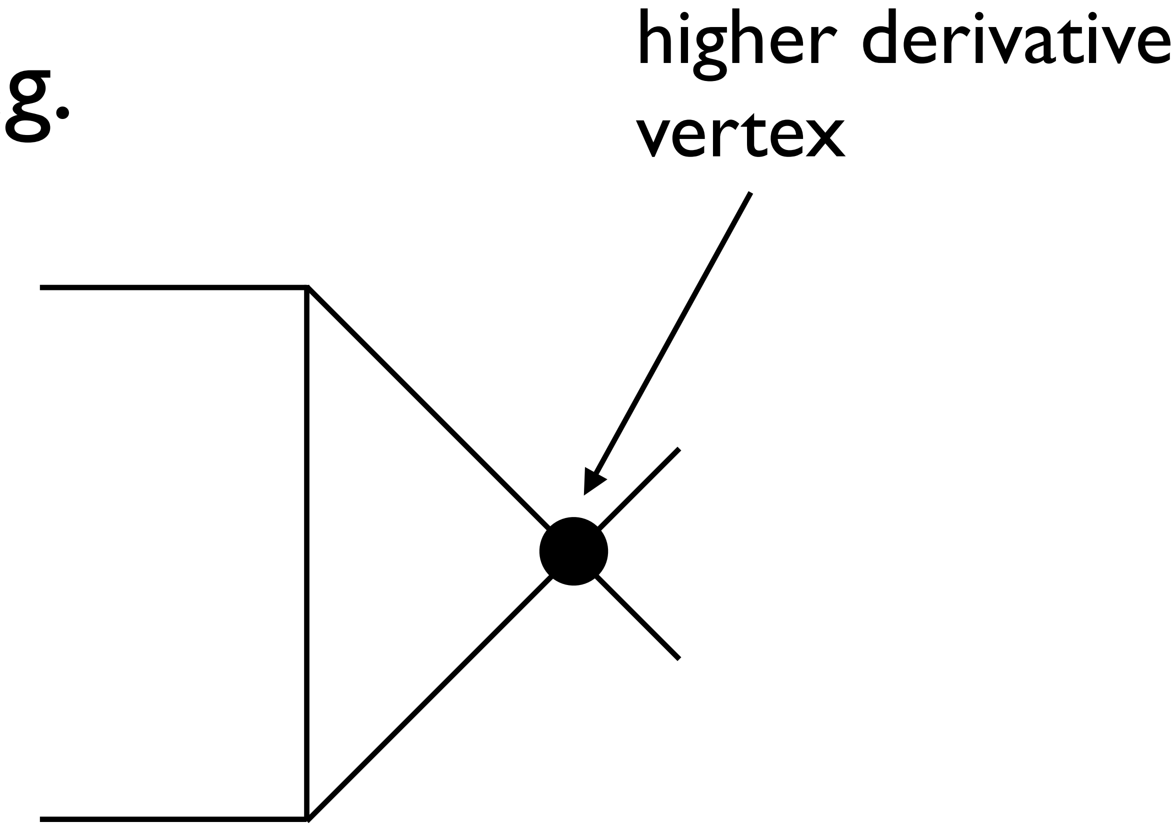
- Calculated the full amplitude covariantly, not only in Euclidean region; still in the process of simplifying the result (and its presentation)
- IR divergencies indeed cancel after adding up all contributions (except for the expected IR divergence from gravitons running in the loop)
- Result can be expressed in terms of $t_8 \mathrm{tr}(R_i R_j R_k R_\ell)$ and $t_8 \mathrm{tr}(R_i R_j) \mathrm{tr}(R_k R_\ell)$

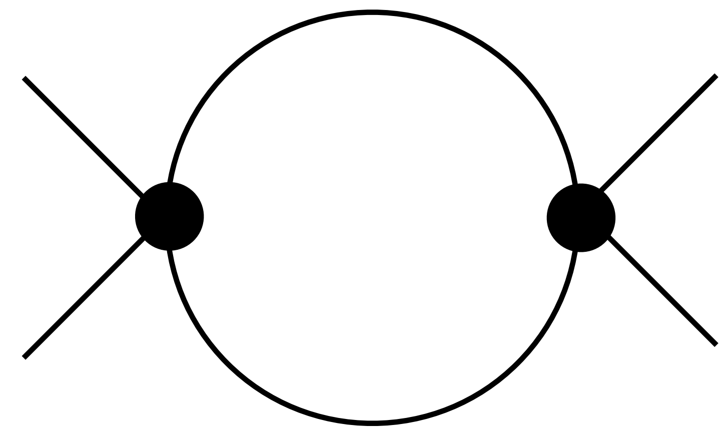
[cf. Bern, Edison, Kosower, Parra-Martinez 2017]

Conclusion & outlook

- Developed new algorithm to evaluate string amplitudes covariantly in low energy limit, using "mass" regularization
- (Parts of it) can be automated
- Should be generalizable, e.g. to
  - ★ less supersymmetry
  - ★ α'-corrections (keeping higher orders in expansions of field theory limit)

E.g.





Σας ευχαριστώ

Thank you!