Asymptotic structure of gravity in the gauge PDE formalism

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Based on: Maxim Grigoriev, M.M. 2310.09637; ongoing work

Background

- Theories defined on manifolds with boundaries, particularly asymptotic boundaries, play an important role in modern field theory. Gauge field theories in such settings are of special interest.

- The study of asymptotic structures in General Relativity has a long history [*Bondi, Metzner, Sachs, Penrose, Geroch, Ashtekar,...*], but has seen a surge of interest recently.

- Since the asymptotic infinity of general relativity is a conformal (Carrollian) manifold, conformal geometry methods become especially effective [*Fefferman, Graham, Gover, Herfray,...*].

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- To work with gauge field theories, it is useful to employ the Batalin–Vilkovisky (BV) formalism. When locality is taken into account, it is helpful to reformulate BV in terms of jet bundles [*Henneaux, Barnich, Brandt,...*].

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- More general object: gauge PDE [*Barnich, Grigoriev 2010; Grigoriev, Kotov 2019*]. They behave well under restriction to submanifolds and boundaries.

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- More general object: gauge PDE [*Barnich, Grigoriev 2010; Grigoriev, Kotov 2019*]. They behave well under restriction to submanifolds and boundaries.

  The aim of this talk is to provide a geometric, Gauge PDE formulation for gauge field theories induced on boundaries and to illustrate it with the example of asymptotically simple gravity.

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Example. Given a smooth manifold X with local coordinates x^μ, one can construct the supermanifold $T[1]X$ with coordinates (x^μ, θ^μ), where the θ^μ are anticommuting: $\theta^\mu \theta^\nu = -\theta^\nu \theta^\mu$. It is equipped with the vector field

$$d_X \equiv \theta^\mu \frac{\partial}{\partial x^\mu}, \quad d_X^2 = 0. \tag{1}$$

One easily sees that that the complex $(C^\infty(T[1]X), d_X)$ is isomorphic to the de Rham complex $(\Lambda(X), d_{\mathrm{dR}})$.

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There is a well-defined notion of equivalence between gauge PDEs. Whenever fiber coordinates forming a contractible pair appear, i.e. w^α, v^α: $Qw^\alpha = v^\alpha$, the subbundle $w^\alpha = v^\alpha = 0$ is equivalent to the original gauge PDE.

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Gauge PDE encodes the complete content of a gauge field theory at the level of the equations of motion.

But where is the actual PDE?

Solution space of a gauge PDE

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Definition

Infinitesimal gauge transformation of a given section σ *is defined as*

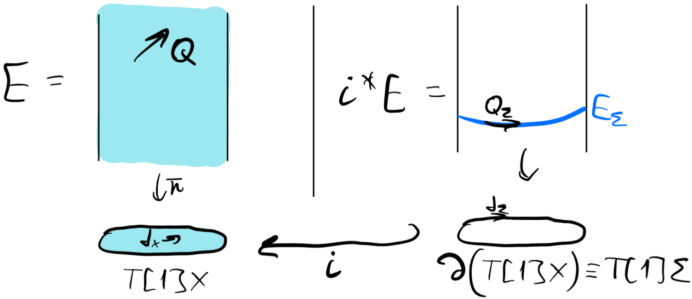
$$\delta\sigma^* = \sigma^*[Q, Y],$$

where Y, $\mathrm{gh}(Y) = -1$ *is a vertical vector field interpreted as a gauge parameter*

Gauge PDE on manifold with boundaries

Definition

A gauge PDE with boundary consists of $(E, Q, T[1]X, E_\Sigma, T[1]\Sigma)$*, where* $(E, Q, T[1]X)$ *is a gPDE on* X *and* $(E_\Sigma, Q_\Sigma, T[1]\Sigma)$ *is a gPDE on the boundary* $\Sigma = \partial X$*, realised as a sub-gPDE of the pull-back* $i^* E$ *with* $i\colon T[1]\Sigma \hookrightarrow T[1]X$*. In particular,* Q_Σ *is the restriction of* Q *to* $E_\Sigma \subset i^* E \subset E$*.*



Conformal-like GR as a gPDE

$E = T[1]X \times F$ Coordinates on F:

$$\{D_{(a)}g_{bc}, D_{(a)}\Omega, D_{(a)}\lambda, D_{(a)}\xi^b\}.$$

The action of Q is induced from

$$Qg_{ab} = \xi^c D_c g_{ab} + g_{cb} D_a \xi^c + g_{ac} D_b \xi^c + 2\lambda g_{bc}, \qquad Q\xi^a = \xi^b D_b \xi^a,$$
$$Q\Omega = \xi^a D_a \Omega + \lambda\Omega, \qquad Q\lambda = \xi^a D_a \lambda$$

and $[Q, D_a] = 0$.

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(Almost) Einstein equations in these terms are given by the D_a-prolongations of

$$\begin{aligned} &\Big(D_b D_c \Omega - \Gamma^d_{bc}[g] D_d \Omega + P_{bc}[g]\Omega\Big)\Big|_{t-f} = 0, \\ &\tfrac{1}{2} D^a \Omega D_a \Omega - \tfrac{1}{D}\Omega\Big(D^a D_a \Omega - g^{bc}\Gamma^a_{bc}[g] D_a \Omega + P[g]\Omega\Big) = \tilde{\Lambda}. \end{aligned} \tag{2}$$

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Note that $D_{(a)} \ln \Omega$ and $D_{(a)}\lambda$ form contractible pairs if $\Omega > 0$.

Reduced conformal-like GR

One can eliminate contractible pairs in the conformal geometry sector[*Boulanger, Erdmenger, 2004*]. Fiber coordinates after:

$$\{g_{ab}, \nabla_{(a)}\Omega, \nabla_{(a)}W^b{}_{cde}\} \cup \{\lambda, \nabla_a\lambda, \xi^a, \nabla_a\xi^b\}.$$

Restricted to the equation manifold, one obtains

$$\nabla_{(c)}(\nabla_a\nabla_b\Omega|_{t-f}) = 0, \quad \frac{1}{2}\nabla_a\Omega\nabla^b\Omega - \frac{1}{D}\Omega\nabla_a\nabla^a\Omega = \tilde{\Lambda}. \tag{3}$$

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Here, the vector fields ∇_a can be interpreted as a covariant total derivative.

$$[\nabla_a, \nabla_b] = -W^d{}_{cab}\Delta^c{}_d - C_{dab}\Gamma^d. \tag{4}$$

On the solution space $\hat{\nabla}_a\sigma^* \equiv \sigma^*\nabla_a$ – conformal covariant derivative introduced in [*Wünsch, 1986*].

Boundary gPDE for gravity

Let $i : T[1]\Sigma \hookrightarrow T[1]X$ be the embedding of the boundary in the spacetime.
Boundary gPDE: subbundle of i^*E specified through the conditions

$$\Omega = 0, \quad \nabla_a \Omega \neq 0, \quad Q\Omega = 0. \tag{5}$$

The first two conditions were already introduced in *[Penrose, 1963]*; the third ensures that Q is tangent.
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Question:

What are the implications of imposing the Einstein equations in the bulk?

Useful tool – vector fields $\mathcal{D}_A$

Splitting the index set $a \equiv \{\Omega, A\}$ one can see that ∇_Ω is not tangent to the surface of the boundary conditions.

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Remarkably, Einstein's equations guarantee that the vector fields

$$\mathcal{D}_A^{(N)} \equiv ad_{\nabla_\Omega}^N(\nabla_A) \equiv [\nabla_\Omega, \ldots, [\nabla_\Omega, \nabla_A]]$$

are tangent and we will use the same symbol $b^*\mathcal{D}_A^{(N)} = \mathcal{D}_A^{(N)} b^*$.

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Hence, any expression of the form $\nabla_\Omega^N \nabla_A f$ in the bulk on the boundary can be rewritten as a sum over «subleadings» $f^{(i)} \equiv b^* \nabla_\Omega^i f$:

$$b^* \nabla_\Omega^N \nabla_A f = \nabla_A f^{(N)} + \sum_{i=0}^{N-1} C_N^i \mathcal{D}_A^{(N-i)} f^{(i)}$$

Boundary calculus

In the case $\Lambda \neq 0$, using the Bianchi identities and introducing

$$T_{AB} \equiv W_{\Omega A \Omega B}, \qquad J_{ABC} \equiv W_{AB\Omega C},$$

one can decompose the covariant jets of the Weyl tensor as

$$\{\nabla_{(a)} W_{bcde}\} = \big\{\nabla_{(A)} W_{BCDE},\ \nabla_{(A)} J_{BCD},\ \nabla_{(A)} T^{(N)}_{BC},\ N \geq 0\big\}.$$

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Theorem (Boundary calculus)

Using the Einstein equations, one can construct an iterative procedure that expresses the set of functions on E_B, $\{T^{(N)}_{AB}, J^{(N)}_{ABC} | N \neq D-3\}$, in terms of conformal geometry on the boundary and, for $N > D-3$, in terms of the coordinates $T^{(D-3)}_{AB}$. The vector fields $\mathcal{D}_A$ are likewise determined through this set of functions and the differential Q.

[Graham, 2008] called the analogues of $T^{(N)}_{AB}$ with $N < D-3$ *extended obstruction tensors*.

On-shell Boundary gPDE for AAdS spacetimes

Using the boundary calculus theorem, the fiber coordinates are

$$\{g_{AB},\, \xi^A,\, \nabla_A \xi^B,\, \lambda,\, \nabla_A \lambda,\, \nabla_{(A)} W^B{}_{CDE}\} \cup \{\nabla_{(A)} T^{(D-3)}_{BC} \mid N \geq 0\}.$$

The only remaining part of the Einstein equations is

$$\sum_{i=0}^{D-4} C^i_{D-4}\, \mathcal{D}_A^{(D-4-i)} J^{(i)A}{}_{BC} = 0, \tag{6}$$

$$\nabla^A T^{(D-3)}_{AB} + \sum_{i=0}^{D-4} C^i_{D-3}\, \mathcal{D}^{(D-3-i)A} T^{(i)}_{AB} = 0. \tag{7}$$

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By an iterative procedure, one finds that for even-dimensional boundaries the first equation (6) is precisely the Fefferman–Graham obstruction equation, expressed entirely in terms of the conformal geometry sector:

$$(\nabla_A \nabla^A)^{\frac{D-5}{2}} B_{BC} + \cdots = 0.$$

Neumann data $T_{AB}^{(D-3)}$

- The second equation (7) is the modified conservation law for the Neumann data $T_{AB}^{(D-3)}$:

$$\nabla^A T_{AB}^{(D-3)} + \cdots = 0,$$

where $\ldots$ depends only on the conformal geometry sector.

- Under Weyl transformations, $T_{AB}^{(D-3)}$ transforms as a field of weight $(D-3)$, but for even-dimensional boundaries an additional inhomogeneous term appears. For instance, for $D=3$ this is the standard Schwarzian term, while for $D=5$ it takes the form

$$\delta T_{AB}^{(2)} = \gamma^C (C_{BCA} + C_{ACB}), \tag{8}$$

where C_{BCA} is the Cotton tensor.

Fields on a Gravitational Background: extended GJMS

- We consider an additional scalar field of weight Δ in the bulk, described by the coordinates $\{\nabla_{(a)}\varphi\}$, subject to conformal-like Klein–Gordon equations. For $\Omega > 0$, the system reduces to just a scalar field obeying the Klein–Gordon equation.

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- Fixing $m^2 = 2\tilde{\Lambda}\Delta(\Delta + 1 - D)$ and applying the same procedure as for gravity, we obtain on the boundary coordinates $\{\nabla_{(A)}\varphi^{(N)} | N \geq 0\}$ subject to

$$c_{\Delta,N}\varphi^{(N)} + (N-1)\tilde{\Lambda}\, b^*\nabla_{\Omega}^{N-2}\nabla_A\nabla^A\varphi = 0, \qquad N \geq 1, \tag{9}$$

where $c_{\Delta,N} \equiv 2\Delta + 1 - D + N$. The interpretation of these equations clearly depends on the zeros of $c_{\Delta,N}$.

Fields on a Gravitational Background: (extended) GJMS

In the case $\Delta = \dfrac{D-1}{2} - l$, $l \in \mathbb{N}$, the boundary theory contains two scalar fields with the corresponding coordinates $\{\nabla_{(A)}\varphi^{(0)}, \nabla_{(A)}\varphi^{(2l)}\}$ and the leading field satisfies

$$b^* \nabla_\Omega^{2l-2} \nabla_A \nabla^A \varphi = 0.$$

For example, for $l = 3$ this expands to

$$(\nabla_A \nabla^A)^3 \varphi^{(0)} + 8\tilde{\Lambda}^{-2} T^{(2)}_{AB} \nabla^A \nabla^B \varphi^{(0)} = 0. \tag{10}$$

At $D = 5$, this equation involves subleading gravitational term $T^{(D-3)}_{AB} \equiv T^{(2)}_{AB}$ and thus is not determined by conformal geometry alone.

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For $D > 5$, we have $T^{(2)}_{AB} = \frac{2\tilde{\Lambda}^2}{D-5} B_{AB}$, and this equation becomes the standard third GJMS equation expressed in terms of Wünsch derivatives.

Conclusions

What has been shown here:

- **Geometric** (in the sense of PDE geometry), **coordinate-independent**, **BV-BRST** approach to the boundary structure of gravity on the level of equations of motion.
- For $\Lambda \neq 0$: **effective calculus** for computing obstruction equations for gravity and fields on a gravitational background, formulated automatically in terms of **conformal covariant derivatives**, usable to obtain (extended) GJMS operators, conformally-invariant (higher) Yang–Mills equations, ...

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What has not been shown here:

- **Lagrangian generalization**: **effective calculus** for obtaining conformally-invariant (first-order) actions for obstruction equations; **BV-BRST version of the covariant phase space formalism**, asymptotic charges, ...
- **Case** $\Lambda = 0$: asymptotic symmetries, **BMS group** *[Grigoriev, M.M. 2310.09637]*.

8D higher conformal Yang-Mills equation

$$\begin{aligned}
&\bar\nabla^{\alpha}(F^{(4)|I}_{\alpha\gamma}) + 2n_\Omega P^{\alpha\beta}(3n_\Omega(C_{\beta\delta\alpha}F^{I|\delta}{}_{\gamma} + C_{\beta\delta\gamma}F^{I|\delta}_{\alpha}) + 3[J^{(1)}_{\beta}, F^{I}_{\alpha\gamma}] + 6\bar\nabla_{\beta}F^{(2)|I}_{\alpha\gamma} + \\
&+6P^{\delta}_{\beta}(\bar\nabla_{\delta}F^{I}_{\alpha\gamma} - 12n_\Omega(g_{\delta\alpha}J^{(1)|I}_{\gamma} - g_{\delta\gamma}J^{(1)|I}_{\alpha})) - 4(g_{\beta\alpha}J^{(3)|I}_{\gamma} - g_{\beta\gamma}J^{(3)|I}_{\alpha}))- \\
&-J^{(3)\beta}{}_{\gamma}{}^{\alpha}F^{I}_{\alpha\beta} + \frac{3}{2}T^{(2)|\beta\alpha}\bar\nabla_{\beta}F^{I}_{\alpha\gamma} + [J^{(3)\alpha}, F_{\alpha\gamma}]^{I} + \\
&+6(-J^{(1)|\sigma}{}_{\gamma}{}^{\alpha}F^{(2)}_{\alpha\sigma} + T^{(2)|\alpha\sigma}(\frac{1}{2}\bar\nabla_{\sigma}F^{I}_{\alpha\gamma} - g_{\sigma\alpha}J^{(1)|I}_{\gamma} + g_{\sigma\gamma}J^{(1)|I}_{\alpha}) + [J^{(1)|\alpha}, F^{(2)}_{\alpha\gamma}]^{I}) = 0
\end{aligned} \tag{11}$$

$$\begin{aligned}
T^{(2)}_{\alpha\beta} &= -\frac{n_\Omega}{2}B_{\alpha\beta}, \\
J^{(1)}_{\gamma\alpha\beta} &= -n_\Omega C_{\gamma\alpha\beta}, \\
J^{(1)|I}_{\beta} &= \frac{n_\Omega}{4}\bar\nabla^{\alpha}F^{I}_{\alpha\beta}, \\
F^{(2)|I}_{\alpha\sigma} &= \bar\nabla_{\alpha}J^{(1)|I}_{\sigma} + P_{\alpha}{}^{\gamma}F^{I}_{\gamma\delta} - (\alpha\leftrightarrow\sigma), \\
J^{(3)|I}_{\beta} &= \frac{3n_\Omega}{2}(\bar\nabla^{\alpha}F^{I(2)}_{\alpha\beta} + n_\Omega P^{\alpha\gamma}\bar\nabla_{\gamma}F^{I}_{\alpha\beta} - \\
&\quad -2n_\Omega P^{\alpha\gamma}(g_{\gamma\alpha}J^{I|(1)}_{\beta} - g_{\gamma\beta}J^{(1)|I}_{\alpha}) + [J^{(1)}_{\alpha}, F^{\alpha}{}_{\beta}]^{I} + n_\Omega C^{\alpha\gamma}{}_{\beta}F^{I}_{\alpha\gamma}), \\
J^{(3)}{}_{\delta\gamma\alpha} &= \frac{3}{2}(\bar\nabla_{\delta}T^{(2)}_{\alpha\gamma} + 2n_\Omega P_{\delta}{}^{\sigma}(J^{(1)}_{\sigma\gamma\alpha} + J^{(1)}_{\sigma\alpha\gamma}) - (\delta\leftrightarrow\sigma)), \\
F^{(4)|I}_{\alpha\gamma} &= \bar\nabla_{\alpha}J^{(3)|I}_{\gamma} + 2n_\Omega(\bar\nabla_{\alpha}J^{(1)|I}_{\gamma} + F^{I}_{\alpha\gamma}) + 3F^{(2)|I}_{\alpha\gamma} + \\
&\quad +3(-J^{(1)}_{\delta\gamma\alpha}J^{(1)|I\delta} + \frac{n_\Omega}{2}T^{(2)}_{\alpha\delta}F^{I\delta}{}_{\gamma} + [J^{(1)}_{\alpha}, J^{(1)}_{\gamma}]^{I}) - (\alpha\leftrightarrow\gamma)
\end{aligned} \tag{12}$$

where $F^{I}_{\alpha\beta}$ is the curvature tensor for the Yang–Mills sector, $P_{\alpha\beta}$, $C_{\alpha\beta\gamma}$, and $B_{\alpha\beta}$ are the Schouten, Cotton, and Bach tensors respectively, $\bar\nabla_{\alpha}$ is the covariant derivative induced from the Levi-Civita and Yang–Mills derivatives, n_Ω is a certain linear function of the cosmological constant, and $[\cdot,\cdot]$ denotes the commutator in the algebra underlying the Yang–Mills theory.