FAKULTA MATEMATIKY,
FYZIKY A INFORMATIKY
Univerzita Komenského
v Bratislave

Bootstrap method in matrix models

Katarína Magdolenová
Supervisor: doc. Mgr. Samuel Kováčik, PhD.

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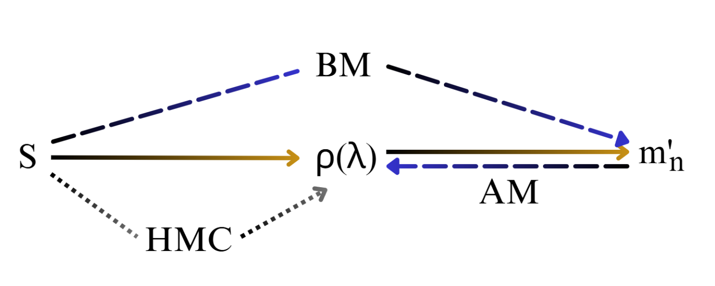
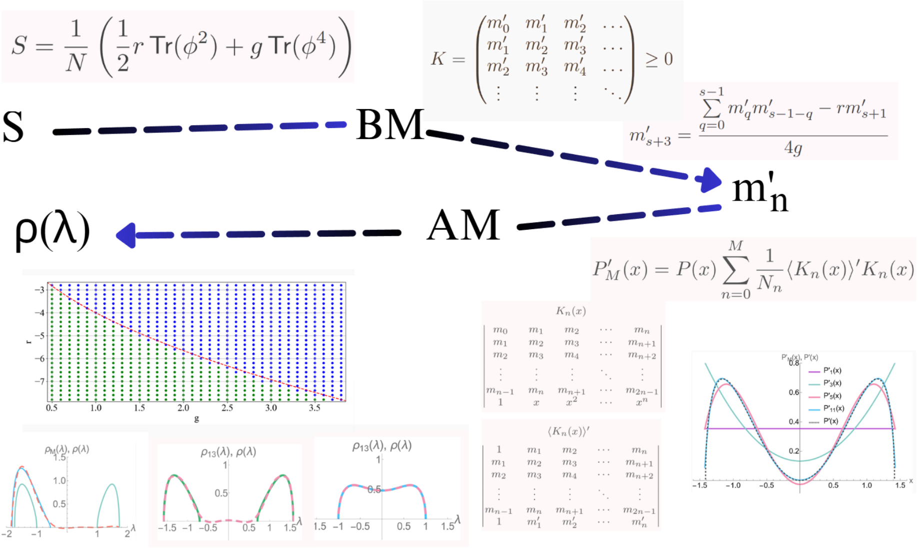
Motivation



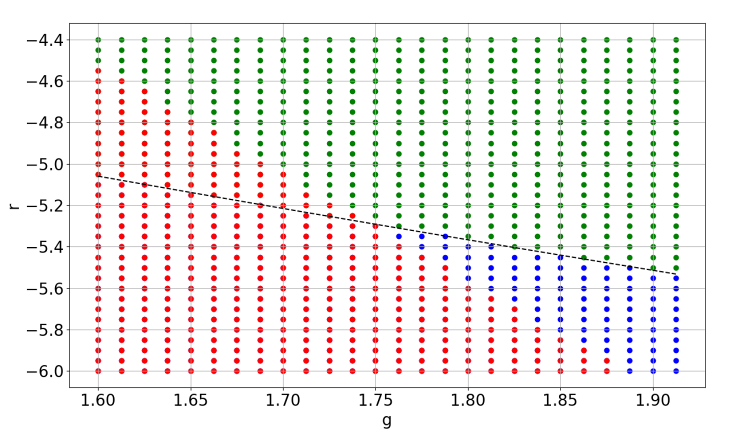
BM - Bootstrap method
HMC - Hamiltonian monte carlo
AM - Approximation method



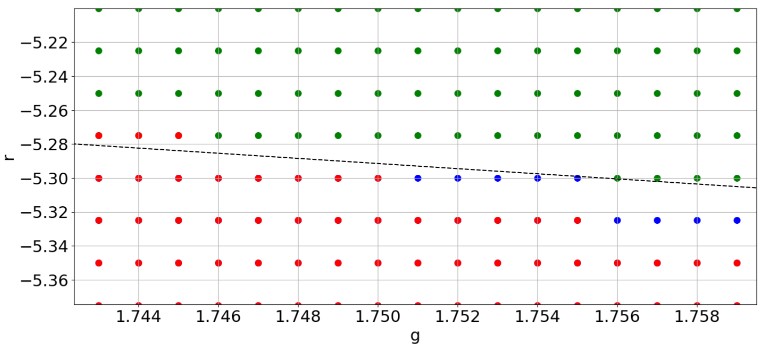
S
BM
m'n
AM
ρ(λ)
Kn(x)
⟨Kn(x)⟩'
P'M(x), P'(x)
P'1(x)
P'3(x)
P'5(x)
P'11(x)
P'(x)
ρM(λ), ρ(λ)
ρ13(λ), ρ(λ)
ρ13(λ), ρ(λ)
g
r

Cubic asymmetric multitrace matrix model

$$S = \frac{1}{N}\left(\frac{1}{2}r\,\mathrm{Tr}(\phi^2) + g\,\mathrm{Tr}(\phi^4)\right) + \frac{1}{N^2}t\,\mathrm{Tr}(\phi)\mathrm{Tr}(\phi^3)$$



Triple point



BM + AM	g=1.75 r=-5.29
HMC	g=1.7706 r=-5.5432
theory	g=1.7325 r=-5.265

Ako by odvodila prezentované vzťahy (1.10) a (1.11)

$$\int w(x) K_n^*(x) K_m(x)\, dx = N_n \delta_{nm} \tag{1.10}$$

$$N_n = \begin{vmatrix} m_0 & m_1 & \cdots & m_{n-1} \\ m_1 & m_2 & \cdots & m_n \\ m_2 & m_3 & \cdots & m_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n-1} & m_n & \cdots & m_{2n-2} \end{vmatrix} \begin{vmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ m_2 & m_3 & \cdots & m_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n-1} & m_n & \cdots & m_{2n-1} \\ m_n & m_{n+1} & \cdots & m_{2n} \end{vmatrix} \tag{1.11}$$

$$K_n(x) = \begin{vmatrix} m_0 & m_1 & m_2 & \cdots & m_n \\ m_1 & m_2 & m_3 & \cdots & m_{n+1} \\ m_2 & m_3 & m_4 & \cdots & m_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{n-1} & m_n & m_{n+1} & \cdots & m_{2n-1} \\ 1 & x & x^2 & \cdots & x^n \end{vmatrix}$$

Orthogonal Polynomials - Gabor Szego (1939) Kapitola 2-3
Všeobecne:
-zoberieme realne, lineárne nezávislé funkcie:
$f_0(x), f_1(x), f_2(x), ..., f_l(x)$
- ortogonalizáciou - Gram-Schmidt - $\phi_0(x), \phi_1(x), ...\phi_l(x)$

$$\phi_n(x) = \frac{1}{\sqrt{D_{n-1}D_n}} D_n(x)$$

$$D_n(x) = \begin{vmatrix} (f_0, f_0) & (f_0, f_1) & \cdots & (f_0, f_n) \\ (f_1, f_0) & (f_1, f_1) & \cdots & (f_1, f_1) \\ \vdots & \vdots & \ddots & \vdots \\ (f_{n-1}, f_0) & (f_{n-1}, f_1) & \cdots & (f_{n-1}, f_n) \\ f_0(x) & f_1(x) & \cdots & f_n(x) \end{vmatrix}.$$

$$D_n = [(f_\nu, f_\mu)]_{\nu,\mu=0,1,2...n} > 0$$

konkretne:

$$m_n = \int_a^b x^n d\alpha(x)$$

Gram-Schmidt - z $1, x, x^2, ...x^n$ dostávame set polynómov $p_0(x), p_1(x), p_2(x), ...p_n(x)$ kde platí ortonormalita: $\int_a^b p_n(x)p_m(x)d\alpha(x) = \delta_{nm}$ - platí aj pre $w(x)dx$, takže ak distribúcia je toho typu vieme o systéme $\sqrt{w(x)}p_n(x)$ tiež povedať, že je ortonormálny.

$$p_n(x) = \frac{1}{\sqrt{D_{n-1}D_n}} \begin{vmatrix} m_0 & m_1 & m_2 & \cdots & m_n \\ m_1 & m_2 & m_3 & \cdots & m_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{n-1} & m_n & m_{n+1} & \cdots & m_{2n-1} \\ 1 & x & x^2 & \cdots & x^n \end{vmatrix} = \frac{1}{\sqrt{D_{n-1}D_n}} K_n(x)$$

$$D_n = [m_{\nu+\mu}]_{\nu,\mu=0,1,2...n} > 0$$

$$\frac{1}{\sqrt{D_{n-1}D_n}}\frac{1}{\sqrt{D_{m-1}D_m}}\int w(x)K_n^*(x)K_m(x)\,dx = \delta_{nm}$$

$$\int w(x)K_n^*(x)K_n(x)\,dx = D_{n-1}D_n$$

$$N_n = D_{n-1}D_n = \begin{vmatrix} m_0 & m_1 & \cdots & m_{n-1} \\ m_1 & m_2 & \cdots & m_n \\ m_2 & m_3 & \cdots & m_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n-1} & m_n & \cdots & m_{2n-2} \end{vmatrix} \begin{vmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ m_2 & m_3 & \cdots & m_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n-1} & m_n & \cdots & m_{2n-1} \\ m_n & m_{n+1} & \cdots & m_{2n} \end{vmatrix}$$

Cubic asymmetric multitrace matrix model

Action for Quartic potential

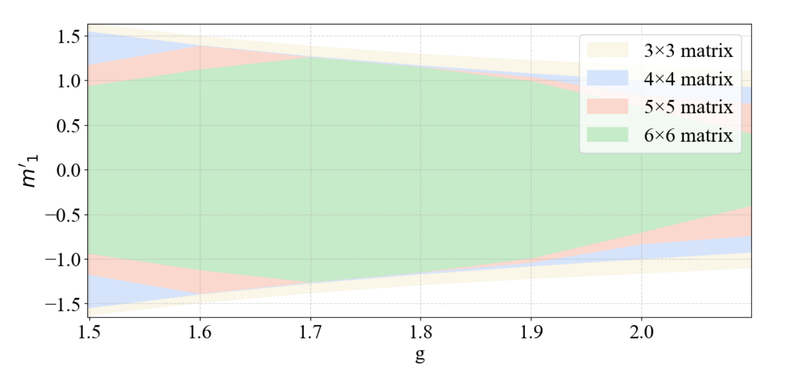
$$S = \frac{1}{N}\left(\frac{1}{2} r\,\mathsf{Tr}(\phi^2) + g\,\mathsf{Tr}(\phi^4)\right) + \frac{1}{N^2} t\mathsf{Tr}(\phi)\mathsf{Tr}(\phi^3)$$

Bootstrap method

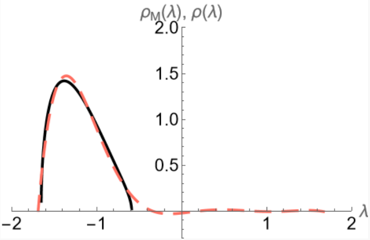
$$m'_{k+3} = \frac{\sum_{q=0}^{k-1} m'_q m'_{k-1-q} - r m'_{k+1} - t\left(m'_3 m'_k - 3 m'_1 m'_{k+2}\right)}{4g}.$$

$$M = \begin{pmatrix} 1 & m'_1 & m'_2 & \cdots \\ m'_1 & m'_2 & -\frac{3t m'_1 m'_2 + r m'_1}{4g+t} & \cdots \\ m'_2 & -\frac{3t m'_1 m'_2 + r m'_1}{4g+t} & \frac{1 - r m'_2 - 4t m'_1 m'_3}{4g} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \succcurlyeq 0$$

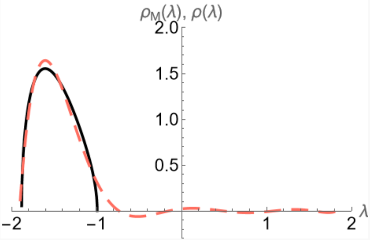
Dependence of the m'_1 on the constant g



Analytic/BM



(a) r=-5.265 g=1.732525

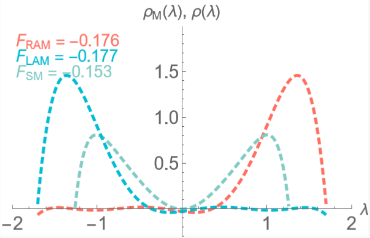


(b) r=-5 g=1.5

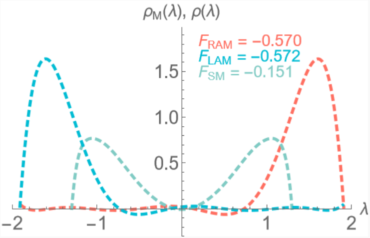
	$g = 1.5\ r = -5$	$g = 1.7325\ r = -5.265$
m'_n	**Analytic/ BM**	
m'_1	-1.48984 / -1.48799	-1.21869 / -1.2187
m'_2	2.267029 / 2.26694999	1.54887 / 1.54889
m'_3	-3.5163478 / -3.51618126	-2.03694 / -2.037
m'_4	5.5483877 / 5.5480629	2.75385 /2.75396

Free energy

$$F = \int_C d\lambda\, V(\lambda)\rho(\lambda) - \iint_{C\times C\backslash \lambda=\lambda'} d\lambda\, d\lambda'\, \rho(\lambda)\rho(\lambda') \log|\lambda - \lambda'|$$



(a) r=-5.265 g=1.732525



(b) r=-5 g=1.5