DIPARTIMENTO DI FISICA E ASTRONOMIA Galileo Galilei

Gaillard-Zumino non-invertible symmetries

Luca Martucci

based on: Fabio Apruzzi & LM arXiv:2509.xxxx

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DUALITY ROTATIONS FOR INTERACTING FIELDS*

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We study the properties of interacting field theories which are invariant under duality rotations which transform a vector field strength into its dual. We consider non-abelian duality groups and find that the largest group for n interacting field strengths is the non-compact Sp($2n$,R), which has U(n) as its maximal compact subgroup. We show that invariance of the equations of motion requires that the lagrangian change in a particular way under duality. We use this property to demonstrate the existence of conserved currents, the invariance of the energy-momentum tensor and the S-matrix, and also in the general construction of the lagrangian. Finally

- Gaillard-Zumino (GZ) 4d Lagrangians: $\mathcal{L}(F, \phi)$

U(1) gauge fields $F^I = \mathrm{d}A^I$, $I = 1, \ldots, n$

ϕ^i neutral sector

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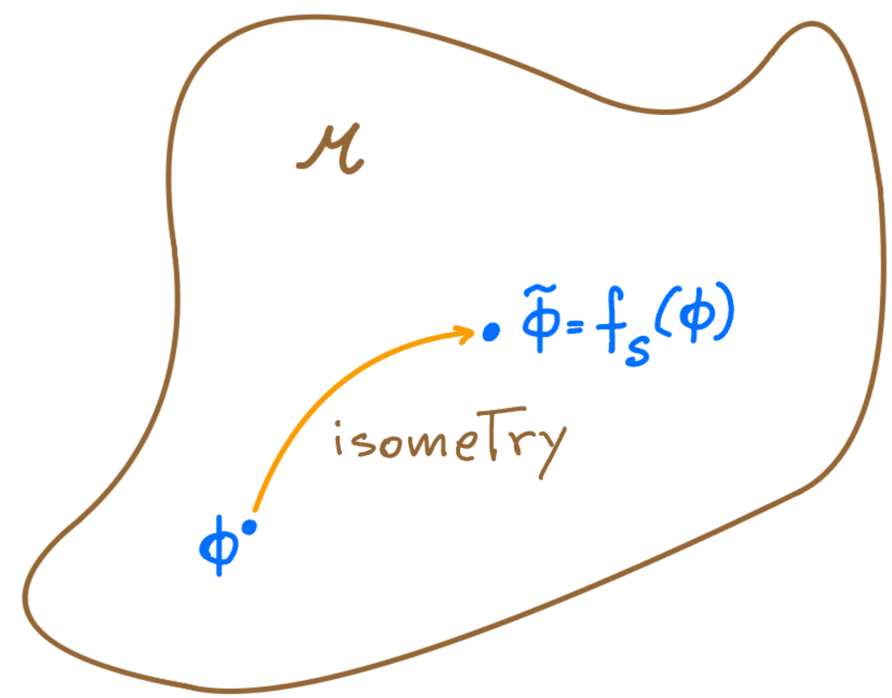
enjoy classical invariance of EoM under continuous group $\mathscr{G}$

$$* \quad \phi^i \longrightarrow \tilde{\phi}^i = f^i_{\mathcal{S}}(\phi)$$

$$\mathcal{S} \in \mathscr{G} \hookrightarrow \mathrm{Sp}(2n, \mathbb{R})$$



$$* \quad \begin{pmatrix} F^I \\ G_J \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{F}^I \\ \tilde{G}_J \end{pmatrix} = \mathcal{S} \cdot \begin{pmatrix} F^I \\ G_J \end{pmatrix}$$

G_J: $2\pi \dfrac{\partial \mathcal{L}}{\partial F^I}$

$\mathcal{S}$: $\begin{pmatrix} \mathrm{A} & \mathrm{B} \\ \mathrm{C} & \mathrm{D} \end{pmatrix}$ continuous symplectic rotations

Examples

- Axion-Maxwell theory: $\mathcal{L} = -\frac{1}{4\pi g^2} F \wedge *F - \frac{M^2}{2} \mathrm{d}\phi \wedge *\mathrm{d}\phi + \frac{1}{8\pi^2} \phi F \wedge F$

$\mathscr{G}$:

- $\phi \ \rightarrow \ \phi + c$, $\mathcal{S} = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in \mathrm{Sp}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})$

- $\begin{pmatrix} \tilde{F} \\ \tilde{G} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}$, $G = -\frac{1}{g^2} * F + \phi F$

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- $\mathcal{L} = -\frac{1}{4\pi} \mathrm{Im}\,\tau\, F \wedge *F - \frac{1}{4\pi} \mathrm{Re}\,\tau\, F \wedge F - M^2 \frac{\mathrm{d}\tau \wedge *\mathrm{d}\bar{\tau}}{(\mathrm{Im}\,\tau)^2}$

$\mathscr{G}$:

- $\tilde{\tau} = f_{\mathcal{S}}(\tau) = \frac{d\tau - c}{a - b\tau}$, $\mathcal{S} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathscr{G} = \mathrm{SL}(2,\mathbb{R})$
- $\begin{pmatrix} \tilde{F} \\ \tilde{G} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}$, $G = -\mathrm{Im}\,\tau\, * F - \mathrm{Re}\,\tau\, F$

Examples

- $\mathcal{N} = 2$ prepotential: $\mathcal{F}(X) = -\frac{1}{3!}\frac{C_{ijk}X^iX^jX^k}{X^0} + \frac{1}{2}\mathsf{y}_{IJ}X^IX^J + \frac{\mathrm{i}}{2}\rho(X^0)^2$

$\mathscr{G}$:

* $\phi^i \equiv \frac{X^i}{X^0} \to \phi^i + \alpha^i$ axionic shifts

* $\mathcal{S}(\alpha) = \begin{pmatrix} \mathsf{A}(\alpha) & 0 \\ \mathsf{C}(\alpha) & \mathsf{A}^{-1\mathrm{t}}(\alpha) \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{R})$

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- … and several other extended supergravities! [Cremmer-Julia ’79, …]

- However…

$$\begin{pmatrix} \tilde{F} \\ \tilde{G} \end{pmatrix} = \mathcal{S} \cdot \begin{pmatrix} F \\ G \end{pmatrix}$$ with $\mathcal{S} \in \mathrm{Sp}(2n, \mathbb{R})$

broken by

$$\frac{1}{2\pi} \oint F^I \in \mathbb{Z}$$
$$\frac{1}{2\pi} \oint G_I \in \mathbb{Z}$$

⇒ only $\mathscr{G}_{\mathbb{Z}} \equiv \mathscr{G} \cap \mathrm{Sp}(2n, \mathbb{Z})$ seems to survive at the quantum level

(“U-duality” group [Hull-Townsend ’94])

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Quantum remnant of non-integral GZ classical symmetries?

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("U-duality" group [Hull-Townsend '94])

- Natural question:

Quantum remnant of non-integral GZ classical symmetries?

- $\mathscr{G}_{\mathbb{Q}} \equiv \mathscr{G} \cap \mathrm{Sp}(2n, \mathbb{Q})$ ⇝ non-invertible symmetries!

Non-invertible symmetries

reviews : [McGreevy, Gomes, Schafer-Nameki, Brennan-Hong, Bhardwaj-Bottini-Fraser-Taliente-Gladden-Gould-Platschorre-Tillim, Shao, Luo-Wang-Wang, Carqueville-Del Zotto-Runkel, Córdova, Del Zotto, Freed, Jordan, Ohmori, …]

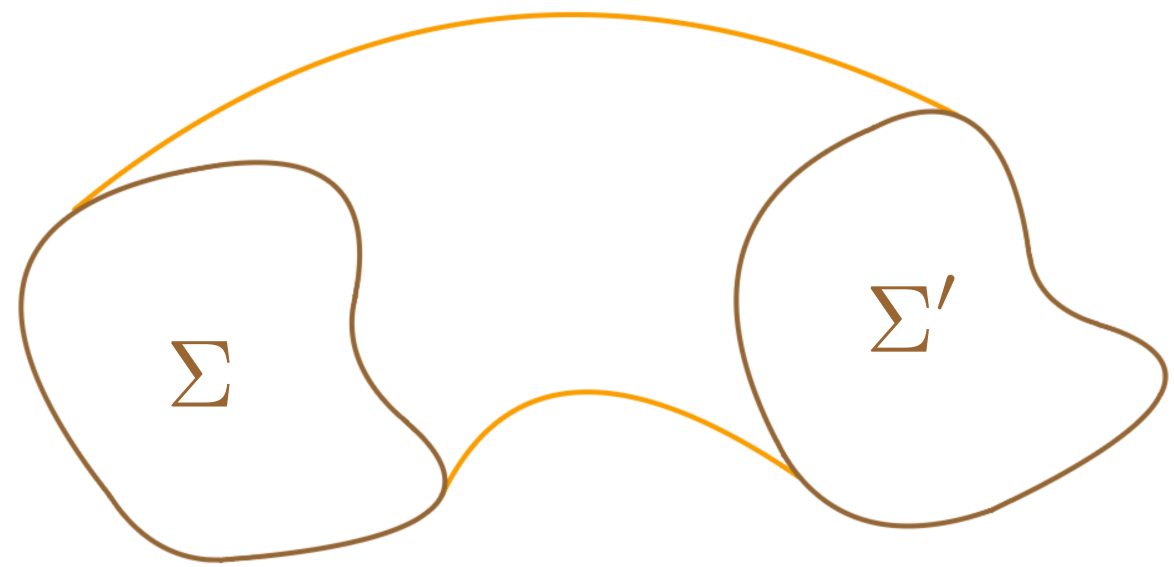
Symmetries ⟷ topological defects/operators

[Gaiotto-Kapustin-Seiberg-Willett '14]

e.g. $\mathscr{G} = \mathrm{U}(1)$, $\mathrm{d}\mathcal{J}_3 = 0$

$$\mathcal{D}_\alpha(\Sigma) = \exp(\mathrm{i}\alpha \oint_\Sigma \mathcal{J}_3)$$

$$\mathcal{D}_\alpha(\Sigma) = \mathcal{D}_\alpha(\Sigma')$$



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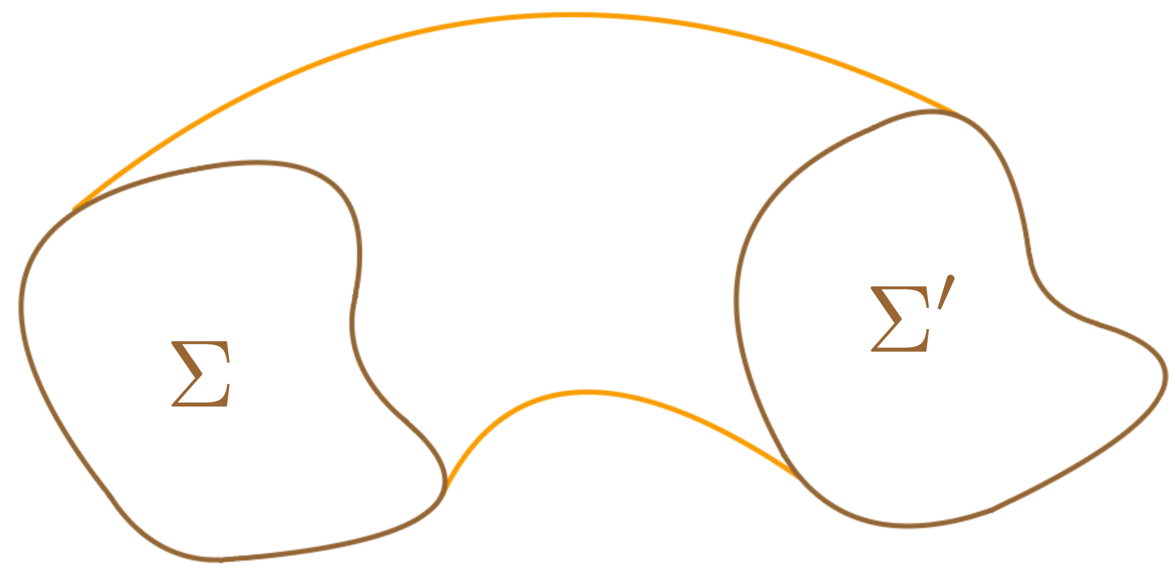


- One may impose: $\mathcal{D}_g(\Sigma) \times \mathcal{D}_{g'}(\Sigma) = \mathcal{D}_{gg'}(\Sigma)$ $\quad g, g' \in \mathscr{G}$

fusion product ~ group law → invertible !

- But topological defects may

  * support world-volume degrees of freedom

  * obey more general fusion rules:

$$\mathcal{D}_a(\Sigma) \times \mathcal{D}_b(\Sigma) = \sum_c \mathcal{T}_c(\Sigma) \otimes \mathcal{D}_c(\Sigma)$$

→ decoupled TQFTs

  * $\nexists\, \mathcal{D}'(\Sigma)$ such that $\mathcal{D}(\Sigma) \times \mathcal{D}'(\Sigma) = \mathbb{1}$

non-invertible!

- They still give non-trivial constraints and selection rules!

- In GZ models, we argue that

$$\mathscr{G}_{\mathbb{Z}} \equiv \mathscr{G} \cap \mathrm{Sp}(2n, \mathbb{Z}) \longrightarrow \text{invertible}$$

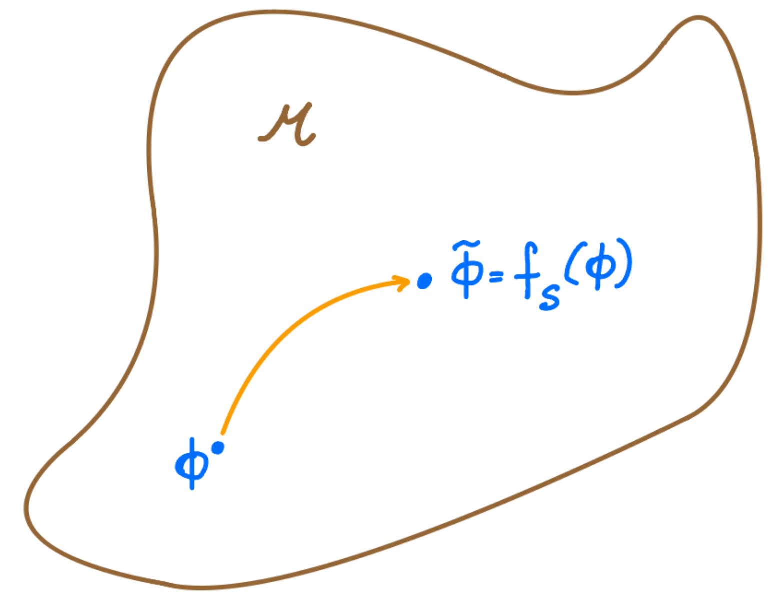
$$\mathscr{G}_{\mathbb{Q}} \equiv \mathscr{G} \cap \mathrm{Sp}(2n, \mathbb{Q}) \longrightarrow \text{non-invertible}$$

GZ non-invertible topological defects

- GZ Lagrangians are not invariant, but is a specific way

GZ-condition

$$\mathcal{L}(F, f_{\mathcal{S}}(\phi)) = \mathcal{L}_{\mathcal{S}}(F, \phi)$$



with $\mathcal{L}_{\mathcal{S}}$ such that

$$\mathcal{L}_{\mathcal{S}}(\tilde{F}, \phi) - \frac{1}{4\pi} \tilde{F}^I \wedge \tilde{G}_I = \mathcal{L}(F, \phi) - \frac{1}{4\pi} F^I \wedge G_I$$

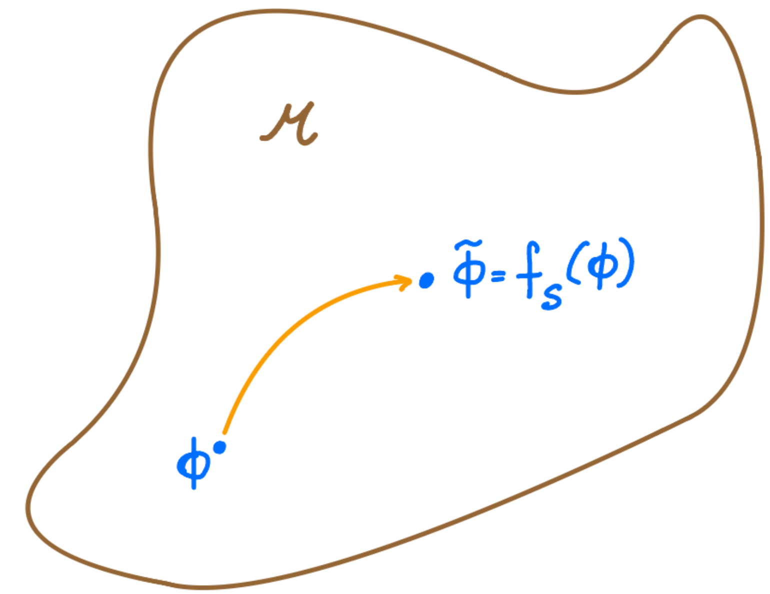
with $\begin{pmatrix} \tilde{F} \\ \tilde{G} \end{pmatrix} = \mathcal{S} \cdot \begin{pmatrix} F \\ G \end{pmatrix}$, $\tilde{G}_I = 2\pi \dfrac{\partial \mathcal{L}_{\mathcal{S}}(\tilde{F}, \phi)}{\partial \tilde{F}^I}$

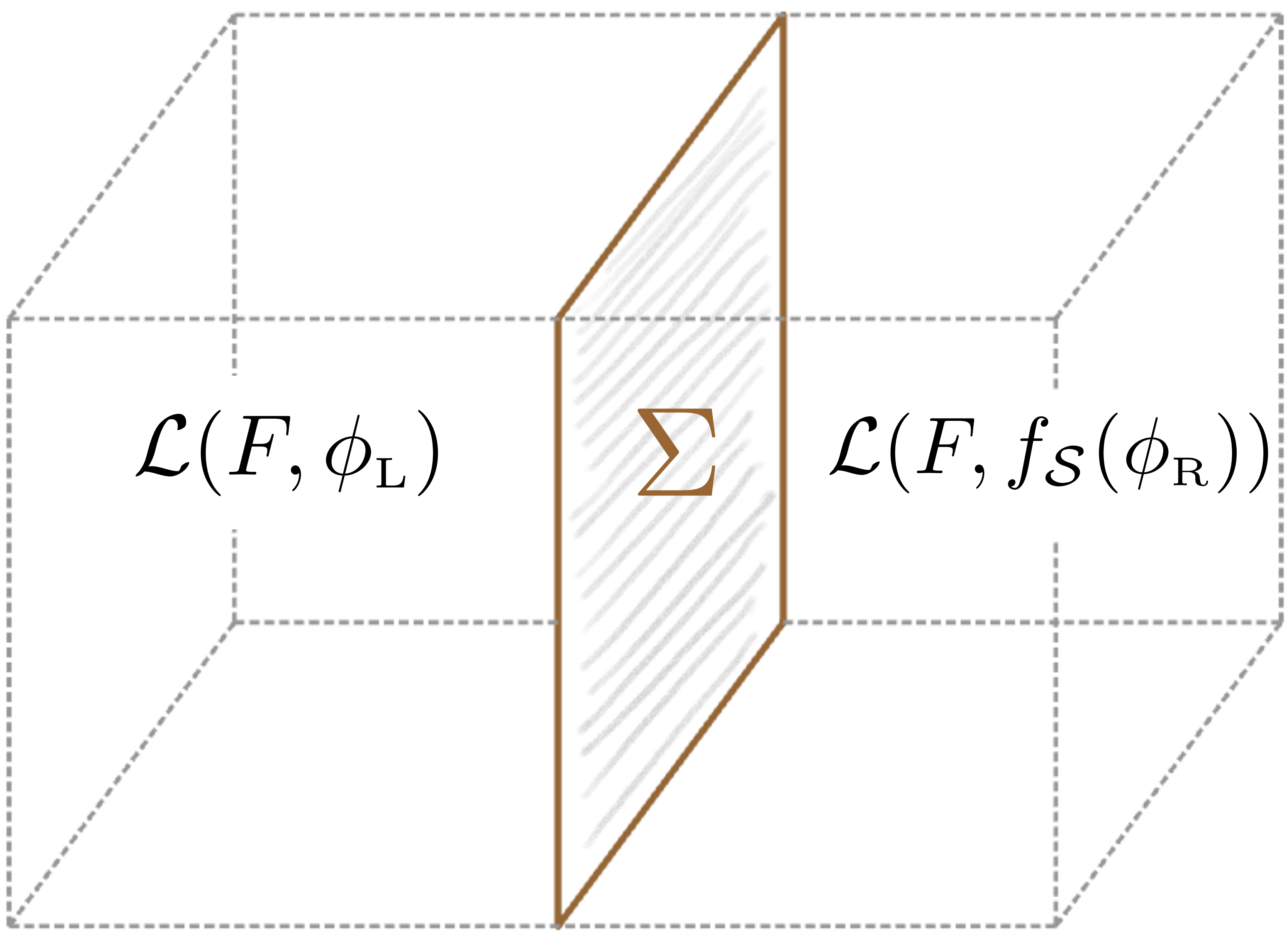
$\mathrm{Sp}(2n, \mathbb{R})$
reparametrization

physically equivalent only if $\mathcal{S} \in \mathrm{Sp}(2n, \mathbb{Z})$!

If $\mathcal{S} \in \mathscr{G}$, we can construct invertible defect

$$\mathcal{U}_{\mathcal{S}}(\Sigma) = \exp\left(i \oint_{\Sigma} J_3\right)$$





$$\mathcal{L}(F, \phi_{\mathrm{L}}) \qquad \mathcal{L}(F, f_{\mathcal{S}}(\phi_{\mathrm{R}})) = \mathcal{L}_{\mathcal{S}}(F, \phi_{\mathrm{R}})$$

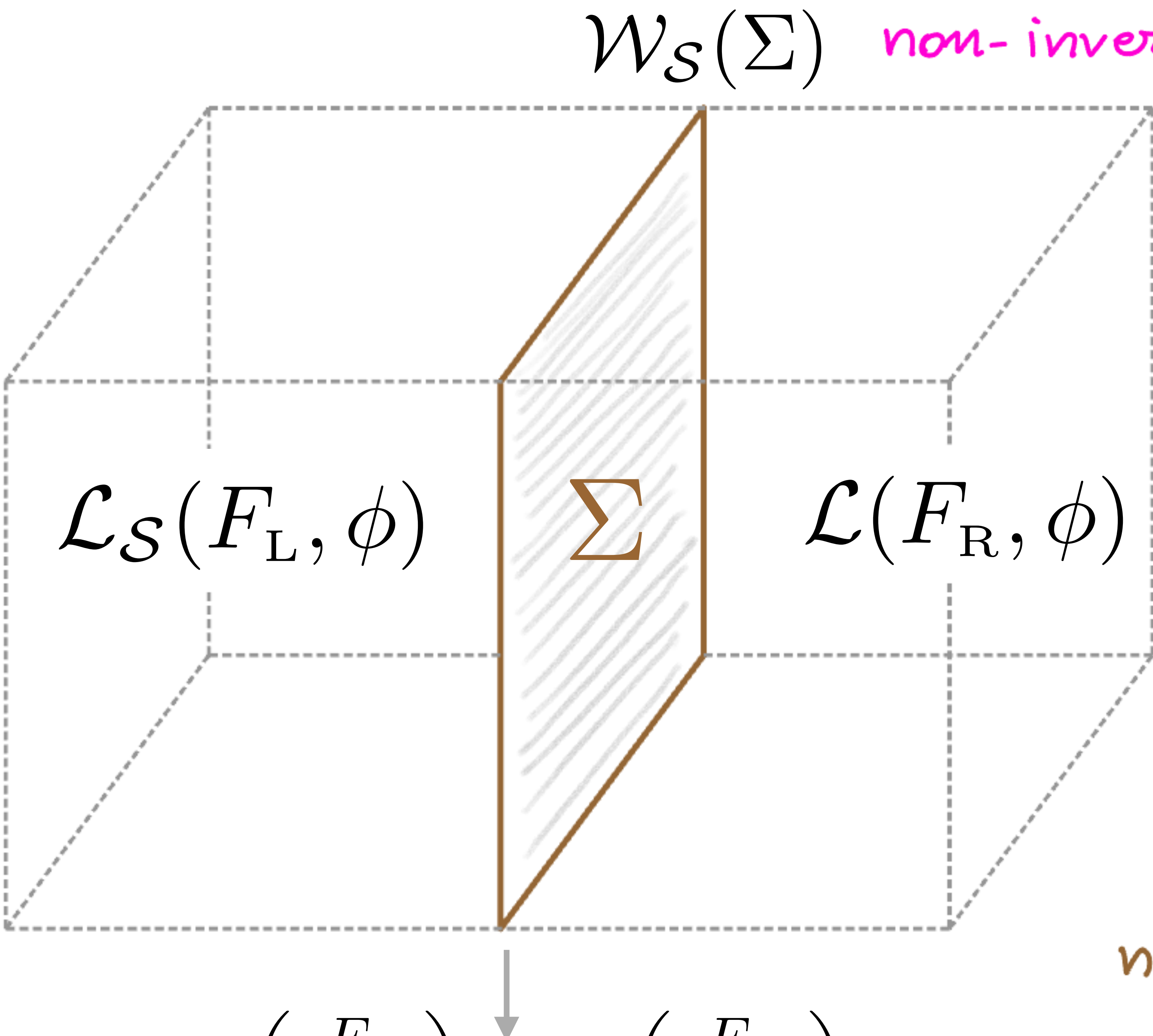
$$\neq \mathcal{L}(F, \phi_{\mathrm{R}})$$

$$\phi_{\mathrm{L}}^{i} = f_{\mathcal{S}}^{i}(\phi_{\mathrm{R}})$$

non-Topological!
if $\mathcal{S} \notin \mathrm{Sp}(2n, \mathbb{Z})$

- If $\boxed{\mathcal{S} \in \mathrm{Sp}(2n, \mathbb{Q})}$, well defined interface ⤳ see later

⤴ non-invertible!



$$\begin{pmatrix} F_{\mathrm{L}} \\ G_{\mathrm{L}} \end{pmatrix} = \mathcal{S} \cdot \begin{pmatrix} F_{\mathrm{R}} \\ G_{\mathrm{R}} \end{pmatrix}$$

non-Topological!

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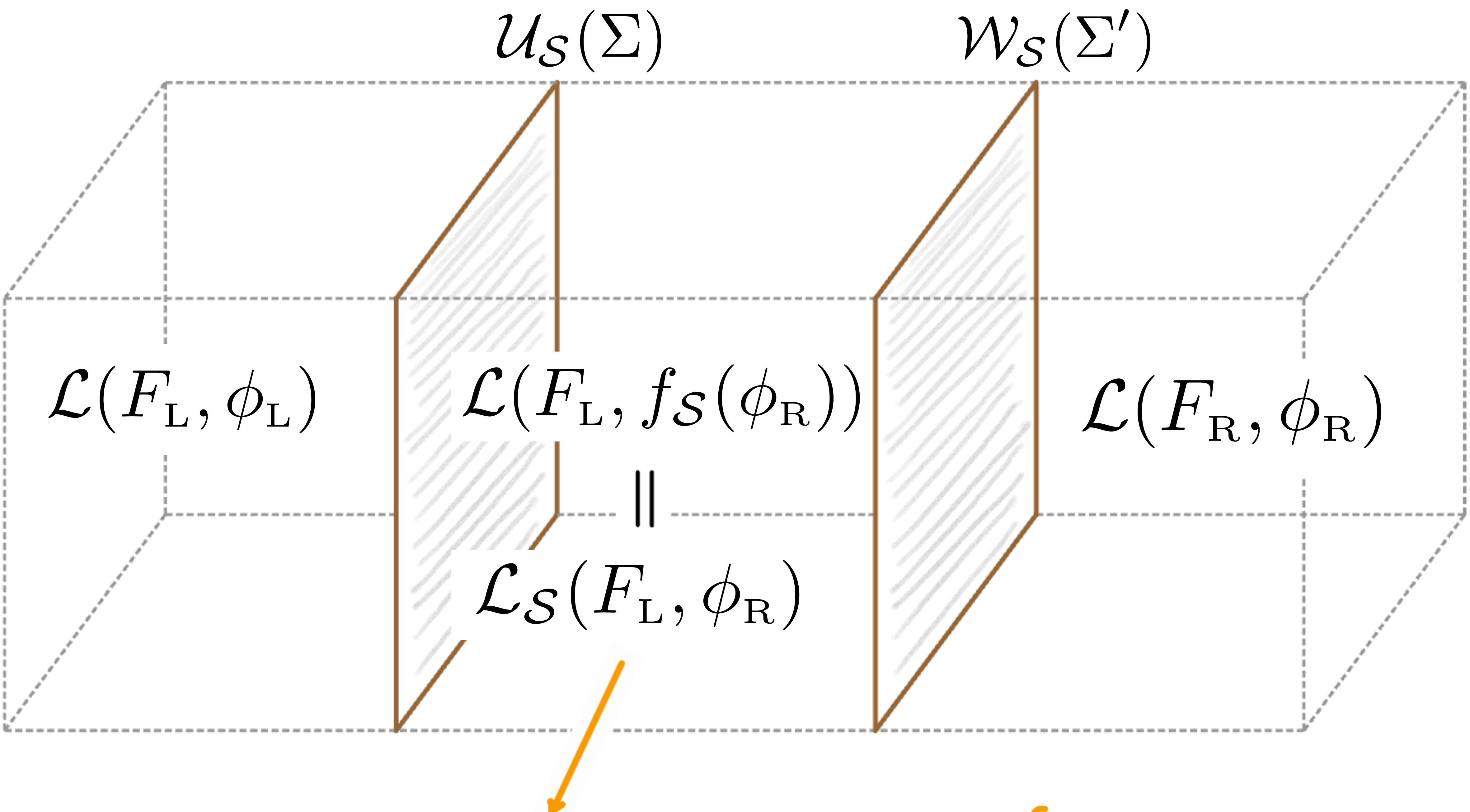
So, if $\mathcal{S} \in \mathscr{G}_{\mathbb{Q}} \equiv \mathscr{G} \cap \mathrm{Sp}(2n,\mathbb{Q})$



GZ-condition

$$\mathcal{L}(F, f_{\mathcal{S}}(\phi)) = \mathcal{L}_{\mathcal{S}}(F, \phi)$$

with

$$\begin{cases} \begin{pmatrix} F_{\rm L} \\ G_{\rm L} \end{pmatrix} = \mathcal{S} \cdot \begin{pmatrix} F_{\rm R} \\ G_{\rm R} \end{pmatrix} \\ \phi^i_{\rm L} = f^i_{\mathcal{S}}(\phi_{\rm R}) \end{cases}$$

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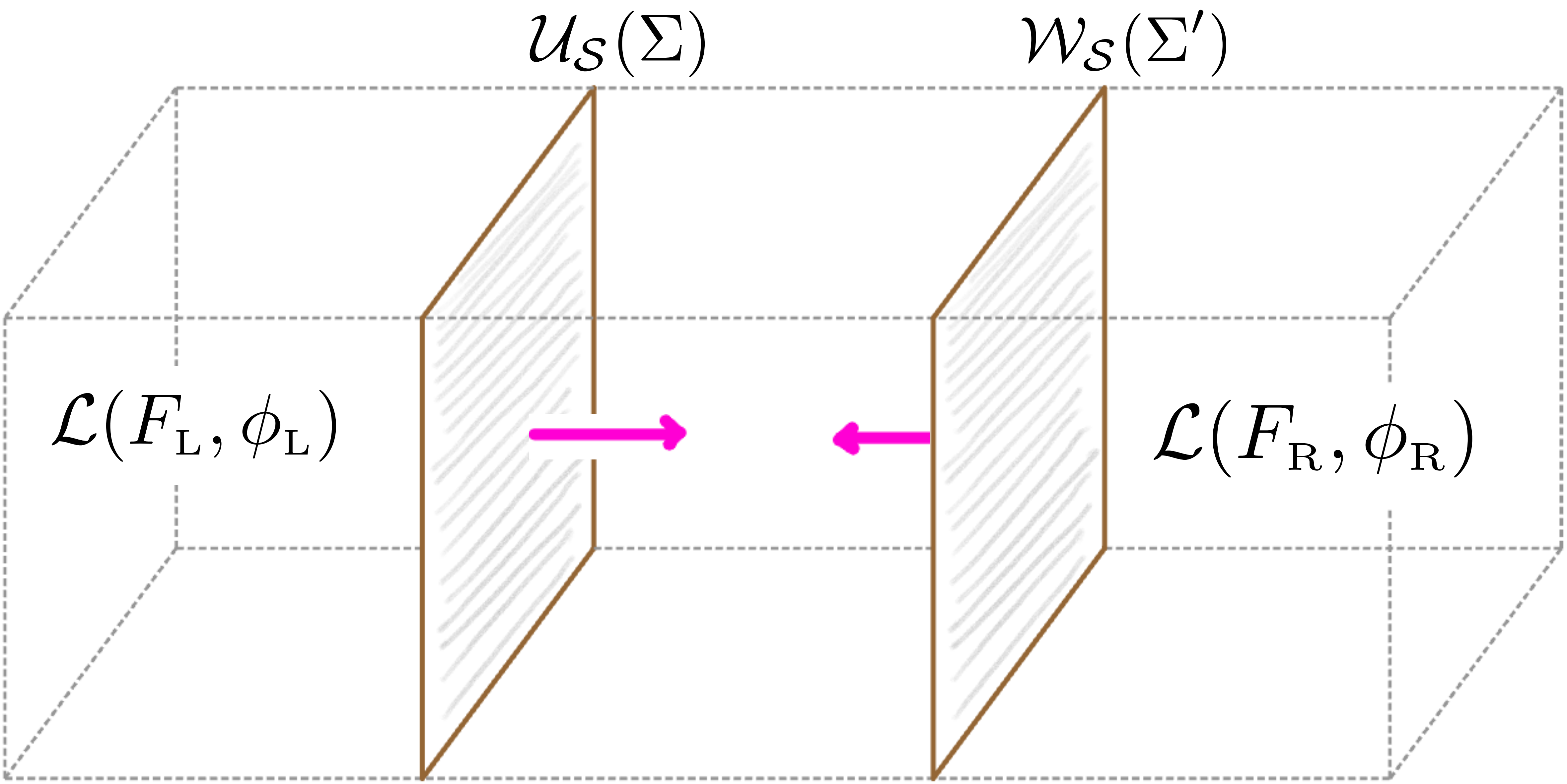

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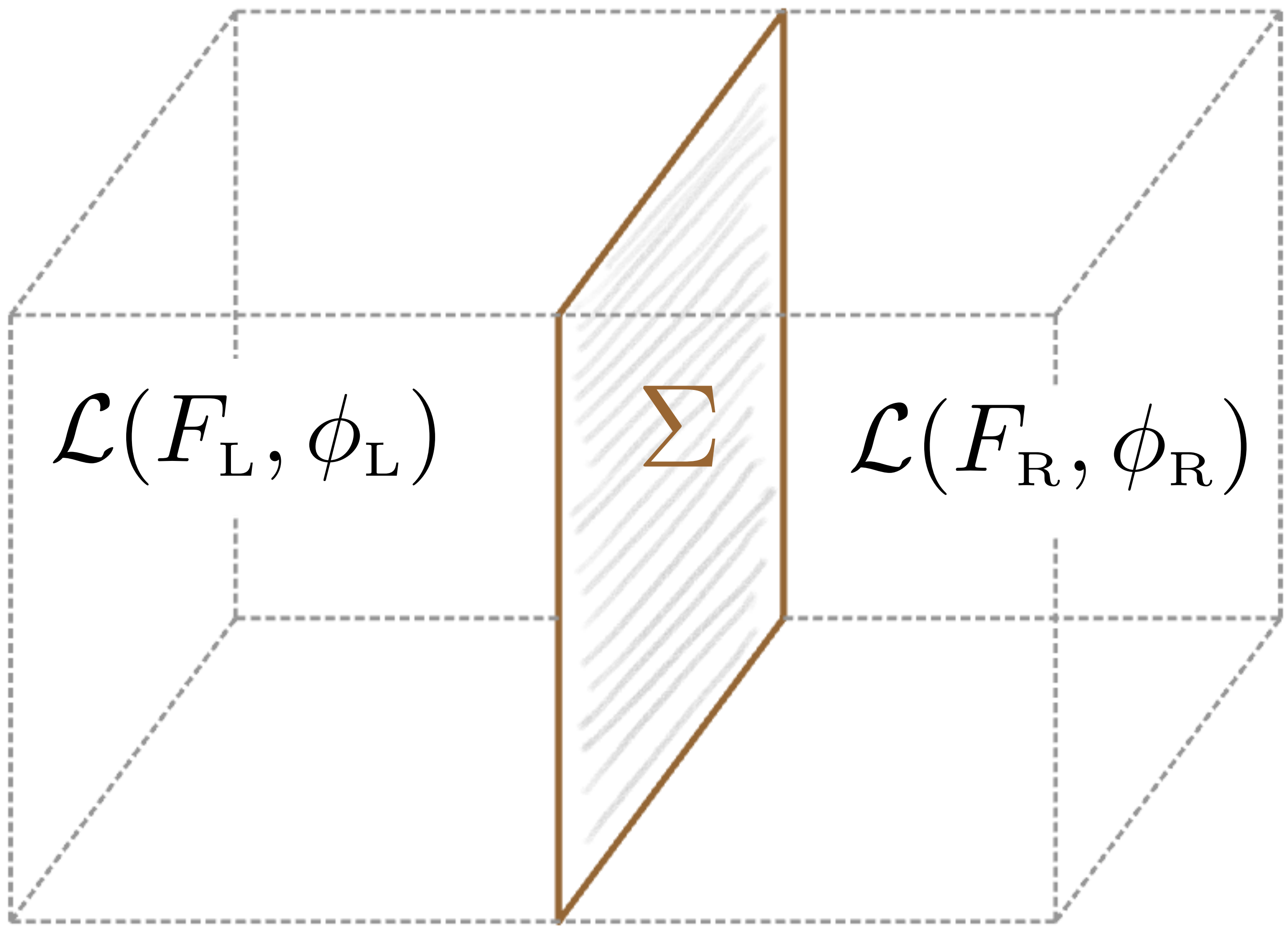
So, if $\mathcal{S} \in \mathscr{G}_{\mathbb{Q}} \equiv \mathscr{G} \cap \mathrm{Sp}(2n, \mathbb{Q})$: GZ topological defect

$$\mathcal{D}_{\mathcal{S}}(\Sigma) \equiv \mathcal{U}_{\mathcal{S}}(\Sigma) \times \mathcal{W}_{\mathcal{S}}(\Sigma)$$



with
$$\begin{cases} \begin{pmatrix} F_{\rm L} \\ G_{\rm L} \end{pmatrix} = \mathcal{S} \cdot \begin{pmatrix} F_{\rm R} \\ G_{\rm R} \end{pmatrix} \\ \phi^i_{\rm L} = f^i_{\mathcal{S}}(\phi_{\rm R}) \end{cases}$$

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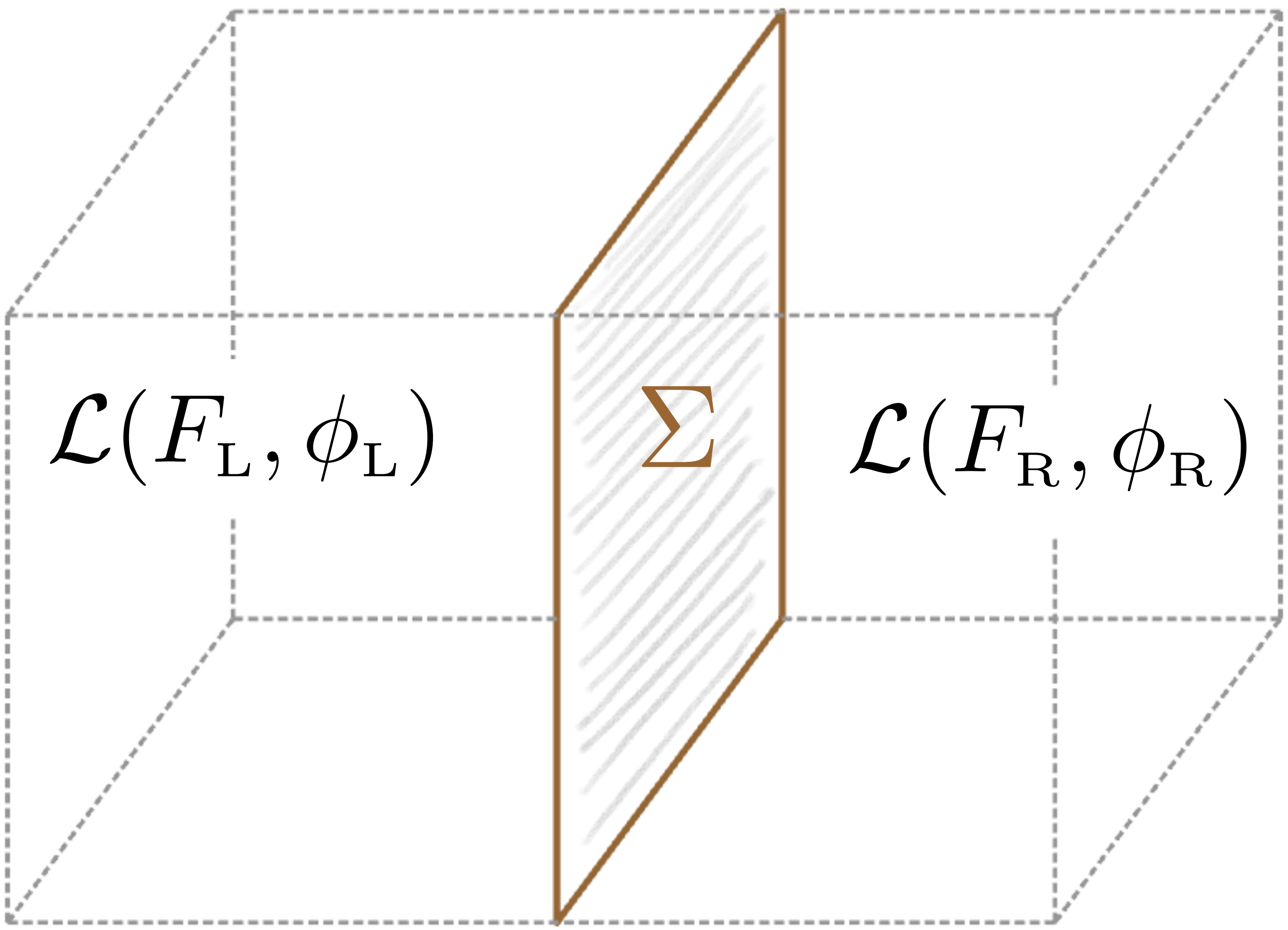

$$T_{\mu\nu}(F_{\mathrm{L}}, \phi_{\mathrm{L}})|_{\Sigma} \overset{GZ}{=} T_{\mu\nu}(F_{\mathrm{R}}, \phi_{\mathrm{R}})|_{\Sigma}$$

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So, if $\mathcal{S} \in \mathscr{G}_{\mathbb{Q}} \equiv \mathscr{G} \cap \mathrm{Sp}(2n, \mathbb{Q})$: GZ topological defect

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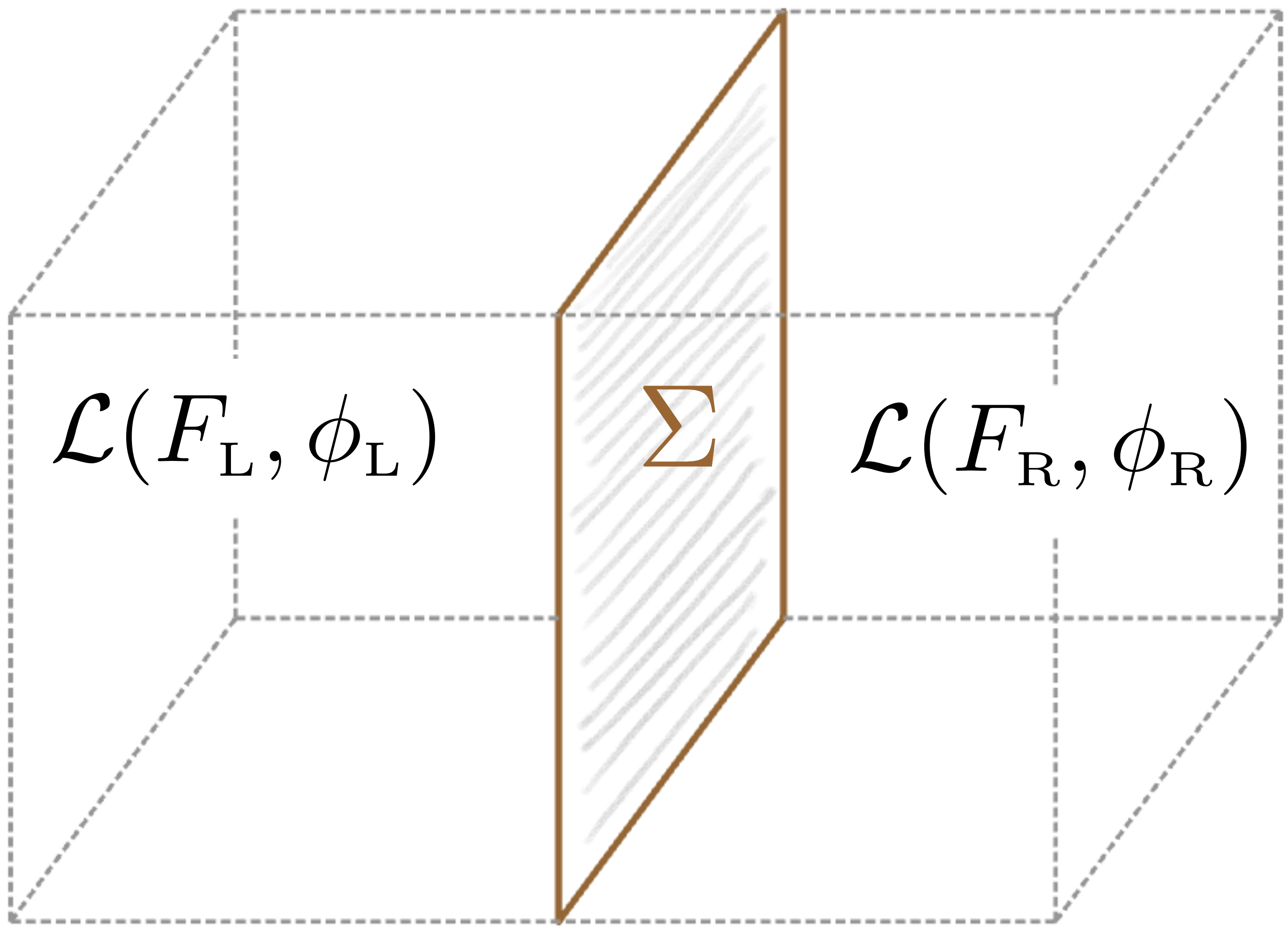


$$T_{\mu\nu}(F_{\rm L}, \phi_{\rm L})|_{\Sigma}$$
$$\Vert \ \text{GZ}$$
$$T_{\mu\nu}(F_{\rm R}, \phi_{\rm R})|_{\Sigma}$$
$$\Downarrow$$

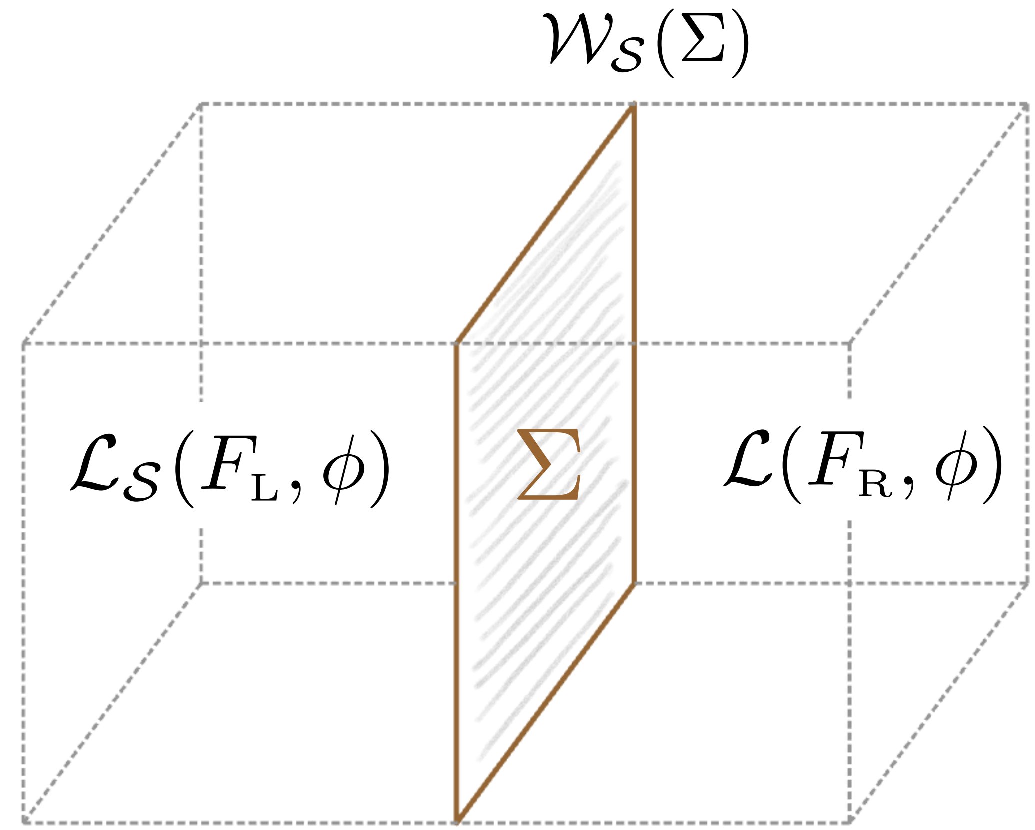
$\mathcal{D}_{\mathcal{S}}(\Sigma)$ Topological

$\Updownarrow$

(non-invertible) symmetry

with
$$\begin{cases} \begin{pmatrix} F_{\rm L} \\ G_{\rm L} \end{pmatrix} = \mathcal{S} \cdot \begin{pmatrix} F_{\rm R} \\ G_{\rm R} \end{pmatrix} \\ \phi^i_{\rm L} = f^i_{\mathcal{S}}(\phi_{\rm R}) \end{cases}$$

Construction of $\mathcal{W}_{\mathcal{S}}$?



$$\begin{pmatrix} F_{\mathrm{L}} \\ G_{\mathrm{L}} \end{pmatrix} = \mathcal{S} \cdot \begin{pmatrix} F_{\mathrm{R}} \\ G_{\mathrm{R}} \end{pmatrix} \qquad \mathcal{S} \in \mathrm{Sp}(2n, \mathbb{Q})$$

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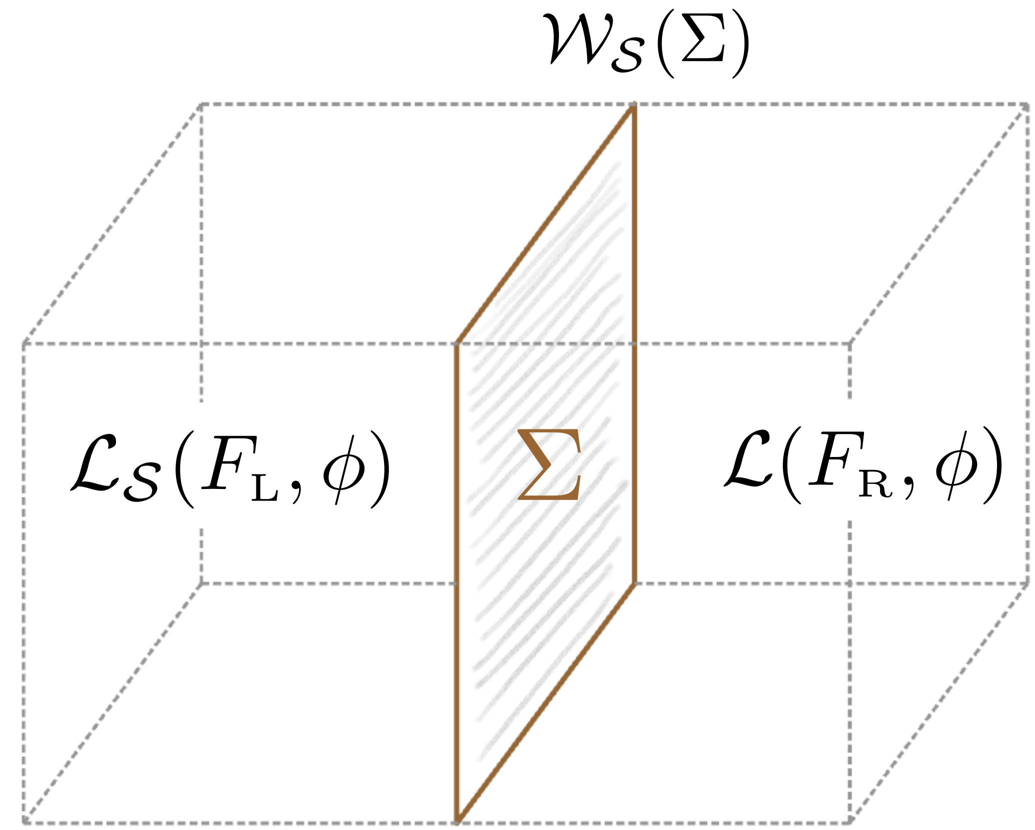

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- We can focus on generators of $\mathrm{Sp}(2n, \mathbb{Q})$

$$\mathcal{S}_{\Omega} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} , \quad \mathcal{S}_{\mathsf{A}} = \begin{pmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{A}^{-1t} \end{pmatrix} , \quad \mathcal{S}_{\mathsf{C}} = \begin{pmatrix} \mathbb{1} & 0 \\ \mathsf{C} & \mathbb{1} \end{pmatrix}$$

$\hookrightarrow \mathsf{A}^{I}{}_{J} \in \mathrm{GL}(n, \mathbb{Q})$

$\hookrightarrow \mathsf{C}_{IJ} = \mathsf{C}_{JI} \in \mathbb{Q}$

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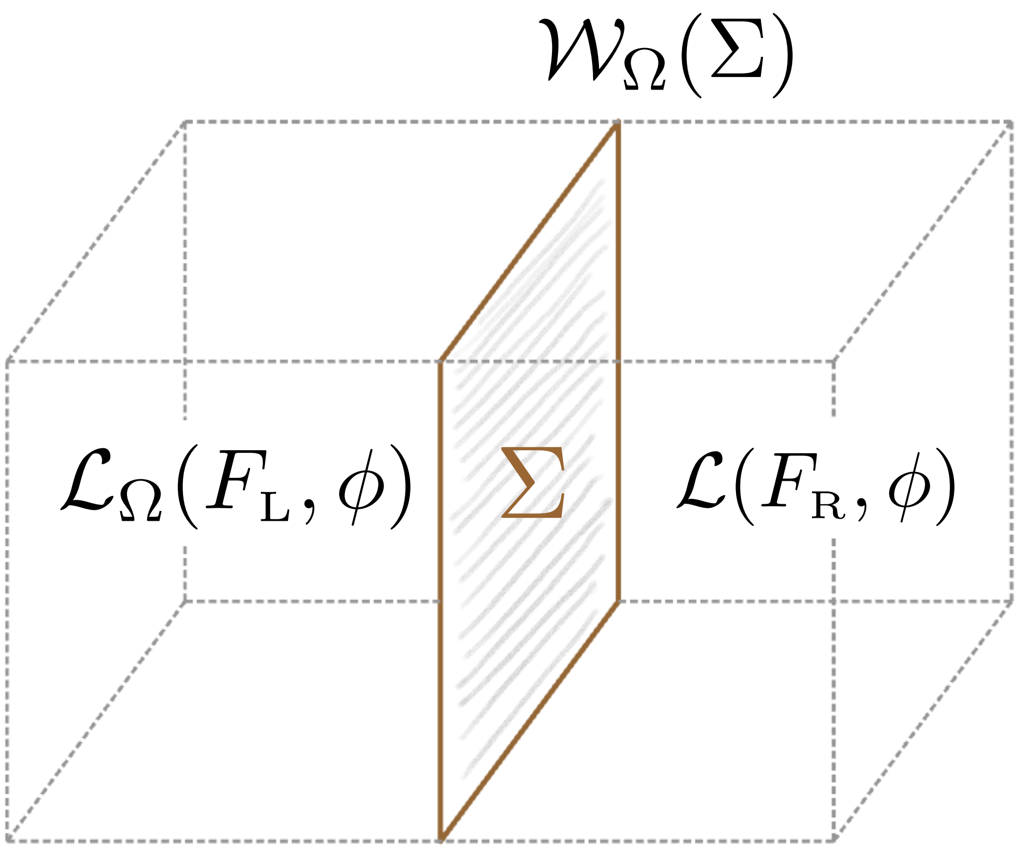

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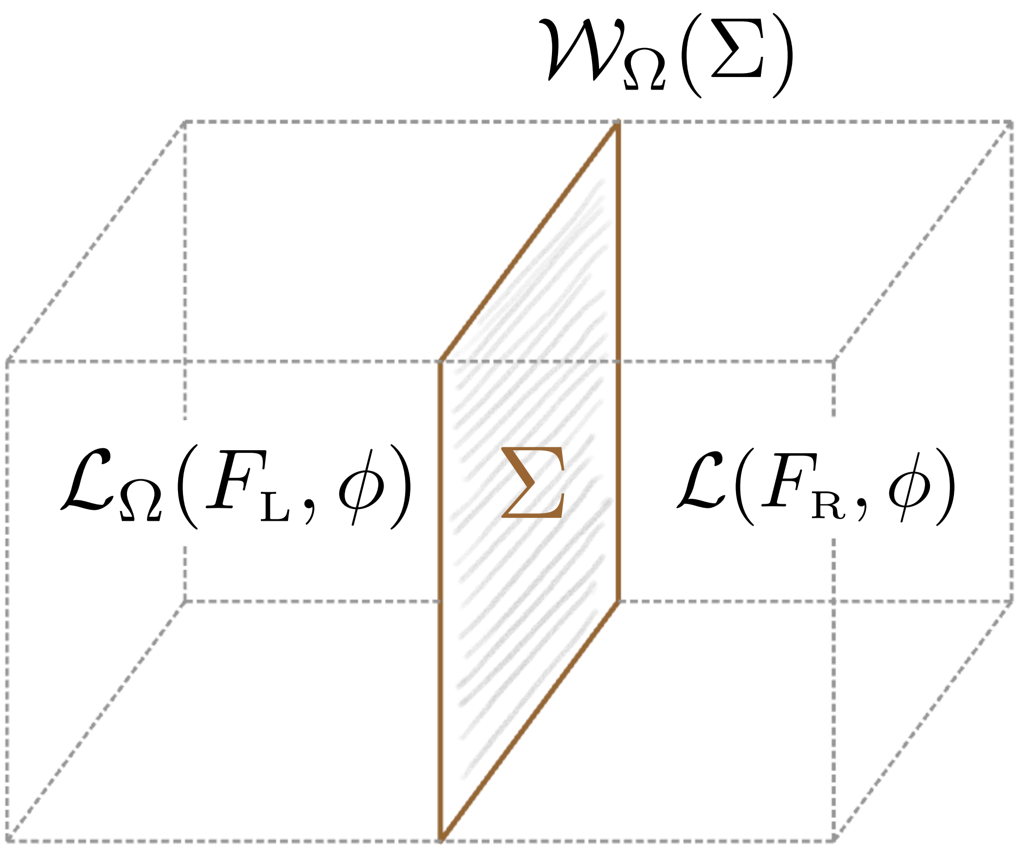
$$\mathcal{L}_\Omega(F_{\rm L}, \phi)$$

||

S-dual of $\mathcal{L}(F_{\rm R}, \phi)$

physically equivalent

$$\mathcal{S}_\Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$



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$$\mathcal{L}_\Omega(F_{\rm L}, \phi)$$
$$||$$
S-dual of $\mathcal{L}(F_{\rm R}, \phi)$

physically equivalent

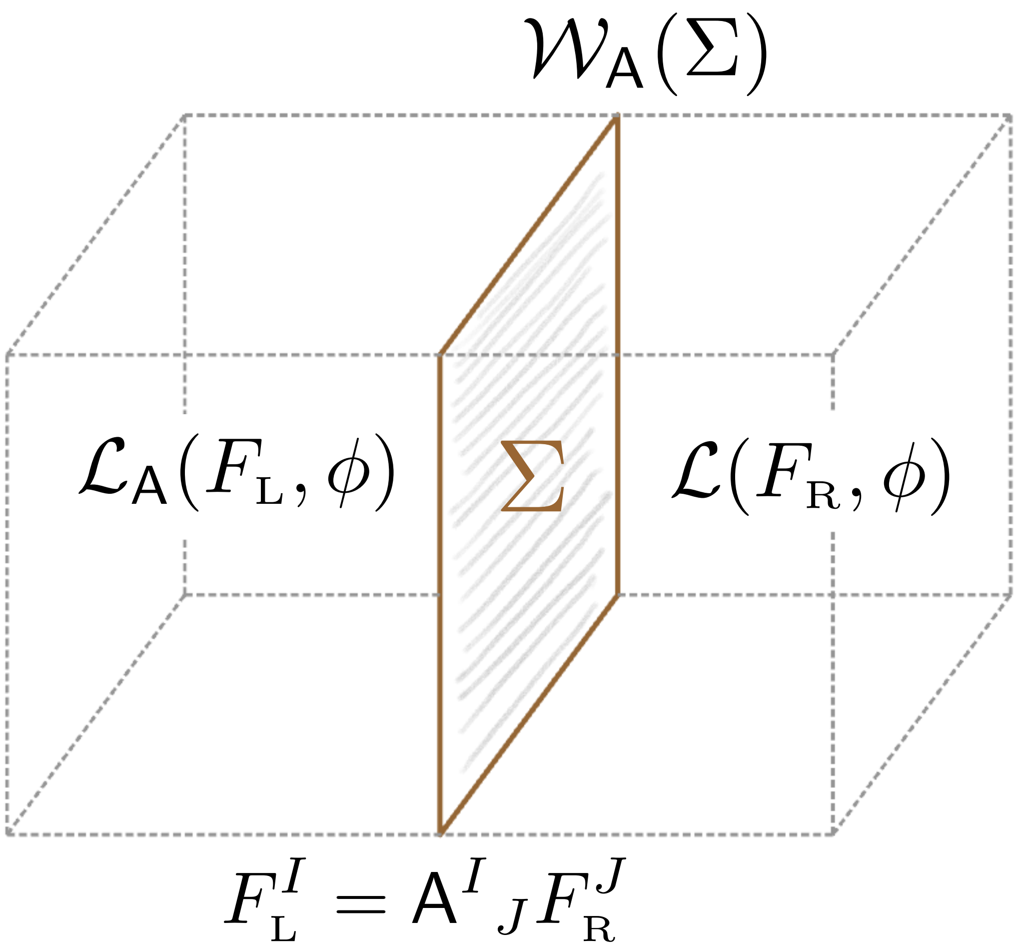
$$\mathcal{W}_\Omega(\Sigma) = \exp\left(\frac{\mathrm{i}}{2\pi} \delta_{IJ} \oint_\Sigma A^I_{\rm L} \wedge F^J_{\rm R} \right)$$

[Ganor '96,
Gaiotto-Witten '08,
Kapustin-Tikhonov '09]

(S-duality wall)

$$\mathcal{S}_{\mathsf{A}} = \begin{pmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{A}^{-1\mathrm{t}} \end{pmatrix} \quad \text{with} \quad \mathsf{A}^{I}{}_{J} \in \mathrm{GL}(n, \mathbb{Q})$$

$$\mathcal{S}_{\mathsf{A}} = \begin{pmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{A}^{-1\mathrm{t}} \end{pmatrix} \quad \text{with} \quad \mathsf{A}^I{}_J \in \mathrm{GL}(n,\mathbb{Q})$$



$$\mathcal{L}_{\mathsf{A}}(\tilde{F},\phi) = \mathcal{L}(F,\phi)$$

$$\tilde{F}^I = \mathsf{A}^I{}_J F^J$$

rotation of $U(1)$ gauge fields

equivalent only if $\mathsf{A}^I{}_J \in \mathrm{GL}(n,\mathbb{Z})$

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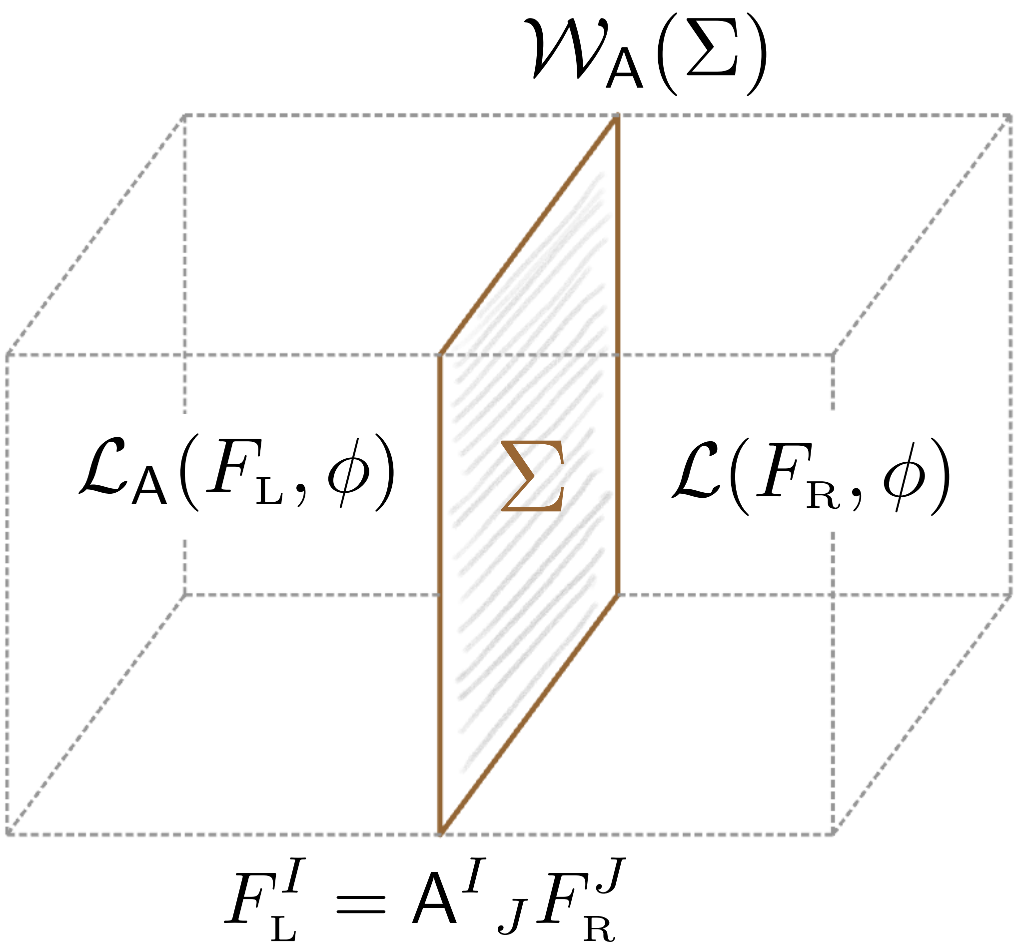

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$$\tilde{F}^I = \mathsf{A}^I{}_J F^J$$

rotation of U(1) gauge fields

equivalent only if $\mathsf{A}^I{}_J \in \mathrm{GL}(n,\mathbb{Z})$

- Pick integral factorization: $\mathsf{A} = \mathsf{M}^{-1}\mathsf{E}\,, \quad \mathsf{M},\mathsf{E} \in \mathrm{Mat}(n,\mathbb{Z})$

$$\mathcal{W}_{\mathsf{A}}(\Sigma) = \int Db\, \exp\left[\frac{\mathrm{i}}{2\pi}\oint_\Sigma b_I \wedge \left(\mathsf{M}^I{}_J F^J_{\mathrm{L}} - \mathsf{E}^I{}_J F^J_{\mathrm{R}}\right)\right]$$

see also [Córdova-Ohmori '23] for $n=1$

$$\mathcal{S}_{\mathsf{C}} = \begin{pmatrix} \mathbb{1} & 0 \\ \mathsf{C} & \mathbb{1} \end{pmatrix} \quad \text{with} \quad \mathsf{C}_{IJ} = \mathsf{C}_{JI} \in \mathbb{Q}$$

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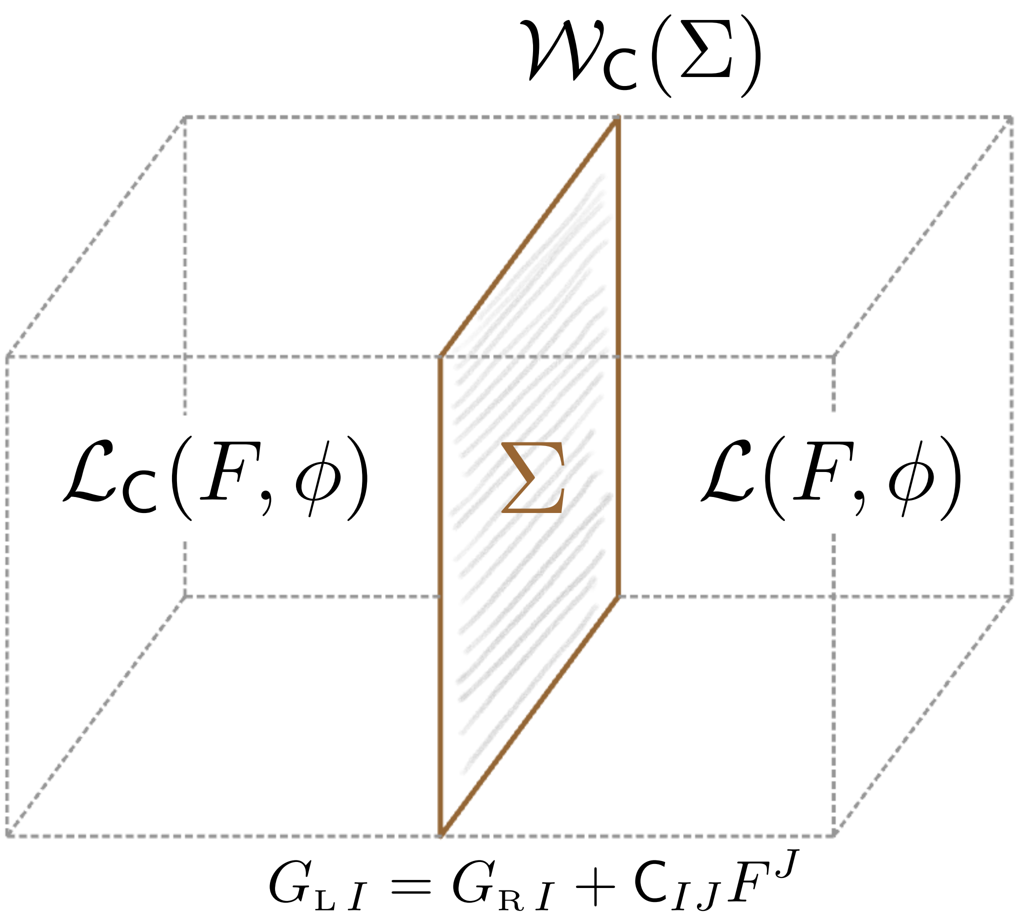


$$\mathcal{L}_{\mathsf{C}}(F, \phi)$$

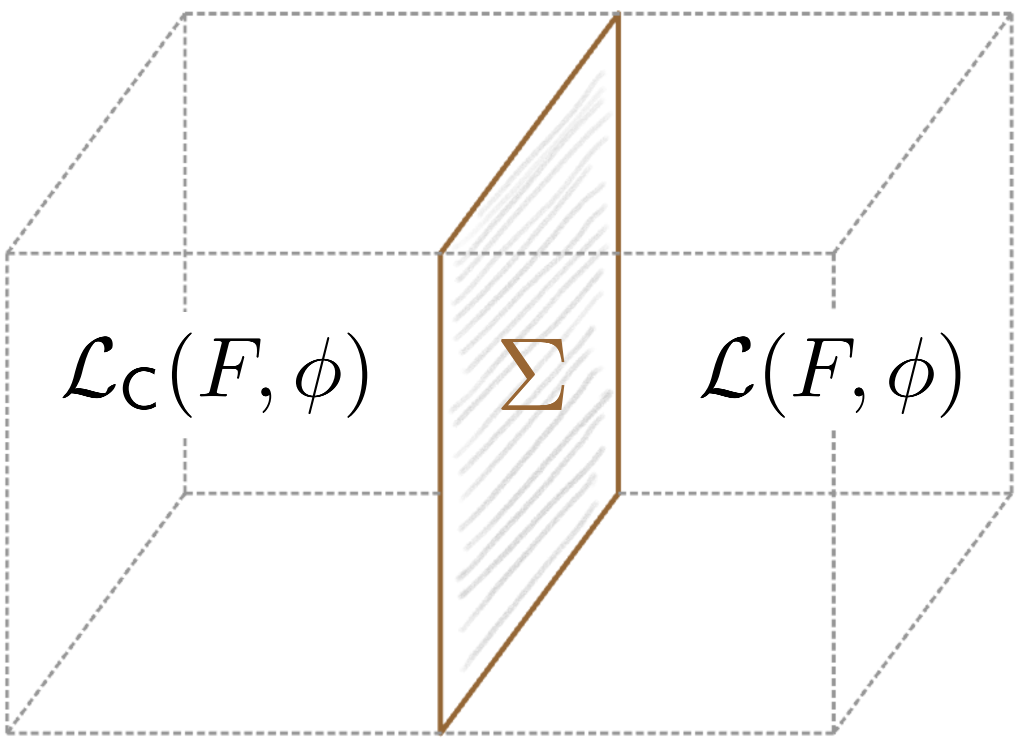
$$||$$

$$\mathcal{L}(F, \phi) + \frac{1}{4\pi} \mathsf{C}_{IJ} F^I \wedge F^J$$

equivalent only if $\mathsf{C}_{IJ} \in \mathbb{Z}$

$$\mathcal{S}_{\mathsf{C}} = \begin{pmatrix} \mathbb{1} & 0 \\ \mathsf{C} & \mathbb{1} \end{pmatrix} \quad \text{with} \quad \mathsf{C}_{IJ} = \mathsf{C}_{JI} \in \mathbb{Q}$$

$\mathcal{W}_{\mathsf{C}}(\Sigma)$



$G_{\mathrm{L}\,I} = G_{\mathrm{R}\,I} + \mathsf{C}_{IJ}F^J$

$$\mathcal{L}_{\mathsf{C}}(F,\phi) = \mathcal{L}(F,\phi) + \frac{1}{4\pi}\mathsf{C}_{IJ}F^I \wedge F^J$$

equivalent only if $\mathsf{C}_{IJ} \in \mathbb{Z}$

- Pick integral factorization:

$$\mathsf{C} = \mathsf{P}\mathsf{N}^{-1}, \quad \mathsf{P},\mathsf{N} \in \mathrm{Mat}(n,\mathbb{Z})$$

$$\mathcal{W}_{\mathsf{C}}(\Sigma) = \int \mathcal{D}a\mathcal{D}c \, \exp\Big[-\frac{\mathrm{i}}{4\pi}(\mathsf{N}^t\mathsf{P})_{IJ} \oint_\Sigma a^I \wedge \mathrm{d}a^J + \frac{\mathrm{i}}{2\pi}\oint_\Sigma c_I \wedge (\mathsf{N}^I{}_J \mathrm{d}a^J + F^I)\Big]$$

see also
[Córdova-Ohmori ’22, Choi-Lam-Shao ’23, Hsin-Lam-Seiberg ’18]

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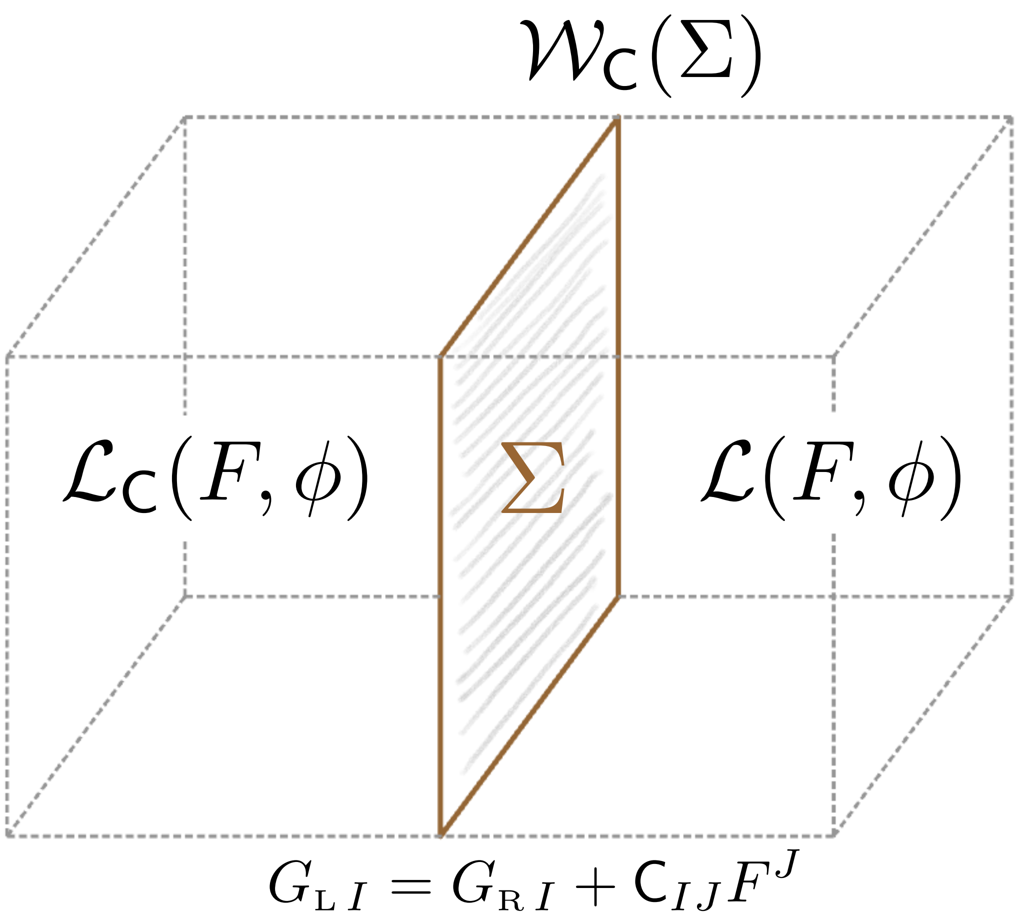

$$\mathcal{L}_{\mathsf{C}}(F,\phi)$$
$$||$$
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↓

$$\mathcal{W}_{\mathsf{C}}(\Sigma) = \mathcal{Z}_{\mathrm{TQFT}}[F]$$

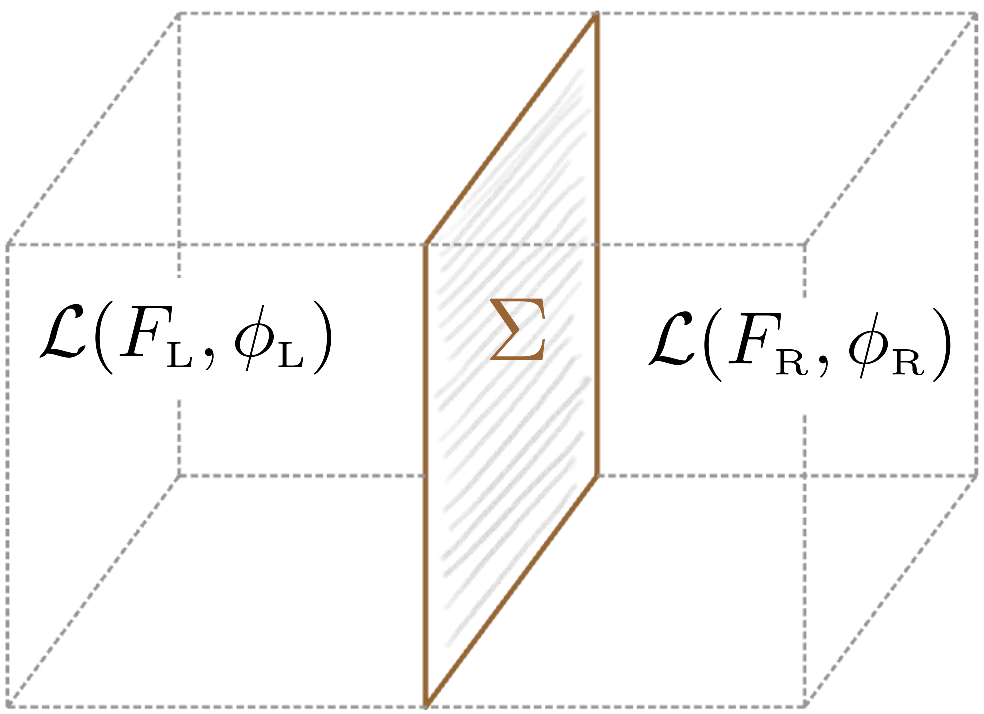
discrete 1-form symmetry anomaly fixed by P, N

see also [Córdova-Ohmori '22, Choi-Lam-Shao '23, Hsin-Lam-Seiberg '18] fusing $\mathcal{W}_{\Omega}(\Sigma)$, $\mathcal{W}_{\mathsf{A}}(\Sigma)$, $\mathcal{W}_{\mathsf{C}}(\Sigma)$ → $\mathcal{W}_{\mathcal{S}}(\Sigma)$ $\forall \mathcal{S} \in \mathrm{Sp}(2n, \mathbb{Q})$

fusing $\mathcal{W}_{\Omega}(\Sigma)\,,\,\mathcal{W}_{\mathsf{A}}(\Sigma)\,,\,\mathcal{W}_{\mathsf{C}}(\Sigma) \longrightarrow \mathcal{W}_{\mathcal{S}}(\Sigma) \quad \forall \mathcal{S} \in \mathrm{Sp}(2n,\mathbb{Q})$

$\forall \mathcal{S} \in \mathscr{G}_{\mathbb{Q}} \equiv \mathscr{G} \cap \mathrm{Sp}(2n,\mathbb{Q})$

$$\mathcal{D}_{\mathcal{S}} = \mathcal{U}_{\mathcal{S}}(\Sigma) \times \mathcal{W}_{\mathcal{S}}(\Sigma)$$



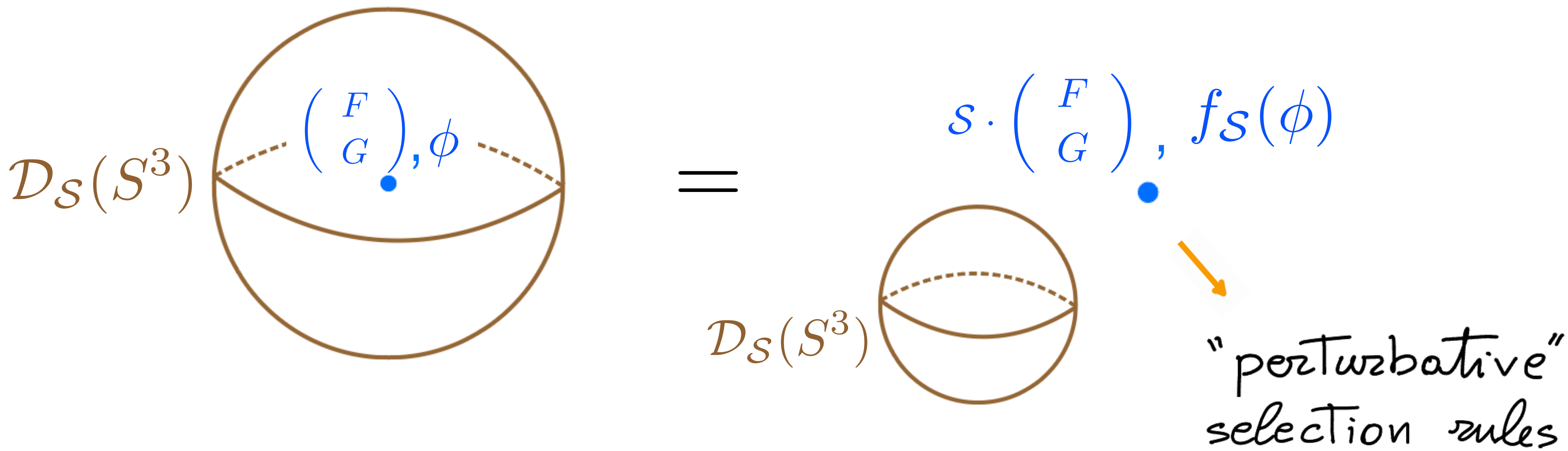
$$\begin{cases} \begin{pmatrix} F_{\mathrm{L}} \\ G_{\mathrm{L}} \end{pmatrix} = \mathcal{S} \cdot \begin{pmatrix} F_{\mathrm{R}} \\ G_{\mathrm{R}} \end{pmatrix} \\ \phi^{i}_{\mathrm{L}} = f^{i}_{\mathcal{S}}(\phi_{\mathrm{R}}) \end{cases}$$

GZ topological defect $\forall \mathcal{S} \in \mathscr{G}_{\mathbb{Q}}$

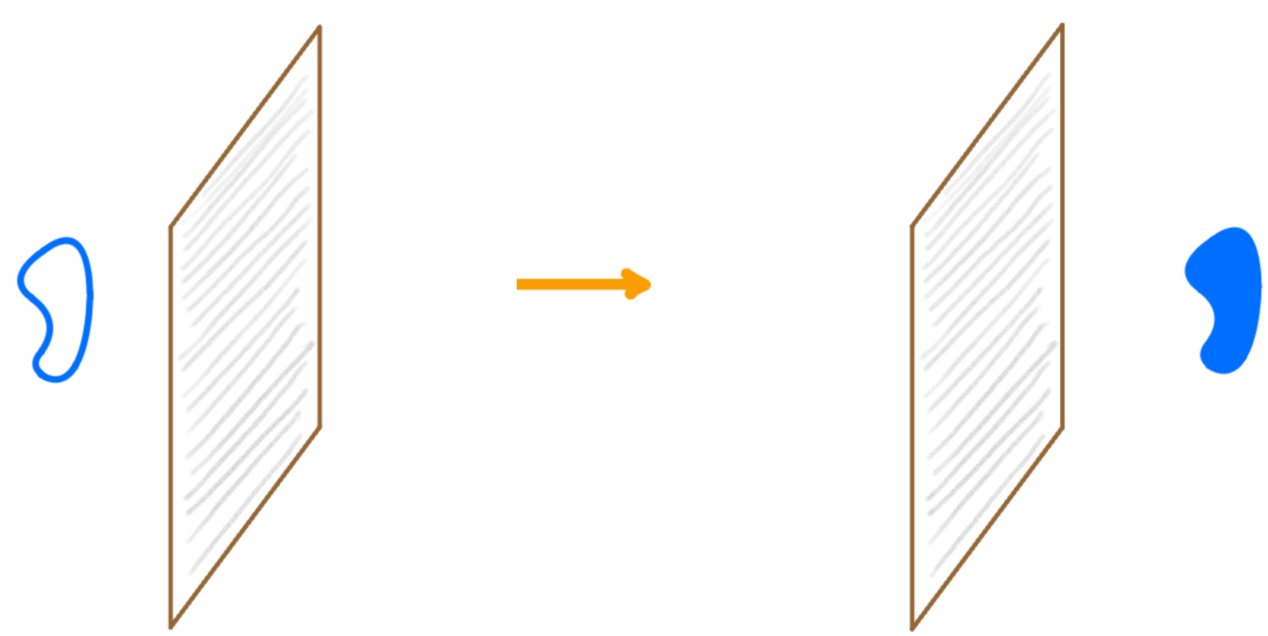
- $\mathcal{D}_{\mathcal{S}} \times \mathcal{D}_{\mathcal{S}^{-1}} = \mathcal{W}_{\mathcal{S}} \times \mathcal{W}_{\mathcal{S}^{-1}} \sim$ "condensation defect"

[Roumpedakis-Seifnashri-Shao'22]

- "Invertible" action on local operators



- Non-invertible action on extended operators



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[Roumpedakis-Seifnashri-Shao'22]

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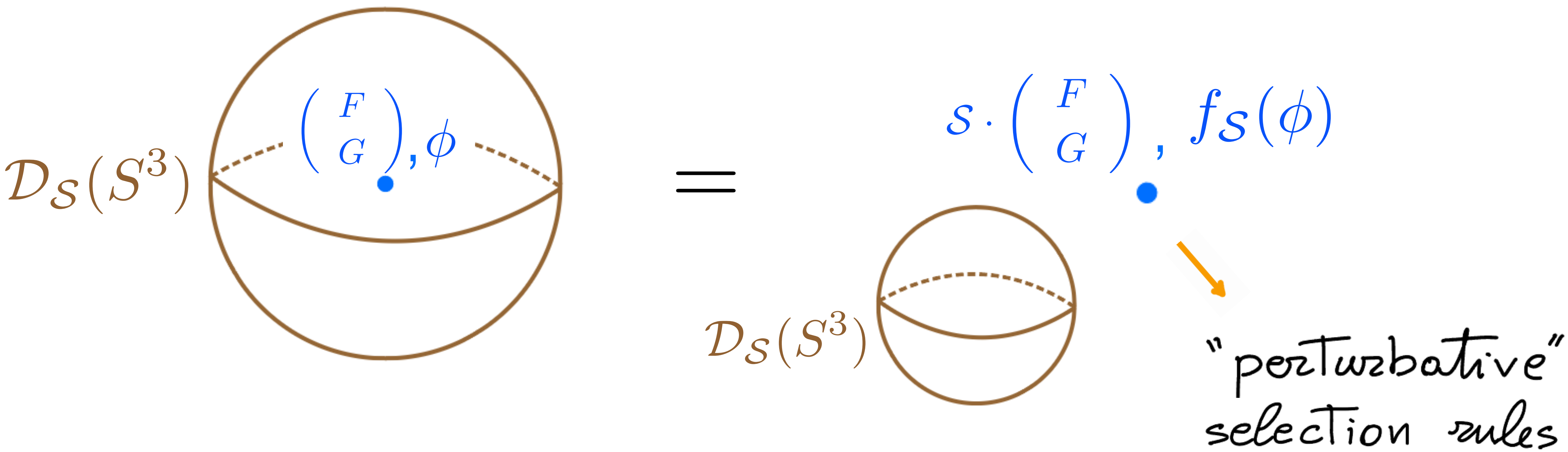

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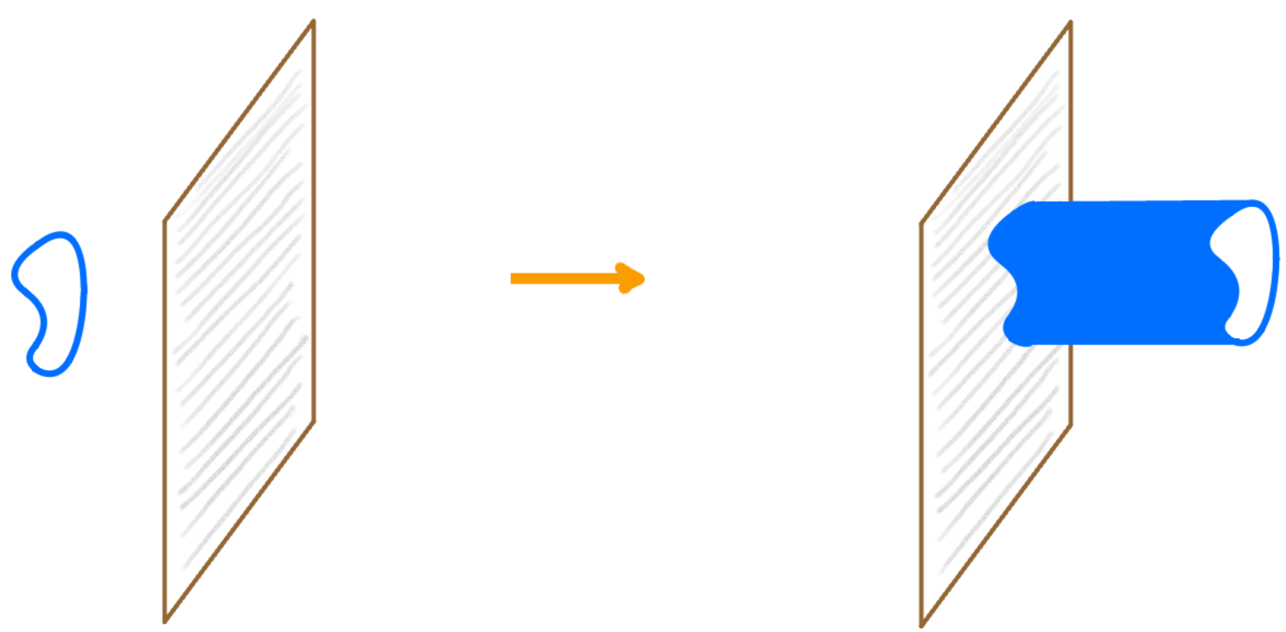


Summary & Outlook

- Proof of principle on existence of GZ non-invertible symmetries

- Large degeneracy: higher structure of GZ symmetries?

  e.g. [Del Zotto-Dell'Acqua-Garding '25] for axion-Maxwell

- GZ-symmetry breaking mechanisms in quantum gravity/ string theory models?

  [Apruzzi-Grieco-LM-Valenzuela] in progress

  * $\mathscr{G}_{\mathbb{Z}} \equiv \mathscr{G} \cap \mathrm{Sp}(2n, \mathbb{Z})$ gauged
  * $\mathscr{G}_{\mathbb{Q}} \equiv \mathscr{G} \cap \mathrm{Sp}(2n, \mathbb{Q})$ broken

  - asymptotic/perturbative regimes?
  - naturalness of small couplings?
  - swampland distance conjecture?

  [Vafa '04, Ooguri-Vafa '06, …, Raman-Vafa '24, Delgado-van de Heisteeg-Raman-Torres-Vafa-Xu '24, Baines-Collazuol-Fraiman-Graña-Waldram '25, …]

Thanks!

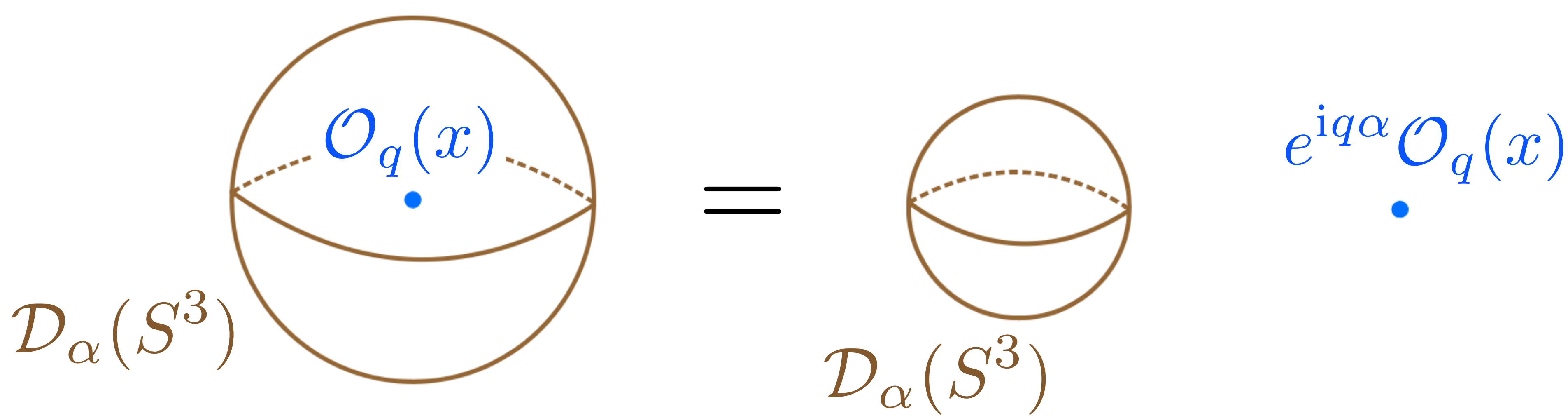
- Symmetries ⟷ topological defects/operators

[Gaiotto-Kapustin-Seiberg-Willett '14]

e.g. $\mathscr{G} = \mathrm{U}(1)$, $\mathrm{d}\mathcal{J}_3 = 0$

$$\mathcal{D}_\alpha(\Sigma) = \exp(\mathrm{i}\alpha \oint_\Sigma \mathcal{J}_3)$$

(0-form symmetry)



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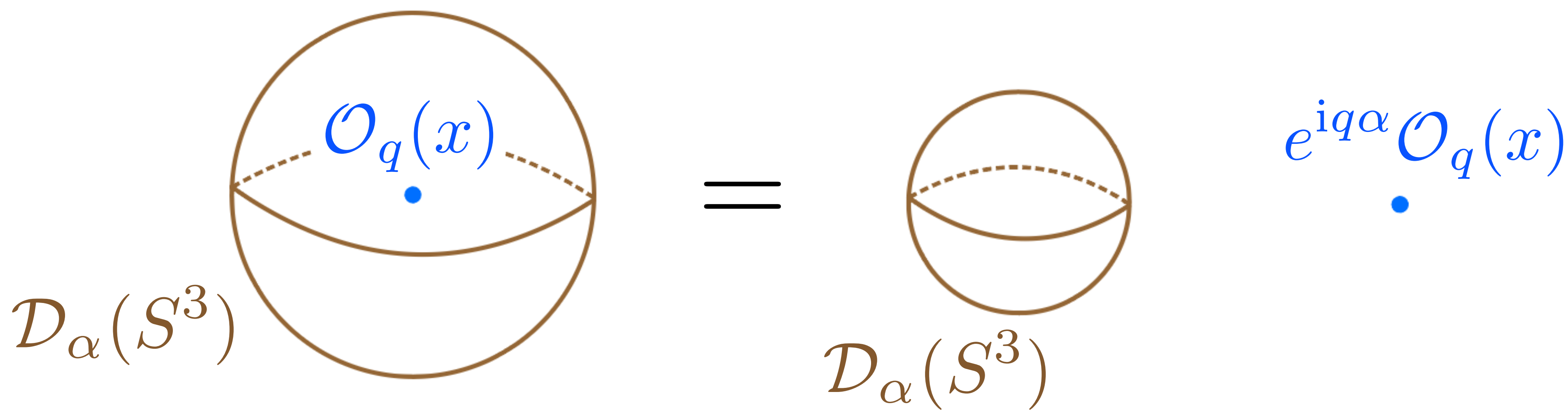


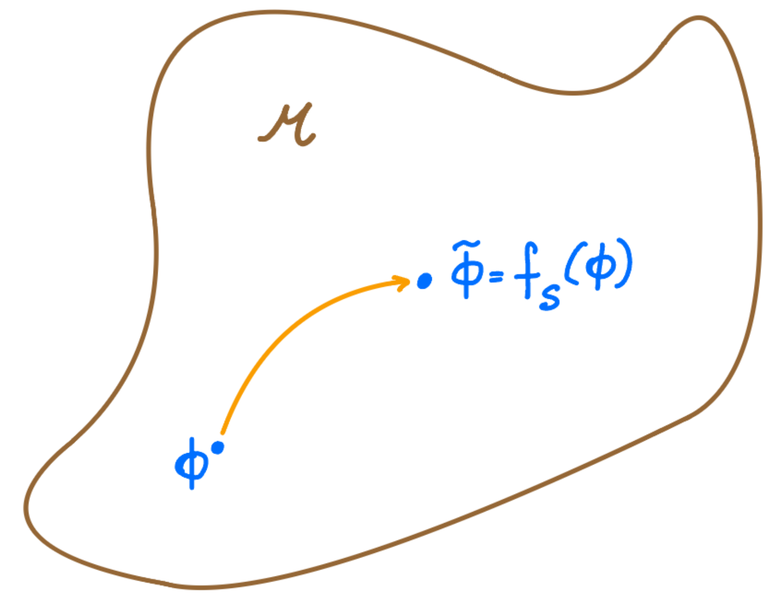
- One may impose: $\mathcal{D}_g(\Sigma) \times \mathcal{D}_{g'}(\Sigma) = \mathcal{D}_{gg'}(\Sigma)$ $\quad g, g' \in \mathscr{G}$

fusion product ~ group law → invertible !

- GZ Lagrangians are not invariant, but is a specific way

GZ-condition

$$\mathcal{L}(F, f_{\mathcal{S}}(\phi)) = \mathcal{L}_{\mathcal{S}}(F, \phi)$$



e.g.

$$\mathcal{L} = -\frac{1}{4\pi}\mathrm{Im}\,\tau\, F \wedge *F - \frac{1}{4\pi}\mathrm{Re}\,\tau\, F \wedge F - M^2\,\frac{\mathrm{d}\tau \wedge *\mathrm{d}\bar{\tau}}{(\mathrm{Im}\,\tau)^2}$$

$$\mathcal{L}_{\mathcal{S}} = -\frac{1}{4\pi}\mathrm{Im}\left(\frac{d\tau - c}{a - b\tau}\right) F \wedge *F - \frac{1}{4\pi}\left(\frac{d\tau - c}{a - b\tau}\right) F \wedge F - M^2\,\frac{\mathrm{d}\tau \wedge *\mathrm{d}\bar{\tau}}{(\mathrm{Im}\,\tau)^2}$$

$$\mathcal{S} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathscr{G} = \mathrm{SL}(2, \mathbb{R})$$

DUAL ONLY IF …

Examples

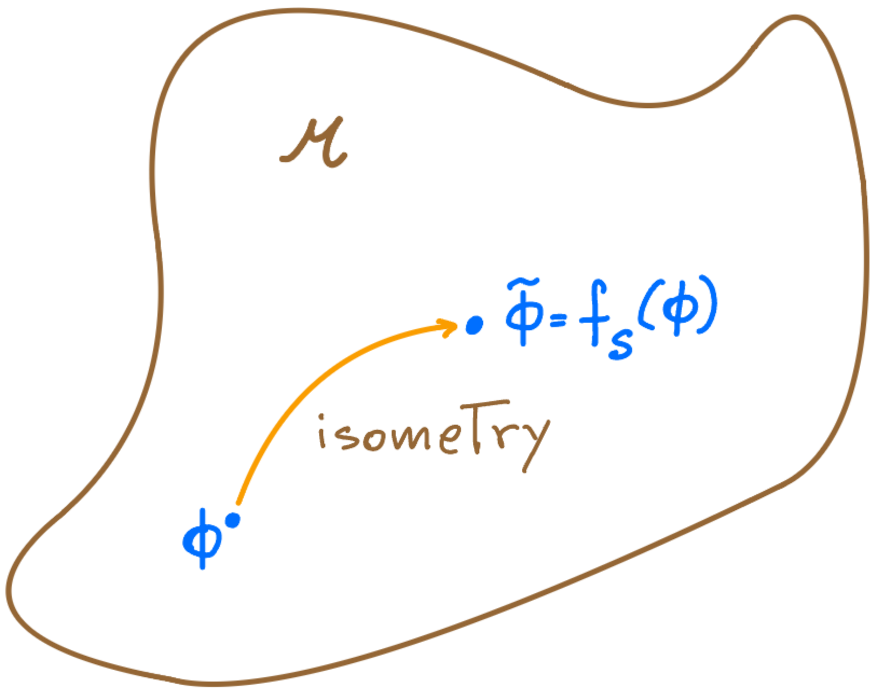
- $$\mathcal{L} = \frac{1}{4\pi}\mathrm{Im}\mathcal{N}_{IJ}(\phi)F^I \wedge * F^J + \frac{1}{4\pi}\mathrm{Re}\mathcal{N}_{IJ}(\phi)F^I \wedge F^J + \ldots$$

* $$G_I = \mathrm{Im}\mathcal{N}_{IJ}(\phi) * F^J + \mathrm{Re}\mathcal{N}_{IJ}(\phi)F^J$$

* $$\mathcal{N}_{IJ}(\tilde{\phi}) = \left(\left[\mathsf{C} + \mathsf{D}\mathcal{N}(\phi)\right]\left[\mathsf{A} + \mathsf{B}\mathcal{N}(\phi)\right]^{-1}\right)_{IJ}$$

$$\mathscr{G} \hookrightarrow \mathrm{Sp}(2n, \mathbb{R})$$

$$\begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{R})$$



e.g. realized in several extended supergravities

[Cremmer-Julia ’79, ...]

- Symmetry is broken in a specific way

$$\mathrm{d}\mathcal{J}_3 = \frac{1}{4\pi}\left(W_{IJ}F^I \wedge F^J - V^{IJ}G_I \wedge G_J - U^I{}_J F^J \wedge G_I + Y_J{}^I F^J \wedge G_I\right)$$

$$\begin{pmatrix} U & V \\ W & Y \end{pmatrix} \in \mathrm{sp}(2n, \mathbb{R})$$

Trivial fluxes can be reabsorbed in redefinition of $\mathcal{J}_3$

- EFT realization and extension of $\mathrm{O}(d,d;\mathbb{Q})$ world-sheet topological lines

[Bachas-Brunner-Roggenkamp '12]

- $\mathcal{W}_{\mathsf{A}}(\Sigma)$, $\mathcal{W}_{\mathsf{C}}(\Sigma)$ can be obtained from half-gauging

~ [Choi-Cordova-Hsin-Lam-Shao '21-'22, Córdova-Ohmori '22]