# On a class of parabolic geometries

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CFT PAN

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- Second part will be about a joint work with Omid Makhmali on *Parabolic quasi-contact cone structures with transversal infinitesimal symmetry*

# Definition: Parabolic Geometries

Let $G/P$ be a homogeneous space.

A Cartan geometry $(\mathcal{G}, \omega)$ of type $(G, P)$ is given by

- a principal bundle $\pi : \mathcal{G} \to M$ with structure group $P$ and
- a Cartan connection, i.e., a 1-form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ with values in $\mathfrak{g} = \mathrm{Lie}(G)$ which satisfies
  - $P$-equivariance: $(r^p)^*\omega = \mathrm{Ad}(p)^{-1} \circ \omega$,
  - $\omega(\zeta_X) = X \ \forall X \in \mathfrak{p}$, where $\zeta_X = \frac{d}{dt}|_{t=0} r^{\exp tX}(u)$
  - $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is an isomorphism $\forall u \in \mathcal{G}$.

A Cartan connection provides an identification

$$\mathcal{G} \times_P \mathfrak{g}/\mathfrak{p} \cong TM, \quad [u, X + \mathfrak{p}] \mapsto T_u\pi\, \omega_u^{-1}(X)$$

The homogeneous model of Cartan geometries of type $(G, P)$ is the principal bundle $\pi : G \to G/P$ equipped with Maurer Cartan form $\omega^G$.

The curvature

$$K = d\omega + \tfrac{1}{2}[\omega, \omega]$$

vanishes iff $(\mathcal{G}, \omega)$ it is locally isomorphic to its homogeneous model.

# Definition: Parabolic Geometries

A parabolic geometry is a Cartan geometry of type $(G, P)$, where $G$ is a semisimple Lie group and $P \subset G$ a parabolic subgroup.

Let $\mathfrak{g}$ be a semisimple Lie algebra with a $|k|$-grading

$$\mathfrak{g} = \underbrace{\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}}_{\mathfrak{g}_-} \oplus \mathfrak{g}_0 \oplus \underbrace{\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k}_{\mathfrak{g}_+}, \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$$

such that $\mathfrak{g}_{-1}$ generates $\mathfrak{g}_-$.

Defining $\mathfrak{g}^i := \mathfrak{g}_i \oplus \mathfrak{g}_{i+1} \oplus \cdots \oplus \mathfrak{g}_k$ one obtains a filtration

$$\mathfrak{g}^k \subset \cdots \subset \mathfrak{g}^0 \subset \cdots \subset \mathfrak{g}^{-k}$$

A subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is parabolic iff it is of the form $\mathfrak{p} = \mathfrak{g}^0 = \mathfrak{g}_0 \oplus \mathfrak{g}_+$ for some $|k|$-grading as above.

## Equivalence of categories (Tanaka, Morimoto,...,Čap-Slovák)

$$\left\{\begin{array}{c}\text{regular normal parabolic}\\ \text{geometries of type } (G,P)\end{array}\right\} \longleftrightarrow \{\text{ underlying structures }\}$$

**Underlying structure** (most cases):

- a filtration of the tangent bundle

$$T^{-1}M \subset \cdots \subset T^{-k}M = TM,$$

  s.t. the symbol algebra $\mathrm{gr}(T_xM) = \bigoplus_i T^i_xM/T^{i+1}_xM$ equipped with bracket induced by Lie bracket of vector fields is isom. to $\mathfrak{g}_-$ $\forall x \in M$, $(\mathfrak{g}^i := \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k \rightsquigarrow T^iM \cong \mathcal{G} \times_P \mathfrak{g}^i/\mathfrak{p}$ via $\omega)$
- reduction of structure group of the graded frame bundle,

$$\mathcal{F}_x = \{\text{graded Lie alg. isomorphism } \phi : \mathfrak{g}_- \to \mathrm{gr}(T_xM)\},$$

  with respect to $\mathrm{Ad} : G_0 \to \mathrm{Aut}_{gr}(\mathfrak{g}_-)$.

## Conformal structure:

equivalence class $[g]$ of metrics of sig. $(p, q)$ $(p + q > 2)$ on $M$, where

$$\hat{g} \sim g \iff \hat{g} = \Omega^2 g, \quad \text{for some} \quad 0 < \Omega \in C^\infty(M, \mathbb{R}).$$

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## Theorem (E. Cartan)

A conformal structure of signature $(p, q)$, $p + q > 2$, determines a canonical parabolic geometry of type $(\mathrm{PO}(p + 1, q + 1), P)$.

**Homogeneous model**: $G/P$, where $G = \mathrm{PO}(p + 1, q + 1)$, $P = P_1$ stabilizer of null-line in $\mathbb{R}^{p+1,q+1}$

$$\mathfrak{so}(p+1, q+1) = \left\{ \begin{pmatrix} \mu & Z^t & 0 \\ Y & M & -Z \\ 0 & -Y^t & -\mu \end{pmatrix} \right\}$$

$$= \mathfrak{g}_{-1} \oplus \underbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1}_{\mathfrak{p}} \qquad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$$

## (2,3,5) distribution:

subbundle $\mathcal{D} \subset TM$ of the tangent bundle of 5-mf $M$ s.t.

$$\mathrm{rank}(\underbrace{\mathcal{D}}_{T^{-1}M}) = 2, \quad \mathrm{rank}(\underbrace{[\mathcal{D},\mathcal{D}]}_{T^{-2}M}) = 3, \quad \mathrm{rank}(\underbrace{[\mathcal{D},[\mathcal{D},\mathcal{D}]]}_{T^{-3}M}) = 5.$$

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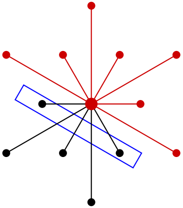
1893 Cartan and Engel (same journal, independent articles): Lie algebra of inf. symmetries of $\mathcal{D} = \mathrm{span}(X_4, X_5)$, where

$$X_4 = \partial_q, \quad X_5 = \partial_x + p\partial_y + q\partial_p + \tfrac{1}{2}q^2\partial_z,$$

is the **exceptional Lie algebra** of type $\mathrm{G}_2$.



**Theorem (Cartan, Tanaka,...)**

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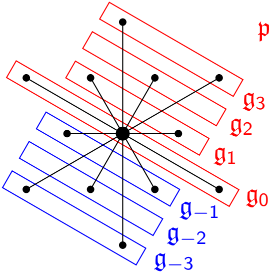
**Homogeneous model**: $\mathrm{G}_2/P$, where $\mathrm{G}_2$ is the split real form of the 14-dim. exceptional Lie group and $P = P_1 \subset G_2$ the parabolic subgroup




$$\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

Consider surfaces $\Sigma_1 \subset \mathbb{R}^3$ and $\Sigma_2 \subset \mathbb{R}^3$ rolling one on another w.o. slipping or twisting. The configuration space is a 5-manifold and rolling motions correspond to curves tangent to a rank 2 distribution $\mathcal{D}$ (the rolling distribution).

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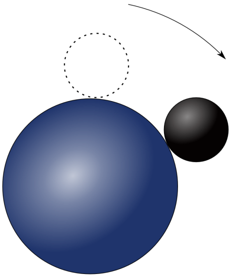

(Picture Bor-Montgomery)

Example: rolling balls

- If $r_1 \neq r_2$ the rolling distribution is $(2,3,5)$.
  The infinitesimal symmetry algebra of $\mathcal{D}$ is $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$,
- except if the ratio of radii is $1:3$, in which case it is the 14-dim. exceptional Lie algebra of type $G_2$.

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subbundle $\mathcal{D} \subset TM$ of the tangent bundle of 6-mf $M$ s.t.

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# Example: $(3,6)$ distributions

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## Theorem (Nurowski, Bryant)

Both $(2,3,5)$ and $(3,6)$ distributions determine canonical conformal structures of signature $(2,3)$ and $(3,3)$, respectively.

Given indef. signature $(M, [g])$, consider the **projectivized null-cone bundle**

$$\pi : \tilde{\mathcal{C}} \to \tilde{M}, \quad \tilde{\mathcal{C}}_x = \{[v] \in \mathbb{P}(T_x\tilde{M}) : g(v,v) = 0\}.$$

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$$T^{-1}_{(x,[v])}\tilde{\mathcal{C}} := \{X \in T_{(x,[v])}\tilde{\mathcal{C}} : \pi_* X \in \mathbb{R}v\}.$$

It splits into the vertical bundle $\tilde{\mathcal{V}}$ and a line bundle $\tilde{\mathcal{E}}$ corresponding to the foliation by lifts of **null-geodesics**.

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Corank 1 distribution $\tilde{\mathcal{H}} = T^{-2}\tilde{\mathcal{C}}$ is **quasi-contact**, i.e.
$\mathcal{L} : \Lambda^2\tilde{\mathcal{H}}^* \to T\tilde{\mathcal{C}}/\tilde{\mathcal{H}}$, $\mathcal{L}(X,Y) = pr([X,Y])$, has max. rank rank $\Longrightarrow$ 1-dim. kernel; $\tilde{\mathcal{E}} = \ker(\mathcal{L}|_{\tilde{\mathcal{H}}})$ is characteristic line bundle.

Similarly, given $\mathcal{D} \subset T\tilde{M}$, consider the **projectivized distribution bundle**

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$(k = 4, 5)$. Here $\tilde{\mathcal{V}}$ is the vertical bundle. The corank 1 distribution

$$\tilde{\mathcal{H}} = T^{-k+1}\tilde{\mathcal{C}}$$

is **quasi-contact** $\implies$ 1-dimensional kernel, $\tilde{\mathcal{E}} = \ker(\mathcal{L}|_{\tilde{\mathcal{H}}})$ (characteristic line bundle); its integral curves are called **abnormal extremals** of $\mathcal{D}$ (Zelenko).

The spaces $\tilde{\mathcal{C}}$ from previous slides are correspondence spaces.

Let $Q \subset P \subset G$ be nested parabolic subgroups ($P/Q$ connected)

Given a Cartan geometry $(\mathcal{G} \to M, \omega)$ of type $(G, P)$, then the correspondence space is

$$\tilde{\mathcal{C}} = \mathcal{G}/Q \cong \mathcal{G} \times_P (P/Q),$$

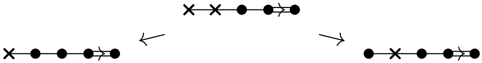
$\tilde{\mathcal{C}} \to M$ is fibre bundle with fibre $P/Q$ and $(\mathcal{G} \to \tilde{\mathcal{C}}, \omega)$ is Cartan geometry of type $(G, Q)$.

## Proposition

The correspondence space construction gives local equivalence between reg. normal parabolic geometries of type $(G, P)$ and reg. normal parabolic geom. of type $(G, Q)$ (see next slide) in the distribution cases and a subclass of such geometries in the conformal case.

On the top the Lie group pairs $(G, Q)$:

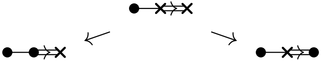
1. $(SO(p+2, q+2), Q = P_{12})$ (odd and even)



   exceptional case: $n = 4$ here 

2. $(G_2, Q = P_{12})$

3. $(SO(3,4), Q = P_{23})$



bottom left: Lie group pairs $(G, P)$ for conformal, $(2, 3, 5)$ and $(3, 6)$.
bottom right: $(G, R)$ corr. to contact gradings

# Parabolic quasi-contact cone structures

Parabolic geometries of type $(G, Q)$ are equivalent to bracket generating distributions with weak derived

$$T^{-1}\tilde{\mathcal{C}} \subset T^{-2}\tilde{\mathcal{C}} \subset \cdots \subset T^{-k}\tilde{\mathcal{C}} = T\tilde{\mathcal{C}}$$

and symbol algebra $\mathfrak{q}_-$. We call these structures parabolic quasi-contact cone structures.

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**Structure of gradings**:

$$\mathfrak{g} = \underbrace{\mathfrak{q}_{-k} \oplus \cdots \oplus \mathfrak{q}_{-1}}_{\mathfrak{q}_-} \oplus \mathfrak{q}_0 \oplus \mathfrak{q}_1 \oplus \cdots \oplus \mathfrak{q}_k, \quad \dim(\mathfrak{q}_{\pm k}) = 1, \text{ where}$$

- $[,]_{-k} : \Lambda^2(\mathfrak{q}_{-k+1} \oplus \cdots \oplus \mathfrak{q}_{-1}) \to \mathfrak{q}_{-k}$ is 2-form of **maximal rank** with 1-dim. kernel $\mathfrak{e}$
- **splitting** $\mathfrak{q}_{-1} = (\mathfrak{q}_{-1} \cap \mathfrak{r}_-) \oplus (\mathfrak{q}_{-1} \cap \mathfrak{r}_0) = \mathfrak{v} \oplus \mathfrak{e}$
- The brackets $\mathfrak{e} \otimes \mathfrak{v} \to \mathfrak{q}_{-2}$, $\mathfrak{e} \otimes \mathfrak{q}_{-i} \to \mathfrak{q}_{-i-1}$, for $i < k-1$, define isomorphisms (and these are all non-triv. brackets on $\mathfrak{q}_-$)

# Symmetry Reduction

Let $\xi$ be a conformal Killing field resp. infinitesimal symmetry of the distribution $\mathcal{D} \subset T\tilde{M}$.

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Question:

What is the induced geometric structure on the local leaf space?

$\rightsquigarrow$ adapt results by Čap and Salač parabolic conformally symplectic structures to our setting

# Parabolic conformally quasi-symplectic structures

A parabolic conformally quasi-symplectic structure (PCQS) is given by:

a) a bracket generating distribution $T^{-1}\mathcal{C}$ with weak derived

$$T^{-1}\mathcal{C} \subset T^{-2}\mathcal{C} \subset \cdots \subset T^{-k+1}\mathcal{C} = T\mathcal{C}$$

and symbol algebra $\mathrm{gr}(T_x\mathcal{C}) =: \mathfrak{s}_- \cong \mathfrak{q}_-/\mathfrak{q}_{-k}$

b) a reduction of structure group of the graded frame bundle with respect to $Q_0 \hookrightarrow Aut_{gr}(\mathfrak{s}_-)$

c) $\rightsquigarrow$ canonical line bundle $\ell \subset \Lambda^2 T^*\mathcal{C}$ s.t. each $\phi_x \in \ell_x$ has max. rank and one further requires that $\ell$ has closed local sections.

**Theorem**

The quotient of a parabolic quasi-contact cone structure by a transversal symmetry $\xi \in \mathfrak{X}(\tilde{\mathcal{C}})$ has a natural PCQS structure. Conversely, locally, any PCQS structure can be realized as a quotient of a quasi-contact cone structure.

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- Let $\alpha \in \Omega^1(\tilde{\mathcal{C}})$ be quasi-cont. form, i.e. $\ker(\alpha) = \tilde{\mathcal{H}}$, s.t. $\alpha(\xi) = 1$.

$$d\alpha(\xi, \eta) = -\alpha([\xi, \eta]) \; \forall \eta \in \Gamma(\tilde{\mathcal{H}}) \implies \iota_\xi d\alpha = 0 \implies \mathcal{L}_\xi d\alpha = 0$$

$\implies$ $d\alpha$ descends to **closed** 2-**form** $\Omega$.

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- For the converse, suppose $\Omega = d\beta \in \Gamma(\ell)$ on $U \subset \tilde{\mathcal{C}}$, define

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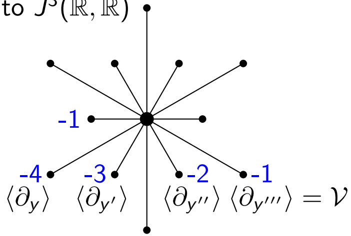
Then $T^{-k}\tilde{\mathcal{C}} := \ker(\alpha)$ is quasi-contact structure, the rest of the filtration lifts and has symmetry $\partial_t$.

Filtration on $\mathcal{C}$ corresponds to 4th order ODE $y'''' = f(x, y, y', y'', y''')$ up to contact transformations:

ODE def. submf $\mathcal{C} \subset J^4(\mathbb{R}, \mathbb{R})$ diff. to $J^3(\mathbb{R}, \mathbb{R})$

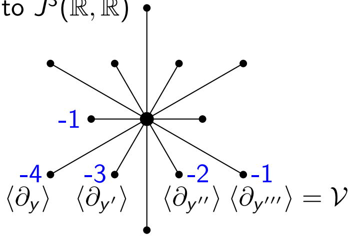
$$\langle \partial_x + y'\partial_y + \cdots + f\partial_{y'''} \rangle = \mathcal{E}$$



Filtration on $\mathcal{C}$ corresponds to 4th order ODE $y'''' = f(x, y, y', y'', y''')$ up to contact transformations:

ODE def. submf $\mathcal{C} \subset J^4(\mathbb{R}, \mathbb{R})$ diff. to $J^3(\mathbb{R}, \mathbb{R})$

$\langle \partial_x + y'\partial_y + \cdots + f\partial_{y'''} \rangle = \mathcal{E}$




Closed 2-form $\Omega$ (with given algebraic structure) $\implies$ ODE is variational, i.e., up to multiple EL equations of non-deg. second order Lagrangian

$$L = L(x, y, y', y''), \quad \frac{\partial^2 L}{\partial y''^2} \neq 0.$$

(Anderson-Thompson, Fels)

### Corollary

The leaf space $\mathcal{C}$, obtained from a $(2,3,5)$ distribution and an inf. symmetry, inherits the structure of a scalar 4th order ODE modulo contact transformations, which is variational. Conversely, any variational scalar 4th order ODE geometry can be locally realized in this way.

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The geometry of a scalar 4th order ODE has 4 (scalar) fundamental invariants:

- $\mathbf{w}_3$, $\mathbf{w}_4$ (Wünschman invariants)
- $\mathbf{c}_3$, $\mathbf{c}_4$

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The 4th order ODE is variational $\iff$ $\mathbf{w}_3 = 0$ and $\mathbf{c}_3 = 0$ (Fels)

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The 4th order ODE is variational $\iff$ $\mathbf{w}_3 = 0$ and $\mathbf{c}_3 = 0$ (Fels)
The corresponding $(2,3,5)$ distribution is flat $\iff$ in addition $\mathbf{w}_4 = 0$.

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